**The Spherical Harmonics in Cartesian Coordinates** PhL 11.4.10

This doc is filed with other angular momentum related docs, currently located here

D:\Work\My Interests\Physics\Quantum Mechanics\angmom_spin math stuff

My interest in this subject arose while I was reading Hobson's work on ellipsoidal coordinates and the Lamé functions. The first step in finding the general form of the Lamé E functions involves expressing the tesseral harmonics in Cartesian coordinates!


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**Overview**

In **Section 1** I dismiss normalization constants, consider range of m, and set up our problem, and discuss the word "tesseral".

For **Section 2** I wrote a Maple program to print out arbitrary tesseral harmonics. Maple does not know these functions a priori, since its LegenderP routine just an evaluator, so I had to do this myself. I then show all the tesseral harmonics for n = 0 through 5.

In **Section 3** I convert a few specific cases of tesserals to Cartesian coordinates, and then I conjecture that all tesseral harmonics when expressed in Cartesian coordinates are homogeneous polynomials of degree n in x,y,z divided by $r^n$. In Section 5 below I show this to be true.

In **Section 4** I write down the $L^2$ angular momentum operator in x,y,z and implement it in Maple. I then apply this $L^2_{x,y,z}$ to two of my Section 3 sample tesseral harmonics f converted to x,y,z to verify that we really do find that $L^2_{x,y,z} f = n(n+1)f$. This was just confidence-building stuff.

In **Section 5** I undertake the general problem of converting an arbitrary tesseral harmonic to Cartesian coordinates. This is not rocket science, because I have all the pieces as follows (last two from GR7)

$$\partial_x^m P_n(x) = a\cos^{n-m}\theta + b\cos^{n-m-2}\theta + c\cos^{n-m-4}\theta + \ldots \qquad x = \cos\theta$$
$$\cos(m\varphi) = a\cos^m\varphi + b\cos^{m-2}\varphi + c\cos^{m-4}\varphi + \ldots.$$
$$\sin(m\varphi) = \sin\varphi\,[d\cos^{m-1}\varphi + e\cos^{m-3}\varphi + f\cos^{m-5}\varphi + \ldots.]$$

I am then able to show that both tesseral harmonics have the following form

$$P_n^m(\cos\theta)\cos(m\varphi) = \text{poly}_n(x/r, y/r, z/r; n, m) = (1/r)^n\,\text{Poly}_n(x,y,z; n,m)$$
$$P_n^m(\cos\theta)\sin(m\varphi) = \text{poly}_n(x/r, y/r, z/r; n, m) = (1/r)^n\,\text{Poly}_n(x,y,z; n,m)$$

Here, $poly_n(a,b,c)$ has terms like $a^n$ and b. The notation $Poly_n$ however means that all terms in the polynomial have the same degree, so Poly means a "homogeneous" polynomial. This of course must be the case since the tesserals are dimensionless. I then combine the above lines to say

$$P_n^m(\cos\theta)\,[\cos(m\varphi), \sin(m\varphi)] \;=\; (1/r)^n\;[\,C_n(x,y,z;\,n,m),\, S_n(x,y,z;\,n,m)\,]$$

where in the last line I just combine the previous lines into one and give names to the homogeneous polynomials. The arguments n,m mean that the polynomial coefficients depend on both n and m. Thus, I have given a formal proof of the conjecture of Section 3 that all tesserals have this nice form (I call it a Theorem).

Next, a bit off the main flow, using $L_z = (-i)(x\partial_y - y\partial_x) = (-i)\partial_\varphi$ I show certain relations between the S and C polynomials:

$$L_z[C_n(x,y,z;\,n,m)/r^n] \;=\; i\,m\,[S_n(x,y,z;\,n,m)]\,/r^n]$$
$$L_z[S_n(x,y,z;\,n,m)/r^n] = -\,i\,m\,[C_n(x,y,z;\,n,m)]\,/r^n]$$

It is pretty obvious for example that $\cos(m\varphi)$ is not an eigenstate of $L_z$, which is why we normally use the $Y_{nm}(\theta,\varphi)$ functions which *are* eigenstates. But of course the tesserals are eigenstates of $L^2$.

I then do some Cartesian conversion examples,

n=1:

$P_1^1(\cos\theta)\cos(1\varphi) = \sin\theta\cos\varphi = (x/r)$ $\qquad$ $P_1^1(\cos\theta)\sin(1\varphi) = \sin\theta\sin\varphi = (x/r)$

$P_1^0(\cos\theta)\cos(0\varphi) = \cos\theta \quad = (z/r)$ $\qquad$ $P_1^0(\cos\theta)\sin(0\varphi) = 0$

n=2:

$P_2^2(\cos\theta)\cos(2\varphi) \sim x^2-y^2$ $\qquad$ $P_2^2(\cos\theta)\sin(2\varphi) \sim xy$

$P_2^1(\cos\theta)\cos(1\varphi) \sim xz$ $\qquad$ $P_2^1(\cos\theta)\sin(1\varphi) \sim zy$

$P_2^0(\cos\theta)\cos(0\varphi) \sim 2z^2-x^2-y^2$ $\qquad$ $P_2^0(\cos\theta)\sin(0\varphi) = 0$

Since the last item always vanishes, we see there are (n+1)*2-1 = 2n+2-1 = (2n+1) functions for each n. In the n=2 case linear combinations give the set { xz, xy, zy, $x^2-y^2$, $z^2- y^2$ }. Each term of course should be divided by $r^2$.

---

**1. Introduction.**

In this document, we are not concerned about the normalizing factors, but more with the general "form" that the spherical harmonics have when expressed in Cartesian Coordinates. So let's start with

$$Y_{nm}(\theta,\varphi) = c_{n,m}\; P_n^m(\cos\theta)\; e^{+im\varphi} \qquad L^2\, Y_{nm}(\theta,\varphi) = n(n+1)\, Y_{nm}(\theta,\varphi) \qquad n = 0,1,2...$$

Since $L_z = (-i)\partial_\varphi$ in our usual Lie Algebra convention, if we want $L_zY = mY$, we must use the positive phase shown. The understanding here is that m ranges from -m to m. It is convenient to deal with the real

and imaginary parts of Y separately, and I tend to refer to these real functions (without $c_{n,m}$) as "the tesseral harmonics",

$$\mathrm{Re}[Y_{nm}(\theta,\varphi)] = c_{n,m}\ P_n^m(\cos\theta)\cos(m\varphi)$$
$$\mathrm{Im}[Y_{nm}(\theta,\varphi)] = c_{n,m}\ P_n^m(\cos\theta)\sin(m\varphi)$$

We know that $P_n^{-m}(\cos\theta) = k_{n.m}\ P_n^m(\cos\theta)$ for some constant $k_{n,m}$. Thus,

$$P_n^{-m}(\cos\theta)\cos(-m\varphi) = k_{n.m}\ P_n^m(\cos\theta)\cos(m\varphi)$$
$$P_n^{-m}(\cos\theta)\sin(-m\varphi) = -k_{n.m}\ P_n^m(\cos\theta)\sin(m\varphi)$$

If we are exploring the "functional form" of our tesseral harmonics, we see that the negative m functions are just constant multiples of the positive m ones, so we might as well restrict our interest to m = 0,1...n. In doing so, we dodge the famous Condon-Shortley phase question for the negative m P functions.

So we now dispense with overall constants, and we shall study these "tesseral harmonics" :

$$P_n^m(\cos\theta)\ [\cos(m\varphi), \sin(m\varphi)] \qquad n = 0,1,2... \qquad m = 0,1..n$$

With this limitation on m, we may use the following expression for the P functions.

$$P_n^m(\cos\theta) = (\sin\theta)^m\ [\partial_x^m\ P_n(x)]^{x=\cos\theta}$$

**History Note**. Here is a wiki quote concerning the tesserals

> A tesseral harmonic is a spherical harmonic of the form $\begin{matrix}\cos\\ \sin\end{matrix}(m\,\phi)\,P_l^m(\cos\theta)$. These harmonics are so named because the curves on which they vanish are $l-m$ parallels of latitude and $2\,m$ meridians, which divide the surface of a sphere into quadrangles whose angles are right angles (Whittaker and Watson 1990, p. 392).

And from ODE

θ **tessella** (tE"sEl@). Pl. **-æ**; rarely **-as**. Also 8 **-ela**.
[L., dim. of tessera.]

θ **tessera** ("tEs@r@). Pl. **tesseræ**.
[L., f. Ionic Gr. σ□ρρεθε|, –θα, = Attic σ□ρρα θε|, –θα four.]
**1.** *Anc. Hist.* A small quadrilateral tablet of wood, bone, ivory, or the like, used for various purposes, as a token, tally, ticket, label, etc.

The induced meaning of a tessella is pretty much a "small tile" with four sides. We speak of "tiling" a surface. More particularly, the tiles of a sphere formed by latitudes and longitudes all have right angles, though the tiles are not planar, because r,θ,φ is an orthogonal coordinate system. A tiled surface and a tessellated surface are the same thing as far as I am concerned, the phrases have loose definitions.

## 2. Maple display of $P_n^m(\cos\theta)$

I don't know of any big tables except some data here which lists n = 0 to 8

http://ciks.cbt.nist.gov/~garbocz/paper134/node10.html

So I decided to get Maple to display $P_n^m(\cos\theta)$ for arbitrary n and m. I first wandered off to Maxima for a while, but then figured out how to Maple do what I wanted. The secret was using a sequence to get strings like x,x,x.. inside the diff function, and then various tricks to print out the functions using sinθ and cosθ appropriately. Here is the code, located in file "Pnm display.mws"

```
restart;
> with(orthopoly):
> p := (n,m,x)-> (1-x^2)^(m/2)*diff(P(n,x),seq(x,i=1..m)):
> for n from 0 to 5 do
>   for m from 0 to n do
>    unassign('x');
>    if (m = 0) then f := P(n,x); else f := p(n,m,x); fi;
>    x := cos(theta);
>    g := subs(sqrt(1-cos(theta)^2)=sin(theta),f):
>    printf("P(%d,%d,x)",n,m);
>    print(subs(1-cos(theta)^2=sin(theta)^2,g));
>   od
> od;
>
```

Here is some output of this little program for n = 0 to 5, presented here in two columns:

P(0,0,x)

$$1$$

P(1,0,x)

$$\cos(\theta)$$

P(1,1,x)

$$\sin(\theta)$$

P(2,0,x)

$$\frac{3}{2}\cos(\theta)^2 - \frac{1}{2}$$

P(2,1,x)

$$3\sin(\theta)\cos(\theta)$$

P(2,2,x)

$$3 - 3\cos(\theta)^2$$

P(3,0,x)

$$\frac{5}{2}\cos(\theta)^3 - \frac{3}{2}\cos(\theta)$$

P(3,1,x)

$$\sin(\theta)\left(\frac{15}{2}\cos(\theta)^2 - \frac{3}{2}\right)$$

P(3,2,x)

$$15\sin(\theta)^2\cos(\theta)$$

P(3,3,x)

$$15\left(\sin(\theta)^2\right)^{\left(\frac{3}{2}\right)}$$

P(4,0,x)

$$\frac{35}{8}\cos(\theta)^4 - \frac{15}{4}\cos(\theta)^2 + \frac{3}{8}$$

P(4,1,x)

$$\sin(\theta)\left(\frac{35}{2}\cos(\theta)^3 - \frac{15}{2}\cos(\theta)\right)$$

P(4,2,x)

$$\sin(\theta)^2\left(\frac{105}{2}\cos(\theta)^2 - \frac{15}{2}\right)$$

P(4,3,x)

$$105\left(\sin(\theta)^2\right)^{\left(\frac{3}{2}\right)}\cos(\theta)$$

P(4,4,x)

$$105\sin(\theta)^4$$

P(5,0,x)

$$\frac{63}{8}\cos(\theta)^5 - \frac{35}{4}\cos(\theta)^3 + \frac{15}{8}\cos(\theta)$$

P(5,1,x)

$$\sin(\theta)\left(\frac{315}{8}\cos(\theta)^4 - \frac{105}{4}\cos(\theta)^2 + \frac{15}{8}\right)$$

P(5,2,x)

$$\sin(\theta)^2\left(\frac{315}{2}\cos(\theta)^3 - \frac{105}{2}\cos(\theta)\right)$$

P(5,3,x)

$$\left(\sin(\theta)^2\right)^{\left(\frac{3}{2}\right)}\left(\frac{945}{2}\cos(\theta)^2 - \frac{105}{2}\right)$$

P(5,4,x)

$$945\sin(\theta)^4\cos(\theta)$$

P(5,5,x)

$$945\left(\sin(\theta)^2\right)^{\left(\frac{5}{2}\right)}$$

**3. Early notes: try to learn something about the nature of the tesserals, a few examples.**

Now let's look at a specific harmonic

$$P_2^2(\cos\theta)\cos(2\varphi) = 3\sin^2\theta\,[\,2\cos^2(\varphi) - 1\,] = 6\sin^2\theta\cos^2(\varphi) - 3\sin^2\theta$$

What patterns do we see in our expressions Maple has printed out?

- If m is even, then P is a polynomial in cos(θ) of degree n

I have not shown this here, but Maple will do expand( cos(8*φ)) stuff and you find that

• if m is even, then cos(mφ) is a polynomial of degree m in cosφ with only even powered terms

Example:

$P_5^4(\cos\theta)\cos(4\varphi) = 945 \sin^4\theta \cos\theta [ 8 \cos^4\varphi - 8\cos^2\varphi + 1 ]$

$= 945 \cos\theta [ 8 \sin^4\theta \cos^4\varphi - 8 \sin^4\theta \cos^2\varphi + \sin^4\theta ]$

$= 945 z [ 8 x^4 - 8 (r^2-z^2) x^2+(r^2-z^2)^2 ] /r^5 = 945 z [ 8 x^4 - 8 (x^2+y^2) x^2+( x^2+y^2)^2 ] /r^5$

$= 945 z [ 8 x^4 - 8 (x^2+y^2) x^2+( x^4 + 2x^2y^2+ y^4) ] /r^5$

$= 945 z [ - 8 (y^2) x^2+( x^4 + 2x^2y^2+ y^4) ] /r^5$

$= 945 z [x^4 -6x^2y^2+ y^4 ] /r^5$

Let's try another one

$P_5^4(\cos\theta)\sin(4\varphi) = 945 \sin^4\theta \cos\theta [ 8\sin\varphi\cos^3\varphi - 4\sin\varphi\cos\varphi ]$

$= 945 \cos\theta [ 8 \sin^4\theta \sin\varphi\cos^3\varphi - 4 \sin^2\theta \sin^2\theta \sin\varphi\cos\varphi ]$

$= 945 \cos\theta [ 8 \sin\theta \sin\varphi \sin^3\theta\cos^3\varphi - 4 \sin^2\theta \sin^2\theta \sin\varphi\cos\varphi ]$

$= 945 z [ 8 y x^3 - 4 (r^2-z^2)xy ] /r^5 = 945 z [ 8 y x^3 - 4 (x^2+y^2)xy ] /r^5$

$= 945 z [ 8 y x^3 - 4x^3y - 4xy^3 ] /r^5 = 4* 945 xyz [ x^2-y^2] = 3780 xyz [ x^2-y^2]$

What happens when m is odd?

$P_5^3(\cos\theta)\cos(3\varphi) = \sin^3\theta (A\cos^2\theta - B)[ 4 \cos\varphi^3-3\cos\varphi]$ A = 945/2 B = 105/2

$= (A\cos^2\theta - B)[ 4 \sin^3\theta \cos\varphi^3-3 \sin^3\theta \cos\varphi]$

$= (Az^2 - Br^2)[ 4x^3-3 (r^2-z^2) x] / r^5$

$P_5^3(\cos\theta)\sin(3\varphi) = \sin^3\theta (A\cos^2\theta - B) [ 4\sin\varphi \cos^2\varphi - \sin\varphi]$

$= (A\cos^2\theta - B) [ 4 \sin^3\theta \sin\varphi \cos^2\varphi - \sin^3\theta \sin\varphi]$

$= (Az^2- Br^2) [ 4x^2y - (r^2-z^2)y]/r^5$

**Question**: Are all tesseral harmonics equal to polynomials in x,y,z of degree n divided by $r^n$? I think the answer is yes. This will be proven below.

**4. Angular Momentum in Cartesian coordinates.**

Here I write down $L^2$ in Cartesians, and then show that some sample tesseral harmonics do in fact satisfy the expected relation $L^2 f = n(n+1)f$. [ When I first did this, I wrongly set r = 1 everywhere and this made things not work since $L^2$ certainly acts on powers of r. This has been quietly all repaired.]

We know how to write the angular momentum operator in Cartesian coordinates

$$- L^2 = (y\partial_z - z\partial_y)^2 + \text{cyclic}$$

We could consider doing the search in Cartesian space for the eigenfunctions

$$L^2 f(x,y,z) = n(n+1)f(x,y,z)$$

The operator $L^2$ is dimensionless in both Cartesian and angular coordinates. Therefore, if f(x,y,z) is a polynomial of degree N, then so is $L^2 f$. Therefore, if you are looking for eigenfunctions of $L^2 f = n(n+1)f$, a polynomial in x,y,z is certainly a good candidate! However, it turns out this is not the form that the eigenfunctions have. As we shall see, the eigenfunction f(x,y,z) is a polynomial in x,y,z of degree n divided by $r^n$. And of course there will be (2n+1) such eigenfunctions.

Let's now check a few of our test cases above using Maple to compute the Cartesian $L^2$. Here is our testing code. We start off with this

```
restart;
Lx := (f)-> y*diff(f,z) - z*diff(f,y);
```

$$Lx := f \to y\left(\frac{\partial}{\partial z} f\right) - z\left(\frac{\partial}{\partial y} f\right)$$

```
Ly := (f)-> z*diff(f,x) - x*diff(f,z);
```

$$Ly := f \to z\left(\frac{\partial}{\partial x} f\right) - x\left(\frac{\partial}{\partial z} f\right)$$

```
Lz := (f)-> x*diff(f,y) - y*diff(f,x);
```

$$Lz := f \to x\left(\frac{\partial}{\partial y} f\right) - y\left(\frac{\partial}{\partial x} f\right)$$

```
r := sqrt(x^2+y^2+z^2);
```

$$r := \sqrt{x^2 + y^2 + z^2}$$

Then we can test specific functions.

First,

$$g = P_5^3(\cos\theta)\cos(3\varphi) = 945\, z\, [\, 8\, x^4 - 8\, (r^2 - z^2)\, x^2 + (r^2 - z^2)^2\, ]\, / r^5$$

and Maple says (we omit leading constants)

```
g := z*(8*x^4 - 8*(r^2-z^2)*x^2 + (r^2-z^2)^2)/r^5;
```

$$g := \frac{z\left(8x^4 - 8\left(x^2+y^2\right)x^2 + \left(x^2+y^2\right)^2\right)}{\left(x^2+y^2+z^2\right)^{\left(\frac{5}{2}\right)}}$$

```
RHS := Lx(Lx(g)) + Ly(Ly(g)) + Lz(Lz(g)): simplify(%);
```

$$-30\,\frac{z\left(x^4 - 6x^2y^2 + y^4\right)}{\left(x^2+y^2+z^2\right)^{\left(\frac{5}{2}\right)}}$$

```
-RHS/g:simplify(%);
```

$$30$$

We have thus shown that $L^2 g = 30g$ and this is our expected result since 5(5+1) = 30.

Second,

$P_5^4(\cos\theta)\sin(4\varphi) = 945\, z\,[\,8\,y\,x^3 - 4\,(r^2-z^2)xy\,]\,/r^5$

```
g := z*(8*y*x^3 - 4*(r^2-z^2)*x*y)/r^5;
```

$$g := \frac{z\left(8yx^3 - 4\left(x^2+y^2\right)xy\right)}{\left(x^2+y^2+z^2\right)^{\left(\frac{5}{2}\right)}}$$

```
RHS := Lx(Lx(g)) + Ly(Ly(g)) + Lz(Lz(g)): simplify(%);
```

$$-120\,\frac{yxz\left(x^2-y^2\right)}{\left(x^2+y^2+z^2\right)^{\left(\frac{5}{2}\right)}}$$

```
-RHS/g:simplify(%);
```

$$30$$

**5. The General Forms of the Tesseral Harmonics in Cartesian coordinates**

Here we figure out the "general forms" both in Cartesian coordinates, and also in Cartesian ratios like x/y since these are what we need when we later change to μ and ν coordinates. For the Cartesian case, we state the result in a Theorem toward the end of this section.

**Question:** How do we know for sure that all the tesseral harmonics $P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$ will have the form $\text{poly}_n/r^n$ ? How would we prove this? We see it is true in our test cases. Well, let's look at the structure of our expressions starting with the cos(mφ) case:

$P_n^m(\cos\theta)\cos(m\varphi) = (\sin\theta)^m \partial_x^m P_n(x) \cos(m\varphi)$

$= \{ \partial_x^m P_n(x) \} \{ (\sin\theta)^m \cos(m\varphi) \}$ $x = \cos\theta$

first piece second piece

The trick is to look at the general nature of these factors. My coefficients a,b,c are generic and if the same a appears in two places, it does not mean the a's are the same. Series end when the exponent is either 1 or 0. So we have:

$P_n(x) = a\, x^n + b\, x^{n-2} + c\, x^{n-4} + ....$ // x means cosθ here, but not later

=> $\partial_x^m P_n(x) = a\, x^{n-m} + b\, x^{n-m-2} + c\, x^{n-m-4} + ...$ // coefficients are functions of n and m

Meanwhile, we have some claims about multiple angle expansions. I just quote the results, you can see the detailed coefficients on page 33 of GR7:

**Fact:** $\cos(m\varphi) = a \cos^m\varphi + b \cos^{m-2}\varphi + c \cos^{m-4}\varphi + ....$ // coefficients are functions of m !

**Fact:** $\sin(m\varphi) = \sin\varphi [d \cos^{m-1}\varphi + e \cos^{m-3}\varphi + f \cos^{m-5}\varphi + .... ]$ // ditto

So what happens when we glue our two pieces together? Well we then get

$P_n^m(\cos\theta)\cos(m\varphi) = \{ a \cos^{n-m}\theta + b \cos^{n-m-2}\theta + c \cos^{n-m-4}\theta + ... \}$
$* \sin^m\theta \{ a \cos^m\varphi + b \cos^{m-2}\varphi + c \cos^{m-4}\varphi + .... \}$

We can write the last two factors this way:

$\sin^m\theta \{ a \cos^m\varphi + b \cos^{m-2}\varphi + c \cos^{m-4}\varphi + .... \}$

$= \{ a \sin^m\theta \cos^m\varphi + b \sin^2\theta \sin^{m-2}\theta \cos^{m-2}\varphi + c \sin^4\theta \sin^{m-4}\theta \cos^{m-4}\varphi + .... \}$

$= \{ a (x/r)^m + b \sin^2\theta (x/r)^{m-2} + c \sin^4\theta (x/r)^{m-4} .... \}$

$= \{ a (x/r)^m + b [1-\cos^2\theta] (x/r)^{m-2} + c [1-\cos^2\theta]^2 (x/r)^{m-4} .... \}$

$= \{ a (x/r)^m + b [1- (z/r)^2] (x/r)^{m-2} + c [1-(z/r)^2]^2 (x/r)^{m-4} .... \}$

$= \{ a (x/r)^m + b [r^2- z^2] (1/r)^2 (x/r)^{m-2} + c [r^2- z^2]^2 (1/r)^4 (x/r)^{m-4} .... \}$

$= (1/r)^m \{ a x^m + b [r^2- z^2] x^{m-2} + c [r^2- z^2]^2 x^{m-4} .... \}$

$= (1/r)^m \{ a x^m + b (x^2+y^2) x^{m-2} + c (x^2+y^2)^2 x^{m-4} .... \}$

$= (1/r)^m [ \text{Poly}_m(x,y; m)] = \text{Poly}_m(x/r,y/r; m)$ // form of the "second piece"'

where the third parameter just reminds us that the coefficients depend on m. I now use Poly to indicate a homogeneous polynomial, meaning all terms have the same degree. We can move in the factors of r to get the second form shown. Every term still has the same degree: $(x/r)^2(y/r)^3$, $(x/r)^1(y/r)^4$ if m = 4.

Now the first piece was this

$$\{ a \cos^{n-m}\theta + b \cos^{n-m-2}\theta + c \cos^{n-m-4}\theta + ... \}$$

$$= \{ a (z/r)^{n-m} + b (z/r)^{n-m-2} + c (z/r)^{n-m-4} + ... \}$$

$$= (1/r)^{n-m} \{ a z^{n-m} + b z^{n-m-2} (1/r)^{-2} + c z^{n-m-4} (1/r)^{-4} + ... \}$$

$$= (1/r)^{n-m} \{ a z^{n-m} + b z^{n-m-2} r^2 + c z^{n-m-4} r^4 + ... \}$$

$$= (1/r)^{n-m} \{ a z^{n-m} + b z^{n-m-2} r^2 + c z^{n-m-4} r^4 + ... \}$$

$$= (1/r)^{n-m} \{ a z^{n-m} + b z^{n-m-2} (x^2+y^2) + c z^{n-m-4} (x^2+y^2)^2 + ... \}$$

$$= (1/r)^{n-m} [ \text{Poly}_{n-m}(x,y,z; n,m) ]$$ // form of the "first piece"

where the n,m reminds us that the coefficients depend on both n and m.

Therefore we have shown that

$$P_n^m(\cos\theta)\cos(m\varphi) = (1/r)^{n-m} [ \text{Poly}_{n-m}(x,y,z; n,m) ] * (1/r)^m [ \text{Poly}_m(x,y; m)]$$

$$= (1/r)^n \text{Poly}_n(x,y,z; n,m)$$ // which was our goal!

Now let's do the sine form which starts off like this:

$$P_n^m(\cos\theta)\cos(m\varphi) = \{ a \cos^{n-m}\theta + b \cos^{n-m-2}\theta + c \cos^{n-m-4}\theta + ... \} * \sin^m\theta \{ \sin\varphi [d \cos^{m-1}\varphi + e \cos^{m-3}\varphi + f \cos^{m-5}\varphi + .... ] \}$$

The "first piece" is of course the same as above. The second piece can be written this way

$$\sin^m\theta \{ \sin\varphi [d \cos^{m-1}\varphi + e \cos^{m-3}\varphi + f \cos^{m-5}\varphi + .... ] \}$$

$$= \sin\theta\sin\varphi [d \sin^{m-1}\theta \cos^{m-1}\varphi + e \sin^{m-1}\theta \cos^{m-3}\varphi + f \sin^{m-1} \cos^{m-5}\varphi + .... ]$$

$$= \sin\theta\sin\varphi [d \sin^{m-1}\theta \cos^{m-1}\varphi + e \sin^2\theta \sin^{m-3}\theta \cos^{m-3}\varphi + f \sin^4\theta \sin^{m-5} \cos^{m-5}\varphi + .... ]$$

$$= (y/r) [d (x/r)^{m-1} + e \sin^2\theta (x/r)^{m-3} + f \sin^4\theta (x/r)^{m-5} + .... ]$$

$$= (y/r) [d (x/r)^{m-1} + e [ 1-(z/r)]^2 (x/r)^{m-3} + f [ 1-(z/r)]^4 (x/r)^{m-5} + .... ]$$

$= (y/r)\,[d\,(x/r)^{m-1} + e\,[\,r^2-z^2]^2\,(1/r)^2(x/r)^{m-3} + f\,[\,r^2-z^2]^4\,(1/r)^4\,(x/r)^{m-5} + \ldots.\,]$

$= (y/r)(1/r)^{m-1}\,[d\,x^{m-1} + e\,[\,r^2-z^2]^2\,x^{m-3} + f\,[\,r^2-z^2]^4\,x^{m-5} + \ldots.\,]$

$= (y/r)(1/r)^{m-1}\,[d\,x^{m-1} + e\,(x^2+y^2)^2\,x^{m-3} + f\,(x^2+y^2)^4\,x^{m-5} + \ldots.\,]$

$= (1/r)^m\,\{\,y\,[d\,x^{m-1} + e\,(x^2+y^2)^2\,x^{m-3} + f\,(x^2+y^2)^4\,x^{m-5} + \ldots.\,]\,\}$

$= (1/r)^m\ \mathrm{Poly}_m(x,y;m)$

But this is exactly the same "form" that our *previous* "second piece" had, and so we end up again with the resulting form $(1/r)^n\ \mathrm{Poly}_n(x,y,z;n,m)$. We have just proved the following theorem:

**Theorem:** $P_n^m(\cos\theta)\cos(m\varphi)$ and $P_n^m(\cos\theta)\sin(m\varphi)$ both have the following form, when converted to Cartesian coordinates:

$$P_n^m(\cos\theta)\,[\cos(m\varphi),\,\sin(m\varphi)] \;=\; (1/r)^n\,[\,C_n(x,y,z;\,n,m),\,S_n(x,y,z;\,n,m)\,]$$

where $C_n$ and $S_n$ are homogeneous polynomials of degree n in x,y,z whose coefficients depend on n and m. Notice that the RHS's are therefore dimensionless, as they must be.

If we then take our Cartesian operator,

$$-\,L^2 = (y\partial_z - z\partial_y)^2 + \text{cyclic}$$

we will find that both our tesseral functions have definite angular momentum magnitude n,

$$L^2\{\,C_n(x,y,z;\,n,m)\ /\ r^n\,\} = n(n+1)\,\{C_n(x,y,z;\,n,m)\ /\ r^n\,\}$$
$$L^2\{\,S_n(x,y,z;\,n,m)\ /\ r^n\,\} = n(n+1)\,\{S_n(x,y,z;\,n,m)\ /\ r^n\,\}$$

If we take the linear combinations

$$P_n^m(\cos\theta)\,[\cos(m\varphi) \pm i\,\sin(m\varphi)] = P_n^m(\cos\theta)\,e^{\pm i m\varphi} \;=\; [C_n(x,y,z;\,n,m) \pm i\,S_n(x,y,z;\,n,m)]\,/r^n$$

and if we take our Cartesian operator

$$L_z = (-i)(x\partial_y - y\partial_x) \qquad\qquad // = (-i)\partial_\varphi$$

we will find that

$$L_z\,\{\,[C_n(x,y,z;\,n,m) \pm i\,S_n(x,y,z;\,n,m)]\,/r^n\} = \pm\,m\,\{\,[C_n(x,y,z;\,n,m) \pm i\,S_n(x,y,z;\,n,m)]\,/r^n\}$$

One implication of this fact is that, equating real and imaginary parts

$$L_z[C_n(x,y,z;\,n,m)/r^n] \;=\; i\,m\,[S_n(x,y,z;\,n,m)]\,/r^n]$$

$$\pm i\, L_z[S_n(x,y,z;\, n,m)/r^n] = \pm m\, [C_n(x,y,z;\, n,m)]\, /r^n]$$

which we can rewrite as

$$L_z[C_n(x,y,z;\, n,m)/r^n] = i\, m\, [S_n(x,y,z;\, n,m)]\, /r^n]$$
$$L_z[S_n(x,y,z;\, n,m)/r^n] = -\, i\, m\, [C_n(x,y,z;\, n,m)]\, /r^n]$$

**Example: n=1**

$P_1^1(\cos\theta)\cos(1\varphi) = \sin\theta\cos\varphi = (x/r)$
$P_1^0(\cos\theta)\cos(0\varphi) = \cos\theta = (z/r)$

$P_1^1(\cos\theta)\sin(1\varphi) = \sin\theta\sin\varphi = (x/r)$
$P_1^0(\cos\theta)\sin(0\varphi) = 0$

Cartesian solutions are thus: $\{ x, y, z \} / r$

**Example: n=2**

$P_2^2(\cos\theta)\cos(2\varphi) = 3\sin^2\theta\, (2\cos^2\varphi - 1) = 6\,(x/r)^2 - 3(1-(z/r)^2) = [6x^2 - 3r^2 + 3z^2]/r^2$
$= [6x^2 - 3x^2 - 3y^2 - 3z^2 + 3z^2]/r^2 = [3x^2 - 3y^2]/r^2 = 3\,(x^2-y^2)/r^2$
$P_2^1(\cos\theta)\cos(1\varphi) = 3\,\sin\theta\cos\theta\,\cos\varphi = 3\,(z/r)(x/r) = 3\,xz/r^2$
$P_2^0(\cos\theta)\cos(0\varphi) = (1/2)[3\cos^2\theta - 1] = (1/2)\,[\,3(z/r)^2 - 1] = (1/2)\,(3z^2-r^2)/r^2$
$= (1/2)\,(3z^2-x^2-y^2-z^2)/r^2 = (1/2)\,(2z^2-x^2-y^2)/r^2$

$P_2^2(\cos\theta)\sin(2\varphi) = 3\sin^2\theta\, 2\,\sin\varphi\,\cos\varphi = 6\,(x/r)(y/r) = 6\,xy/r^2$
$P_2^1(\cos\theta)\sin(1\varphi) = 3\,\sin\theta\cos\theta\,\sin\varphi = 3\,(z/r)(y/r) = 3\,zy\,/\,r^2$
$P_2^0(\cos\theta)\sin(0\varphi) = 0$

This suggests five functions, where now I ignore constants and I ignore the $/r^2$ part

$P_2^2(\cos\theta)\cos(2\varphi) \sim x^2-y^2$ $\qquad$ $P_2^2(\cos\theta)\sin(2\varphi) \sim xy$
$P_2^1(\cos\theta)\cos(1\varphi) \sim xz$ $\qquad$ $P_2^1(\cos\theta)\sin(1\varphi) \sim zy$
$P_2^0(\cos\theta)\cos(0\varphi) \sim 2z^2-x^2-y^2$ $\qquad$ $P_2^0(\cos\theta)\sin(0\varphi) = 0$

If we add the last two and divide by 2 we get $z^2 - y^2$ so we could take these instead

$$xz \qquad xy \qquad zy \qquad x^2-y^2 \qquad z^2-y^2$$

You can see that this gets messier as n increases. For example, back in Section 3 we had

$$P_5^4(\cos\theta)\cos(4\varphi) = 945\, z\, [\, 8\, x^4 - 8\,(r^2-z^2)\, x^2 + (r^2-z^2)^2\, ]\, /r^5$$
$$= 945\, z\, [\, 8\, x^4 - 8\,(x^2+y^2)\, x^2 + (x^2+y^2)^2\, ]\, /r^5$$
$$= 945\, z\, [\, 8\, x^4 - 8x^4 - 8\, x^2y^2 + x^4 + 2x^2y^2 + y^4\, ]\, /r^5$$

$$= 945\, z\, [x^4 + 6\, x^2y^2 + y^4\,]\, /r^5$$

$$P_5^4(\cos\theta)\, \sin(4\varphi) = 945\, z\, [\, 8\, y\, x^3 - 4\, (r^2-z^2)xy\,]\, /r^5$$
$$= 945\, z\, [\, 8\, y\, x^3 - 4\, (x^2+y^2)xy\,]\, /r^5$$
$$= 945\, z\, [\, 8\, y\, x^3 - 4x^3y - 4xy^3\,]\, /r^5$$
$$= 945*4\, zxy\, [\, 2\, x^2 - x^2 - y^2\,]\, /r^5$$
$$= -\, 945*4\, xyz(x-y)^2\, /r^5$$

I think I could make Maple generate these functions if I really wanted (see next section).

**5. Maple program to display the tesseral harmonics in Cartesian coordinates.**

Maple is always a pain but I was finally able to get it to do what I wanted. There is one program and you set it either to cos(mφ) or sin(mφ) with two simple edits. The problem is getting Maple to simplify things correctly. The program is in the same file "Pnm display.mws", so I won't replicate that code here. You can see that the functions are all homogeneous polynomials of degree n.

**Display of $r^n\, P_n^m(\cos\theta)\, \cos(m\varphi)$ for n=0 to 5 and m = 0 to n**

```
r^0 P(0,0,z)cos(0 phi)
```

$$1$$

```
r^1 P(1,0,z)cos(0 phi)
```

$$z$$

```
r^1 P(1,1,z)cos(1 phi)
```

$$x$$

```
r^2 P(2,0,z)cos(0 phi)
```

$$z^2-\frac{1}{2}x^2-\frac{1}{2}y^2$$

```
r^2 P(2,1,z)cos(1 phi)
```

$$3zx$$

```
r^2 P(2,2,z)cos(2 phi)
```

$$3x^2-3y^2$$

```
r^3 P(3,0,z)cos(0 phi)
```

$$-\frac{1}{2}z(-2z^2+3x^2+3y^2)$$

```
r^3 P(3,1,z)cos(1 phi)
```

$$-\frac{3}{2}(-4z^2+x^2+y^2)x$$

```
r^3 P(3,2,z)cos(2 phi)
```

$$-15(-x^2+y^2)z$$

```
r^3 P(3,3,z)cos(3 phi)
```

$$-15x(-x^2+3y^2)$$

```
r^4 P(4,0,z)cos(0 phi)
```

$$z^4-3z^2x^2-3z^2y^2+\frac{3}{8}x^4+\frac{3}{4}x^2y^2+\frac{3}{8}y^4$$

```
r^4 P(4,1,z)cos(1 phi)
```

$$-\frac{5}{2}z(-4z^2+3x^2+3y^2)x$$

```
r^4 P(4,2,z)cos(2 phi)
```

$$45z^2x^2-45z^2y^2-\frac{15}{2}x^4+\frac{15}{2}y^4$$

```
r^4 P(4,3,z)cos(3 phi)
```

$$-105zx(-x^2+3y^2)$$

```
r^4 P(4,4,z)cos(4 phi)
```

$$105x^4-630x^2y^2+105y^4$$

```
r^5 P(5,0,z)cos(0 phi)
```

$$\frac{1}{8}z(8z^4-40z^2x^2-40z^2y^2+15x^4+30x^2y^2+15y^4)$$

```
r^5 P(5,1,z)cos(1 phi)
```

$$\frac{15}{8}(8z^4-12z^2x^2-12z^2y^2+x^4+2x^2y^2+y^4)x$$

```
r^5 P(5,2,z)cos(2 phi)
```

$$\frac{105}{2}(2z^2x^2-2z^2y^2-x^4+y^4)z$$

```
r^5 P(5,3,z)cos(3 phi)
```

$$\frac{105}{2}x(8z^2x^2-24z^2y^2-x^4+2x^2y^2+3y^4)$$

```
r^5 P(5,4,z)cos(4 phi)
```

$$945(x^4-6x^2y^2+y^4)z$$

```
r^5 P(5,5,z)cos(5 phi)
```

$$945x(x^4-10x^2y^2+5y^4)$$

**Display of $r_n P_n^m(\cos\theta) \sin(m\varphi)$ for n=0 to 5 and m = 0 to n**

```
r^0 P(0,0,z)sin(0 phi)
```

$$0$$

```
r^1 P(1,0,z)sin(0 phi)
```

$$0$$

```
r^1 P(1,1,z)sin(1 phi)
```

$$y$$

```
r^2 P(2,0,z)sin(0 phi)
```

$$0$$

```
r^2 P(2,1,z)sin(1 phi)
```

$$3zy$$

```
r^2 P(2,2,z)sin(2 phi)
```

$$6yx$$

```
r^3 P(3,0,z)sin(0 phi)
```

$$0$$

```
r^3 P(3,1,z)sin(1 phi)
```

$$-\frac{3}{2}(-4z^2+x^2+y^2)y$$

```
r^3 P(3,2,z)sin(2 phi)
```

$$30zyx$$

```
r^3 P(3,3,z)sin(3 phi)
```

$$-15y(-3x^2+y^2)$$

```
r^4 P(4,0,z)sin(0 phi)
```

$$0$$

```
r^4 P(4,1,z)sin(1 phi)
```

$$-\frac{5}{2}z(-4z^2+3x^2+3y^2)y$$

```
r^4 P(4,2,z)sin(2 phi)
```

$$-15(-6z^2+x^2+y^2)yx$$

```
r^4 P(4,3,z)sin(3 phi)
```

$$-105zy(-3x^2+y^2)$$

```
r^4 P(4,4,z)sin(4 phi)
```

$$-420(-x^2+y^2)yx$$

```
r^5 P(5,0,z)sin(0 phi)
```

$$0$$

```
r^5 P(5,1,z)sin(1 phi)
```

$$\frac{15}{8}(8z^4-12z^2x^2-12z^2y^2+x^4+2x^2y^2+y^4)y$$

```
r^5 P(5,2,z)sin(2 phi)
```

$$-105(-2z^2+x^2+y^2)zyx$$

```
r^5 P(5,3,z)sin(3 phi)
```

$$\frac{105}{2}y(24z^2x^2-8z^2y^2-3x^4-2x^2y^2+y^4)$$

```
r^5 P(5,4,z)sin(4 phi)
```

$$-3780(-x^2+y^2)zyx$$

```
r^5 P(5,5,z)sin(5 phi)
```

$$945y(5x^4-10x^2y^2+y^4)$$