

Block Matrix Sinc-Galerkin Solution of the Wind-Driven Current Problem

Sanoe Koonprasert and Kenneth L. Bowers

*Department of Mathematical Sciences, Montana State University
Bozeman, MT 59717, USA*

Abstract

A block matrix formulation is presented for the Sinc-Galerkin technique applied to the wind-driven current problem from oceanography. The block matrix form is used to determine an approximate solution for the coupled system of differential equations describing the wind-driven current problem. The approximate solution determined by this new Sinc-Galerkin scheme is compared to the approximate solution for the original complex-valued Sinc-Galerkin approach to ocean current models. The approximate solution is also compared to exact solutions to illustrate the exponential convergence rate of the method.

Key words: Block matrix, Sinc-Galerkin method, wind-driven currents

1 Introduction

In recent years, the complex-valued Sinc-Galerkin method for wind-driven current models has been applied to the model formulated as a complex-valued ordinary differential equation. However, a Sinc-Galerkin technique can also be applied to a formulation of the model that consists of two real-valued coupled ordinary differential equations. The block matrix depends on the sinc basis functions, the differential equations, and the boundary conditions. The complex-valued Sinc-Galerkin method results in a smaller discrete system but requires complex arithmetic at all steps. The block formulation eliminates the need for any complex-valued calculations at the expense of a slightly larger matrix system.

In this paper, we perform a numerical study on a model of wind-driven currents in the ocean. This model is found in [1]. There the Sinc-Galerkin technique was applied to a model describing wind-driven subsurface currents in coastal regions and semi-enclosed seas where the vertical eddy viscosity coefficient is

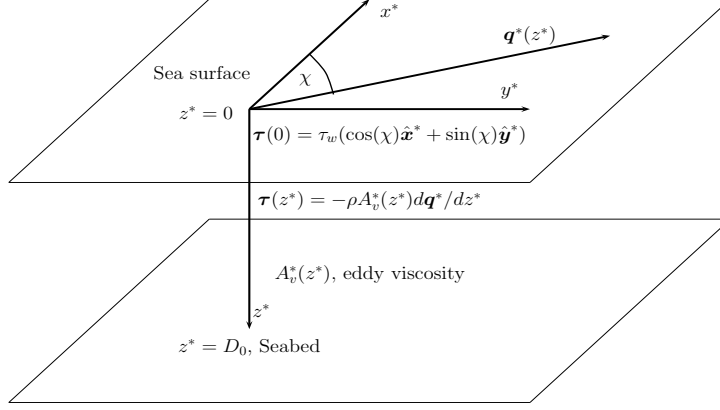


Fig. 1. The general physical model of the depth-dependent eddy viscosity oceanography problem

represented as a continuously differentiable function of depth. Our goal is to construct the block matrix that represents this Sinc-Galerkin formulation and use it for solving a coupled linear system.

2 Governing Equations

To develop a mathematical model, we first construct a right-handed coordinate system with the vertical coordinate z^* directed positive downward from the free surface, and with x^* and y^* directed northward and eastward, respectively. A plane at $z^* = D_0$ corresponds to the impermeable boundary at the seabed. For the purpose of illustrating the Sinc-Galerkin method, we employ several of the simplifying assumptions invoked in one-dimensional wind-drift studies - the ocean depth, D_0 , and ocean mass density, ρ , are assumed constant, and the effects of tides, inertial terms, free surface slope, and variations in atmospheric pressure are neglected. Currents are driven by a tangential surface wind stress (the surface is at $z^* = 0$) of magnitude τ_w , represented as $\boldsymbol{\tau}(0) = \tau_w(\cos(\chi)\hat{\mathbf{x}}^* + \sin(\chi)\hat{\mathbf{y}}^*)$, with χ being the angle between the positive x^* -axis and the wind direction in Figure 1. Here $\hat{\mathbf{x}}^*$ and $\hat{\mathbf{y}}^*$ represent unit vectors in the directions of the positive x^* -axis and the positive y^* -axis, respectively. In this model, called a specified eddy viscosity model, internal frictional stresses are parameterized as $\boldsymbol{\tau}(z^*) = -\rho A_v^*(z^*) d\mathbf{q}^*/dz^*$, where the specified effective vertical eddy viscosity coefficient $A_v^*(z^*)$ is a continuously differentiable function of $z^* \in (0, D_0)$. Here $\mathbf{q}^*(z^*) = U^*(z^*)\hat{\mathbf{x}}^* + V^*(z^*)\hat{\mathbf{y}}^*$ represents the horizontal wind-drift current which is the difference between the total velocity and the geostrophic current in [2]. The Coriolis force acting on the moving water is balanced by a horizontal pressure gradient force.

Under the present assumptions, the conservation of linear momentum equations express a balance between the Coriolis force and the internal friction associated with turbulence. The wind-drift current $\mathbf{q}^*$ is determined by solving the boundary-value problem

$$\frac{d}{dz^*} \left(A_v^*(z^*) \frac{d\mathbf{q}^*}{dz^*} \right) = -f \hat{\mathbf{z}}^* \times \mathbf{q}^*, \quad 0 < z^* < D_0, \quad (1)$$

where the stress condition at the sea surface, $z^* = 0$, is the tangential surface wind stress

$$-\rho A_v^*(0) \frac{d\mathbf{q}^*(0)}{dz^*} = \tau_w (\cos(\chi) \hat{\mathbf{x}}^* + \sin(\chi) \hat{\mathbf{y}}^*) \quad (2)$$

while at the seabed, $z^* = D_0$, the frictional stress is linearly proportional to the current, hence

$$-\rho A_v^*(D_0) \frac{d\mathbf{q}^*(D_0)}{dz^*} = k_f \rho \mathbf{q}^*(D_0). \quad (3)$$

Here $f \equiv 2\Omega \sin \theta$ is the Coriolis parameter at latitude θ where $\Omega = 7.29 \times 10^{-5} \text{ rad s}^{-1}$ is the angular speed of rotation of the earth. Though $-\pi/2 < \theta < \pi/2$, for the rest of our work we assume we are in the northern hemisphere and hence $0 < \theta < \pi/2$ ($f > 0$). The parameter k_f is the linear slip bottom stress coefficient. If $\mathbf{q}^*(D_0) = \mathbf{0}$, (3) is called a zero-velocity (no-slip) condition.

From (1), we can simplify the boundary-value problem as follows

$$\begin{aligned} \frac{d}{dz^*} \left(A_v^*(z^*) \frac{d\mathbf{q}^*}{dz^*} \right) &= \frac{d}{dz^*} \left(A_v^*(z^*) \frac{dU^*(z^*)}{dz^*} \right) \hat{\mathbf{x}}^* + \frac{d}{dz^*} \left(A_v^*(z^*) \frac{dV^*(z^*)}{dz^*} \right) \hat{\mathbf{y}}^* \\ &= -f \hat{\mathbf{z}}^* \times \mathbf{q}^* \\ &= -f \hat{\mathbf{z}}^* \times [U^*(z^*) \hat{\mathbf{x}}^* + V^*(z^*) \hat{\mathbf{y}}^*] \\ &= -f (U^*(z^*) \hat{\mathbf{y}}^* - V^*(z^*) \hat{\mathbf{x}}^*), \quad 0 < z^* < D_0. \end{aligned}$$

Then it can be written in component form as

$$-\frac{d}{dz^*} \left(A_v^*(z^*) \frac{dU^*(z^*)}{dz^*} \right) = -f V^*(z^*), \quad 0 < z^* < D_0 \quad (4)$$

and

$$-\frac{d}{dz^*} \left(A_v^*(z^*) \frac{dV^*(z^*)}{dz^*} \right) = f U^*(z^*), \quad 0 < z^* < D_0. \quad (5)$$

The stress condition at the sea surface, $z^* = 0$, separates as

$$-\rho A_v^*(0) \frac{dU^*(0)}{dz^*} = \tau_w \cos(\chi), \quad -\rho A_v^*(0) \frac{dV^*(0)}{dz^*} = \tau_w \sin(\chi). \quad (6)$$

At the seabed, $z^* = D_0$, the frictional stress separates as

$$-\rho A_v^*(D_0) \frac{dU^*(D_0)}{dz^*} = k_f \rho U^*(D_0), \quad -\rho A_v^*(D_0) \frac{dV^*(D_0)}{dz^*} = k_f \rho V^*(D_0). \quad (7)$$

To nondimensionalize the model equations we begin with a measure of near surface turbulent eddy viscosity, $A_0 \equiv A_v^*(0)$, and define a nominal “upper-layer” Ekman depth by $D_E \equiv \sqrt{2A_0/f}$. Also define a current speed in units of $U_0 = \tau_w D_E / (\rho A_0) = \sqrt{2} \tau_w / (\rho \sqrt{A_0 f})$. U_0 is the natural velocity scale in an infinitely deep sea with uniform eddy viscosity in the steady-state. A nondimensional form of the equations of motion can be expressed with the introduction of the nondimensional variables

$$z \equiv \frac{z^*}{D_0}, \quad A_v(z) \equiv \frac{A_v^*(z^*)}{A_v^*(0)}, \quad \mathbf{q}(z) \equiv \frac{\mathbf{q}^*(z^*)}{U_0} \equiv U(z) \hat{\mathbf{x}} + V(z) \hat{\mathbf{y}} \quad (8)$$

together with two nondimensional constants (a depth ratio κ and a bottom friction parameter σ) given by

$$\kappa \equiv \frac{D_0}{D_E} = D_0 \sqrt{\frac{f}{2A_0}}, \quad \sigma \equiv \frac{A_0 A_v(1)}{k_f D_0} = \frac{A_v^*(D_0)}{k_f D_0}. \quad (9)$$

We apply (8) and (9) to nondimensionalize the equations (4) and (5). For (4), we have

$$-\frac{1}{D_0} \frac{d}{dz} \left(A_0 A_v(z) \frac{1}{D_0} \frac{d(U_0 U(z))}{dz} \right) = -f U_0 V(z), \quad 0 < z < 1,$$

which results in

$$-\frac{d}{dz} \left(A_v(z) \frac{dU(z)}{dz} \right) = -2\kappa^2 V(z), \quad 0 < z < 1. \quad (10)$$

Similarly, the result from (5) is

$$-\frac{d}{dz} \left(A_v(z) \frac{dV(z)}{dz} \right) = 2\kappa^2 U(z), \quad 0 < z < 1. \quad (11)$$

The surface boundary conditions in (6) are nondimensionalized to produce

$$-\rho \frac{A_0}{D_0} \frac{d(U_0 U(0))}{dz} = \tau_w \cos(\chi), \quad -\rho \frac{A_0}{D_0} \frac{d(U_0 V(0))}{dz} = \tau_w \sin(\chi),$$

which leads to

$$\frac{dU(0)}{dz} = -\kappa \cos(\chi), \quad \frac{dV(0)}{dz} = -\kappa \sin(\chi). \quad (12)$$

The seabed conditions in (7) are also nondimensionalized and become

$$U(1) + \sigma \frac{dU(1)}{dz} = 0, \quad V(1) + \sigma \frac{dV(1)}{dz} = 0. \quad (13)$$

In the special no-slip case, we can set $\sigma = 0$ in (13).

We first transform the nonhomogenous boundary conditions to homogeneous boundary conditions by using the linear transformations

$$U(z) = u(z) + \kappa(1 + \sigma - z) \cos(\chi), \quad V(z) = v(z) + \kappa(1 + \sigma - z) \sin(\chi). \quad (14)$$

The first derivative of each transformation in (14) yields

$$\frac{dU(z)}{dz} = \frac{du(z)}{dz} - \kappa \cos(\chi), \quad \frac{dV(z)}{dz} = \frac{dv(z)}{dz} - \kappa \sin(\chi),$$

so the resulting boundary-value problem for $u(z)$ satisfies

$$-\frac{d}{dz} \left(A_v(z) \frac{du}{dz} \right) + \frac{d}{dz} (A_v(z) \kappa \cos(\chi)) = -2\kappa^2 [v(z) + \kappa(1 + \sigma - z) \sin(\chi)],$$

$$0 < z < 1,$$

which gives

$$-\frac{d}{dz} \left(A_v(z) \frac{du}{dz} \right) + \kappa \cos(\chi) A'_v(z) = -2\kappa^2 v(z) - 2\kappa^3 (1 + \sigma - z) \sin(\chi),$$

$$0 < z < 1. \quad (15)$$

Similarly the boundary-value problem for $v(z)$ satisfies

$$-\frac{d}{dz} \left(A_v(z) \frac{dv}{dz} \right) + \kappa \sin(\chi) A'_v(z) = 2\kappa^2 u(z) + 2\kappa^3 (1 + \sigma - z) \cos(\chi),$$

$$0 < z < 1. \quad (16)$$

The surface boundary conditions (12) become

$$\frac{du(0)}{dz} = 0, \quad \frac{dv(0)}{dz} = 0 \quad (17)$$

while the seabed boundary conditions (13) become

$$u(1) + \sigma \frac{du(1)}{dz} = 0, \quad v(1) + \sigma \frac{dv(1)}{dz} = 0. \quad (18)$$

For the purpose of illustrating the exposition of the Sinc-Galerkin technique, we define

$$\mathcal{L}u(z) \equiv -\frac{d}{dz} \left(A_v(z) \frac{du}{dz} \right), \quad \mathcal{L}v(z) \equiv -\frac{d}{dz} \left(A_v(z) \frac{dv}{dz} \right). \quad (19)$$

Then (15) and (16) are now given by the coupled u and v equation systems

$$\mathcal{L}u(z) + 2\kappa^2 v(z) = F_1(z), \quad 0 < z < 1, \quad (20)$$

$$\mathcal{L}v(z) - 2\kappa^2 u(z) = F_2(z), \quad 0 < z < 1, \quad (21)$$

where

$$F_1(z) = -2\kappa^3(1 + \sigma - z) \sin(\chi) - \kappa \cos(\chi) A_v'(z), \quad (22)$$

$$F_2(z) = 2\kappa^3(1 + \sigma - z) \cos(\chi) - \kappa \sin(\chi) A_v'(z). \quad (23)$$

The surface boundary conditions are

$$\frac{du(0)}{dz} = 0, \quad \frac{dv(0)}{dz} = 0. \quad (24)$$

The seabed boundary conditions are

$$u(1) + \sigma \frac{du(1)}{dz} = 0, \quad v(1) + \sigma \frac{dv(1)}{dz} = 0. \quad (25)$$

3 The Sinc Function and the Sinc-Galerkin Method

We first review sinc function properties, sinc quadrature rules, and the Sinc-Galerkin method. These are discussed thoroughly in [3] and [4]. The sinc function is defined for all $z \in \mathcal{C}$ by

$$\text{sinc}(z) \equiv \begin{cases} \frac{\sin(\pi z)}{\pi z}, & \text{if } z \neq 0 \\ 1, & \text{if } z = 0 \end{cases}.$$

For $h > 0$ and $k = 0, \pm 1, \pm 2, \pm 3, \dots$ the translated sinc functions with evenly spaced nodes are given by

$$S(k, h)(z) \equiv \text{sinc}\left(\frac{z - kh}{h}\right) \equiv \begin{cases} \frac{\sin(\pi \frac{z - kh}{h})}{\pi \frac{z - kh}{h}}, & \text{if } z \neq kh \\ 1, & \text{if } z = kh \end{cases}.$$

The Sinc-Galerkin procedure for the coupled linear system in (20)-(21) begins by selecting composite sinc functions appropriate to the interval $(0, 1)$ as the basis functions for the expansion of approximate solutions for the current components $u(z)$ and $v(z)$. We introduce the conformal mapping function

$$w = \phi(z) = \ln\left(\frac{z}{1 - z}\right), \quad (26)$$

which is a conformal mapping from $\mathcal{D}_E$, the eye-shaped domain in the z -plane, onto the infinite strip in the w -plane, $\mathcal{D}_S$, where

$$\mathcal{D}_E = \left\{ z = x + iy : \left| \arg\left(\frac{z}{1 - z}\right) \right| < d \leq \frac{\pi}{2} \right\},$$

$$\mathcal{D}_S = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$

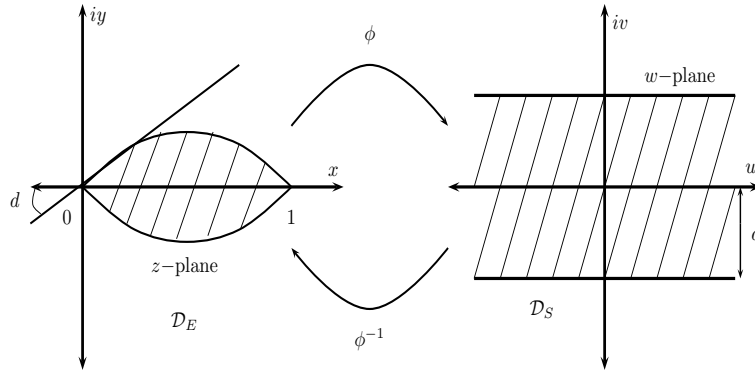


Fig. 2. The relationship between the eye-shaped domain, $\mathcal{D}_E$, and the infinite strip, $\mathcal{D}_S$

This is shown in Figure 2. The basis functions are derived from the composite translated sinc functions,

$$S_k(z) \equiv S(k, h) \circ \phi(z) = \text{sinc}\left(\frac{\phi(z) - kh}{h}\right), \quad (27)$$

for $z \in \mathcal{D}_E$. These are shown in Figure 3 for real values, x .

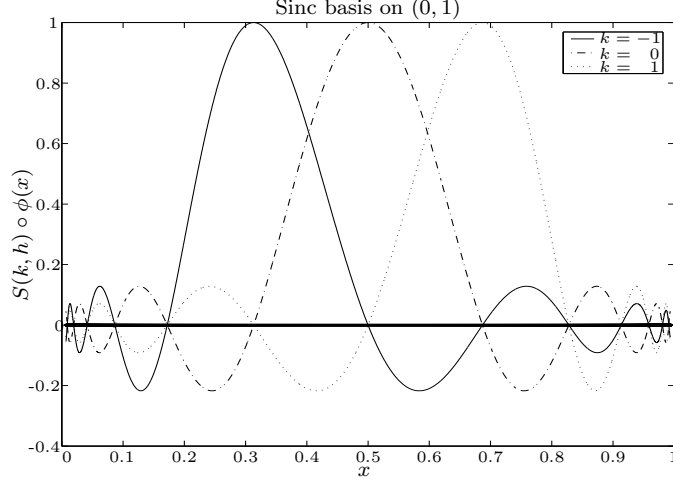


Fig. 3. Three adjacent members $S(k, h) \circ \phi(x)$ when $k = -1, 0, 1$ and $h = \frac{\pi}{8}$ of the mapped sinc basis on the interval $(0, 1)$

For the Neumann or radiation boundary conditions (24)-(25), the sinc basis functions in (27) do not have a derivative when z tends to 0 or 1. Thus we modify the sinc basis functions as

$$\frac{S(k, h) \circ \phi(z)}{\phi'(z)} \equiv \frac{\text{sinc}\left(\frac{\phi(z) - kh}{h}\right)}{\phi'(z)}. \quad (28)$$

These are shown in Figure 4 for real values, x . Note that the derivatives of the modified sinc basis functions are defined as z approaches 0 or 1. We also add boundary basis functions that are Hermite polynomials. Thus the derivatives at $z = 0$ and 1 for these basis functions are defined. These Hermite polynomials are given by

$$B_0(z) = (2z + 1)(1 - z)^2, \quad B_1(z) = (1 - z)z^2 + \sigma(3 - 2z)z^2. \quad (29)$$

Sinc rules for a special class of functions $B(\mathcal{D}_E)$ have been developed. A discussion of the properties of functions in $B(\mathcal{D}_E)$ is found in [3] and [4].

Definition 1 Let $\phi : \mathcal{D}_E \rightarrow \mathcal{D}_S$ be a conformal map of $\mathcal{D}_E$ to $\mathcal{D}_S$ with inverse ψ . Let $\Gamma = \{\psi(u) \in \mathcal{D}_E : -\infty < u < \infty\} = (0, 1)$. Then $B(\mathcal{D}_E)$ is the class of functions F which are analytic in $\mathcal{D}_E$, satisfy

$$\int_{\psi(t+L)} |F(z)| dz \rightarrow 0 \quad , \quad t \rightarrow \pm\infty,$$

where $L = \left\{iv : |v| < d \leq \frac{\pi}{2}\right\}$, and on the boundary of $\mathcal{D}_E$, denoted $\partial\mathcal{D}_E$, satisfy

$$\mathcal{N}(F) \equiv \int_{\partial\mathcal{D}_E} |F(z)| dz < \infty \quad .$$

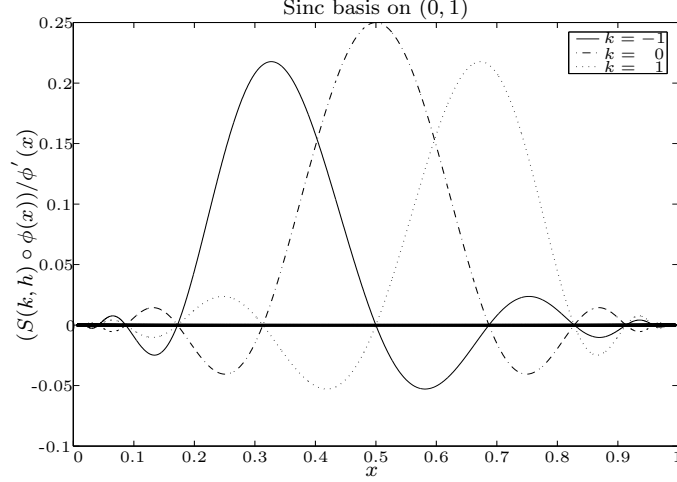


Fig. 4. Three adjacent members $\frac{S(k, h) \circ \phi(x)}{\phi'(x)}$ when $k = -1, 0, 1$ and $h = \frac{\pi}{8}$ of the modified sinc basis on the interval $(0, 1)$

The proofs of the following theorems are found in [3] and [4].

Theorem 1 *If $\phi' F \in B(\mathcal{D}_E)$ then for all $z \in \Gamma$,*

$$F(z) = \sum_{j=-\infty}^{\infty} F(z_j) S(j, h) \circ \phi(z) + E_F$$

where

$$E_F \equiv \frac{\sin\left(\frac{\pi\phi(z)}{h}\right)}{2\pi i} \int_{\partial\mathcal{D}_E} \frac{\phi'(w) F(w) dw}{(\phi(w) - \phi(z)) \sin(\pi\phi(w)/h)} .$$

Theorem 2 *If $F \in B(\mathcal{D}_E)$ then*

$$\int_0^1 F(z) dz = h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} + I_F$$

where

$$I_F \equiv \frac{i}{2} \int_{\partial\mathcal{D}_E} \frac{F(z) \kappa(\phi, h)(z)}{\sin(\pi\phi(z)/h)} dz$$

with

$$\kappa(\phi, h)(z) = \exp\left[\frac{i\pi\phi(z)}{h} \operatorname{sgn}(\mathcal{I}(\phi(z)))\right].$$

The infinite quadrature rule appearing in Theorem 2 can be evaluated directly, but in general it must be truncated to a finite sum for the Sinc-Galerkin method. The following theorem indicates the conditions under which exponential convergence results.

Theorem 3 Let $F \in B(\mathcal{D}_E)$ and ϕ be a conformal map with constants α, β , and C so that

$$\left| \frac{F(z)}{\phi'(z)} \right| \leq C \begin{cases} \exp(-\alpha|\phi(z)|), & z \in \Gamma_a \\ \exp(-\beta|\phi(z)|), & z \in \Gamma_b \end{cases}$$

where

$$\Gamma_a \equiv \{z \in \Gamma : \phi(z) = x \in (-\infty, 0)\} = \left(0, \frac{1}{2}\right),$$

$$\Gamma_b \equiv \{z \in \Gamma : \phi(z) = x \in [0, \infty)\} = \left[\frac{1}{2}, 1\right).$$

Then the sinc trapezoidal quadrature rule is

$$\begin{aligned} \int_0^1 F(z)dz &= h \sum_{j=-M}^N \frac{F(z_j)}{\phi'(z_j)} + \mathcal{O}(\exp(-\alpha Mh)) + \mathcal{O}(\exp(-\beta Nh)) \\ &\quad + \mathcal{O}(\exp(-2\pi d/h)). \end{aligned} \quad (30)$$

Hence, make the selections

$$N = \left\lceil \left\lfloor \frac{\alpha}{\beta} M + 1 \right\rfloor \right\rceil, \quad h = \sqrt{2\pi d/(\alpha M)},$$

where $\lceil \cdot \rceil$ denotes the greatest integer, and the exponential order of the sinc trapezoidal quadrature rule in (30) is $\mathcal{O}(\exp(-\sqrt{2\pi d\alpha M}))$.

Corollary 1 An important special case of Theorem 3 occurs when the integrand has the form $G(z)S(l, h) \circ \phi(z)$. Due to the interpolation

$$S(l, h) \circ \phi(z_j) = S(l, h)(jh) = \delta_{jl}^{(0)},$$

the sinc quadrature rule is a weighted point evaluation to the order of the method,

$$\int_0^1 G(z)S(l, h) \circ \phi(z)dz = h \frac{G(z_l)}{\phi'(z_l)} + \mathcal{O}(\exp(-2\pi d/h)). \quad (31)$$

In the Sinc-Galerkin technique, we first assume the approximate solutions for $u(z)$ and $v(z)$ in (20)-(21), subject to the mixed conditions (24) and (25), are represented by

$$u_a(z) = c_{-N-1}B_0(z) + u_h(z) + c_{N+1}B_1(z), \quad (32)$$

$$v_a(z) = d_{-N-1}B_0(z) + v_h(z) + d_{N+1}B_1(z), \quad (33)$$

where choosing $M = N$ gives

$$u_h(z) = \sum_{j=-N}^N c_j \frac{S_j(z)}{\phi'(z)}, \quad v_h(z) = \sum_{j=-N}^N d_j \frac{S_j(z)}{\phi'(z)}. \quad (34)$$

Both approximate solutions clearly satisfy the boundary conditions (24)-(25). The $m = 2N + 3$ coefficients $\{c_j\}_{j=-N-1}^{N+1}$ and the m coefficients $\{d_j\}_{j=-N-1}^{N+1}$ are determined by orthogonalizing the residual $\mathcal{L}u_a(z) + 2\kappa^2 v_a(z) - F_1(z)$ and $\mathcal{L}v_a(z) - 2\kappa^2 u_a(z) - F_2(z)$ with respect to the sinc basis functions $\{S_j\}_{j=-N-1}^{N+1}$ in (27). The inner product

$$(F, G) = \int_0^1 F(z)G(z)w(z)dz,$$

uses a weight function $w(z) = 1/\sqrt{\phi'(z)} = \sqrt{z(1-z)}$. So this yields the discrete Sinc-Galerkin system, for $j = -N - 1, \dots, N + 1$

$$(\mathcal{L}u_a + 2\kappa^2 v_a - F_1, S_j) = 0, \quad (35)$$

$$(\mathcal{L}v_a - 2\kappa^2 u_a - F_2, S_j) = 0. \quad (36)$$

This leads to

$$c_{-N-1}(\mathcal{L}B_0, S_j) + (\mathcal{L}u_h, S_j) + c_{N+1}(\mathcal{L}B_1, S_j) + (2\kappa^2 v_a, S_j) = (F_1, S_j), \quad (37)$$

$$d_{-N-1}(\mathcal{L}B_0, S_j) + (\mathcal{L}v_h, S_j) + d_{N+1}(\mathcal{L}B_1, S_j) + (2\kappa^2 u_a, S_j) = (F_2, S_j). \quad (38)$$

With the sinc quadrature rule in (31) we evaluate the inner products in (37)-(38). First, we evaluate $(\mathcal{L}B_0, S_j)$ and $(\mathcal{L}B_1, S_j)$ as

$$(\mathcal{L}B_0, S_j) = \int_0^1 \mathcal{L}B_0(z) \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \approx h \frac{\mathcal{L}B_0(z_j)}{(\phi'(z_j))^{3/2}}, \quad (39)$$

$$(\mathcal{L}B_1, S_j) = \int_0^1 \mathcal{L}B_1(z) \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \approx h \frac{\mathcal{L}B_1(z_j)}{(\phi'(z_j))^{3/2}}. \quad (40)$$

The nodal points $z_j = e^{jh}/(1 + e^{jh})$. The numerators in these terms are

$$\begin{aligned}
\mathcal{L}B_0(z) &= -\frac{d}{dz} \left(A_v(z) \frac{dB_0(z)}{dz} \right) = 6 \frac{d}{dz} \left(A_v(z) z(1-z) \right) \\
&= 6 \left(A_v(z)(1-2z) + A'_v(z) z(1-z) \right), \\
\mathcal{L}B_1(z) &= -\frac{d}{dz} \left(A_v(z) \frac{dB_1(z)}{dz} \right) = -\frac{d}{dz} \left(A_v(z)(2-3z+6\sigma-6\sigma z)z \right) \\
&= - \left(A_v(z)(2+6\sigma-6z-12z) + A'_v(z)(2-3z+6\sigma-6\sigma z)z \right).
\end{aligned}$$

The inner products $(\mathcal{L}u_h, S_j)$ and $(\mathcal{L}v_h, S_j)$ in (37)-(38) are evaluated by integrating by parts twice. Thus since

$$\mathcal{L}u_h(z) = \sum_{k=-N}^N c_k \left(-A_v(z) \left(\frac{S_k(z)}{\phi'(z)} \right)' \right)', \quad (41)$$

the inner product involving $\mathcal{L}u_h$ is

$$(\mathcal{L}u_h, S_j) = \sum_{k=-N}^N c_k \left(- \left(A_v \left(\frac{S_k}{\phi'} \right)' \right)', S_j \right). \quad (42)$$

The inner product is evaluated by integrating by parts twice to get

$$\begin{aligned}
\left(\left(-A_v \left(\frac{S_k}{\phi'} \right)' \right)', S_j \right) &= \int_0^1 \left(-A_v(z) \left(\frac{S_k(z)}{\phi'(z)} \right)' \right)' \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \\
&= B_T - \int_0^1 \frac{S_k(z)}{\phi'(z)} A_v(z) \left(\frac{S_j(z)}{\sqrt{\phi'(z)}} \right)'' dz \\
&\quad - \int_0^1 \frac{S_k(z)}{\phi'(z)} A'_v(z) \left(\frac{S_j(z)}{\sqrt{\phi'(z)}} \right)' dz,
\end{aligned} \quad (43)$$

where

$$B_T = -A_v(z) \left(\frac{S_k(z)}{\phi'(z)} \right)' \frac{S_j(z)}{\sqrt{\phi'(z)}} \Big|_0^1 + A_v(z) \frac{S_k(z)}{\phi'(z)} \left(\frac{S_j(z)}{\sqrt{\phi'(z)}} \right)' \Big|_0^1.$$

B_T tends to zero as shown in [1].

Since $\phi(z) = \ln \left(\frac{z}{1-z} \right)$, then $\frac{1}{\phi'(z)} = z(1-z)$. This leads to the result

$\frac{1}{\sqrt{\phi'(z)}} \left(\frac{1}{\sqrt{\phi'(z)}} \right)' = \frac{1}{2}(1-2z)$, $\frac{1}{(\phi'(z))^{3/2}} \left(\frac{1}{\sqrt{\phi'(z)}} \right)'' = -\frac{1}{4}$. The derivative in (43) are

$$\begin{aligned} \left(\frac{S_j(z)}{\sqrt{\phi'(z)}} \right)' &= \frac{dS_j(z)}{d\phi} \sqrt{\phi'(z)} + S_j(z) \left(\frac{1}{\sqrt{\phi'(z)}} \right)' \\ &= \left(\frac{dS_j(z)}{d\phi} + \frac{1}{2}(1-2z)S_j(z) \right) \sqrt{\phi'(z)}, \end{aligned} \quad (44)$$

and

$$\begin{aligned} \left(\frac{S_j(z)}{\sqrt{\phi'(z)}} \right)'' &= \frac{d^2 S_j(z)}{d\phi^2} (\phi'(z))^{3/2} + \frac{dS_j(z)}{d\phi} \left(\frac{\phi''(z)}{\sqrt{\phi'(z)}} + 2\phi'(z) \frac{-\phi''(z)}{2(\phi'(z))^{3/2}} \right) \\ &\quad + S_j(z) \left(\frac{1}{\sqrt{\phi'(z)}} \right)'' \\ &= \left(\frac{d^2 S_j(z)}{d\phi^2} - \frac{1}{4} S_j(z) \right) (\phi'(z))^{3/2}. \end{aligned} \quad (45)$$

Applying the sinc quadrature rule, (43) becomes

$$\begin{aligned} \left(\left(-A_v \left(\frac{S_k}{\phi'} \right)' \right)', S_j \right) &= - \int_0^1 \frac{S_k(z)}{\phi'(z)} A_v(z) \left(\frac{d^2 S_j(z)}{d\phi^2} - \frac{1}{4} S_j(z) \right) (\phi'(z))^{3/2} dz \\ &\quad - \int_0^1 \frac{S_k(z)}{\phi'(z)} A'_v(z) \left(\frac{dS_j(z)}{d\phi} + \frac{1}{2}(1-2z)S_j(z) \right) \sqrt{\phi'(z)} dz \\ &\approx h \left\{ \frac{-1}{h^2} \delta_{jk}^{(2)} + \frac{1}{4} \delta_{jk}^{(0)} \right\} A_v(z_k) (\phi'(z_k))^{-1/2} \\ &\quad + h \left\{ \frac{-1}{h} \delta_{jk}^{(1)} + \frac{1}{2} (2z_k - 1) \delta_{jk}^{(0)} \right\} A'_v(z_k) (\phi'(z_k))^{-3/2}. \end{aligned}$$

So

$$\begin{aligned} (\mathcal{L}u_h, S_j) &\approx \sum_{k=-N}^N \left\{ \frac{-1}{h^2} \delta_{jk}^{(2)} + \frac{1}{4} \delta_{jk}^{(0)} \right\} A_v(z_k) (\phi'(z_k))^{-1/2} c_k \\ &\quad + \sum_{k=-N}^N \left\{ \frac{-1}{h} \delta_{jk}^{(1)} + \frac{1}{2} (2z_k - 1) \delta_{jk}^{(0)} \right\} A'_v(z_k) (\phi'(z_k))^{-3/2} c_k. \end{aligned} \quad (46)$$

Here the r^{th} derivative of $S(j, h) \circ \phi(z)$, with respect to ϕ , evaluated at the

nodal point z_k is denoted by

$$\frac{1}{h^r} \delta_{jk}^{(r)} \equiv \frac{d^r}{d\phi^r} [S(j, h) \circ \phi(z)] \Big|_{z=z_k}. \quad (47)$$

Similarly

$$\begin{aligned} (\mathcal{L}v_h, S_j) &\approx \sum_{k=-N}^N \left\{ \frac{-1}{h^2} \delta_{jk}^{(2)} + \frac{1}{4} \delta_{jk}^{(0)} \right\} A_v(z_k) (\phi'(z_k))^{-1/2} d_k \\ &\quad + \sum_{k=-N}^N \left\{ \frac{-1}{h} \delta_{jk}^{(1)} + \frac{1}{2} (2z_k - 1) \delta_{jk}^{(0)} \right\} A'_v(z_k) (\phi'(z_k))^{-3/2} d_k. \end{aligned} \quad (48)$$

Next, applying the sinc quadrature rule (31), the inner products (u_a, S_j) and (v_a, S_j) are

$$(u_a, S_j) = \int_0^1 u_a(z) \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \approx h \frac{1}{(\phi'(z_j))^{3/2}} u_a(z_j), \quad (49)$$

$$(v_a, S_j) = \int_0^1 v_a(z) \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \approx h \frac{1}{(\phi'(z_j))^{3/2}} v_a(z_j), \quad (50)$$

where the solutions $u_a(z_j)$ and $v_a(z_j)$ are given by

$$u_a(z_j) = c_{-N-1} B_0(z_j) + c_j \frac{1}{\phi'(z_j)} + c_{N+1} B_1(z_j), \quad (51)$$

$$v_a(z_j) = d_{-N-1} B_0(z_j) + d_j \frac{1}{\phi'(z_j)} + d_{N+1} B_1(z_j). \quad (52)$$

Last, applying the sinc quadrature rule (31), the inner products (F_1, S_j) and (F_2, S_j) are

$$(F_1, S_j) = \int_0^1 F_1(z) \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \approx h \frac{1}{(\phi'(z_j))^{3/2}} F_1(z_j), \quad (53)$$

$$(F_2, S_j) = \int_0^1 F_2(z) \frac{S_j(z)}{\sqrt{\phi'(z)}} dz \approx h \frac{1}{(\phi'(z_j))^{3/2}} F_2(z_j), \quad (54)$$

where F_1 and F_2 are given in (22)-(23). Substituting (39), (40), (46), (48), (51), (52), (53), and (54) into (37)-(38), the result will be linear equations whose solutions give the coefficients for $u_a(z)$ and $v_a(z)$. For $-N-1 \leq j \leq N+1$,

$$\begin{aligned}
& \frac{\mathcal{L}B_0(z_j)}{(\phi'(z_j))^{3/2}} c_{-N-1} + \sum_{k=-N}^N \left\{ \frac{-1}{h^2} \delta_{jk}^{(2)} + \frac{1}{4} \delta_{jk}^{(0)} \right\} A_v(z_k) (\phi'(z_k))^{-1/2} c_k \\
& + \sum_{k=-N}^N \left\{ \frac{-1}{h} \delta_{jk}^{(1)} + \frac{1}{2} (2z_k - 1) \delta_{jk}^{(0)} \right\} A'_v(z_k) (\phi'(z_k))^{-3/2} c_k \\
& + \frac{\mathcal{L}B_1(z_j)}{(\phi'(z_j))^{3/2}} c_{N+1} + 2\kappa^2 (\phi'(z_j))^{-3/2} v_a(z_j) = \frac{F_1(z_j)}{(\phi'(z_j))^{3/2}}
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
& \frac{\mathcal{L}B_0(z_j)}{(\phi'(z_j))^{3/2}} d_{-N-1} + \sum_{k=-N}^N \left\{ \frac{-1}{h^2} \delta_{jk}^{(2)} + \frac{1}{4} \delta_{jk}^{(0)} \right\} A_v(z_k) (\phi'(z_k))^{-1/2} d_k \\
& + \sum_{k=-N}^N \left\{ \frac{-1}{h} \delta_{jk}^{(1)} + \frac{1}{2} (2z_k - 1) \delta_{jk}^{(0)} \right\} A'_v(z_k) (\phi'(z_k))^{-3/2} d_k \\
& + \frac{\mathcal{L}B_1(z_j)}{(\phi'(z_j))^{3/2}} d_{N+1} - 2\kappa^2 (\phi'(z_j))^{-3/2} u_a(z_j) = \frac{F_2(z_j)}{(\phi'(z_j))^{3/2}}.
\end{aligned} \tag{56}$$

Introduce the column vectors $\mathbf{c}$ and $\mathbf{d}$ that represent the coefficients of the approximate solutions in (32)-(33),

$$\mathbf{c} \equiv [c_{-N-1} \ c_{-N} \ \cdots \ c_N \ c_{N+1}]^T, \quad \mathbf{d} \equiv [d_{-N-1} \ d_{-N} \ \cdots \ d_N \ d_{N+1}]^T,$$

and define

$$\begin{aligned}
\mathbf{F}_1 & \equiv [F_1(z_{-N-1}) \ F_1(z_{-N}) \ \cdots \ F_1(z_N) \ F_1(z_{N+1})]^T, \\
\mathbf{F}_2 & \equiv [F_2(z_{-N-1}) \ F_2(z_{-N}) \ \cdots \ F_2(z_N) \ F_2(z_{N+1})]^T, \\
\mathbf{u}_a & \equiv [u_a(z_{-N-1}) \ u_a(z_{-N}) \ \cdots \ u_a(z_N) \ u_a(z_{N+1})]^T, \\
\mathbf{v}_a & \equiv [v_a(z_{-N-1}) \ v_a(z_{-N}) \ \cdots \ v_a(z_N) \ v_a(z_{N+1})]^T,
\end{aligned}$$

where $F_1(z)$ and $F_2(z)$ are given in (22)-(23). The expressions in (47) for each j and k can be stored in a matrix $I^{(r)} = [\delta_{jk}^{(r)}]$. For $r = 0, 1, 2$, we have

$$\delta_{jk}^{(0)} \equiv [S(j, h) \circ \phi(z)]|_{z=z_k} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j, \end{cases} \tag{57}$$

$$\delta_{jk}^{(1)} \equiv h \frac{d}{d\phi} [S(j, h) \circ \phi(z)]|_{z=z_k} = \begin{cases} 0 & , \text{ if } k = j \\ \frac{(-1)^{k-j}}{k-j} & , \text{ if } k \neq j \end{cases}$$

$$\delta_{jk}^{(2)} \equiv h^2 \frac{d^2}{d\phi^2} [S(j, h) \circ \phi(z)]|_{z=z_k} = \begin{cases} \frac{-\pi^2}{3} & , \text{ if } k = j \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & , \text{ if } k \neq j. \end{cases}$$

The following matrices will be some examples for $I^{(0)}, I^{(1)}, I^{(2)}$. Given $-N - 1 \leq j \leq N + 1$, ($m = 2N + 3$), the $m \times m$, square matrices $I^{(0)}, I^{(1)}, I^{(2)}$ are given by

$$I^{(0)} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} = I, \quad I^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \dots & \frac{(-1)^{m-1}}{m-1} \\ 1 & \ddots & \ddots & \ddots & \vdots \\ -\frac{1}{2} & \ddots & \ddots & \ddots & \frac{1}{2} \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \frac{(-1)^m}{m-1} & \dots & -\frac{1}{2} & 1 & 0 \end{bmatrix}, \quad (58)$$

$$I^{(2)} = \begin{bmatrix} -\frac{\pi^2}{3} & 2 & -\frac{2}{2^2} & \dots & \frac{-2(-1)^{m-1}}{(m-1)^2} \\ 2 & \ddots & \ddots & \ddots & \vdots \\ -\frac{2}{2^2} & \ddots & \ddots & \ddots & -\frac{2}{2^2} \\ \vdots & \ddots & \ddots & \ddots & 2 \\ \frac{-2(-1)^{m-1}}{(m-1)^2} & \dots & -\frac{2}{2^2} & 2 & -\frac{\pi^2}{3} \end{bmatrix}. \quad (59)$$

When $-N \leq j \leq N$, we remove the first and last columns of $I^{(0)}, I^{(1)}$, and $I^{(2)}$ in (58)-(59), to arrive at the $m \times n$, ($n = 2N + 1$), non-square matrices

$$I_z^{(0)} = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ 0 & \dots & 0 \end{bmatrix}, \quad I_z^{(1)} = \begin{bmatrix} -1 & \frac{1}{2} & \dots & \frac{(-1)^{m-2}}{m-2} \\ 0 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \frac{1}{2} \\ -\frac{1}{2} & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-1)^{m-1}}{m-2} & \dots & -\frac{1}{2} & 1 \end{bmatrix}, \quad (60)$$

$$(61)$$

$$I_z^{(2)} = \begin{bmatrix} 2 & -\frac{2}{2^2} & \cdots & \frac{-2(-1)^{m-2}}{(m-2)^2} \\ \frac{-\pi^2}{3} & \ddots & \ddots & \vdots \\ 2 & \ddots & \ddots & -\frac{2}{2^2} \\ -\frac{2}{2^2} & \ddots & \ddots & 2 \\ \vdots & \ddots & \ddots & \frac{-\pi^2}{3} \\ \frac{-2(-1)^{m-2}}{(m-2)^2} & \cdots & -\frac{2}{2^2} & 2 \end{bmatrix}. \quad (62)$$

The $n \times n$ square diagonal matrix $\mathcal{D}_n(f)$ is written as

$$D_n(f) = \begin{bmatrix} f(z_{-N}) & & & 0 \\ & \ddots & & \\ & & f(z_0) & \\ & & & \ddots \\ 0 & & & & f(z_N) \end{bmatrix}. \quad (63)$$

From (55) and (56) we arrive at the discrete system

$$\begin{aligned} \mathcal{B}_b \mathbf{c} + 2\kappa^2 \mathcal{D}_m \left(\frac{1}{(\phi')^{3/2}} \right) \mathcal{E}_b \mathbf{d} &= \mathcal{D}_m \left(\frac{1}{(\phi')^{3/2}} \right) \mathbf{F}_1 \\ \mathcal{B}_b \mathbf{d} - 2\kappa^2 \mathcal{D}_m \left(\frac{1}{(\phi')^{3/2}} \right) \mathcal{E}_b \mathbf{c} &= \mathcal{D}_m \left(\frac{1}{(\phi')^{3/2}} \right) \mathbf{F}_2, \end{aligned}$$

where $\mathcal{B}_b$ is defined by the $m \times m$ bordered matrix

$$\mathcal{B}_b \equiv [\mathbf{a}_{-N-1} \mid \mathcal{A}_{ns} \mid \mathbf{a}_{N+1}]. \quad (64)$$

The non-square $m \times n$ matrix

$$\begin{aligned} \mathcal{A}_{ns} &\equiv \left\{ \frac{-1}{h^2} I_z^{(2)} + \frac{1}{4} I_z^{(0)} \right\} \mathcal{D}_n \left(\frac{A_v}{(\phi')^{1/2}} \right) \\ &+ \left\{ \frac{-1}{h} I_z^{(1)} + \frac{1}{2} I_z^{(0)} \mathcal{D}_n (2z - 1) \right\} \mathcal{D}_n \left(\frac{A'_v}{(\phi')^{3/2}} \right) \end{aligned} \quad (65)$$

and the $m \times 1$ column vectors $\mathbf{a}_{-N-1}$ and $\mathbf{a}_{N+1}$ have j^{th} component, $-N-1 \leq j \leq N+1$,

$$[\mathbf{a}_{-N-1}]_j \equiv \frac{\mathcal{L}B_0(z_j)}{(\phi'(z_j))^{3/2}}, \quad [\mathbf{a}_{N+1}]_j \equiv \frac{\mathcal{L}B_1(z_j)}{(\phi'(z_j))^{3/2}}. \quad (66)$$

The $m \times m$ evaluator matrix $\mathcal{E}_b$ is defined by

$$\mathcal{E}_b \equiv \left[\begin{array}{c|c|c} \mathbf{b}_{-N-1} & I^{(0)}\mathcal{D}_n\left(\frac{1}{\phi'}\right) & \mathbf{b}_{N+1} \end{array} \right] \quad (67)$$

and the $m \times 1$ column vectors $\mathbf{b}_{-N-1}$ and $\mathbf{b}_{N+1}$ have j^{th} component

$$[\mathbf{b}_{-N-1}]_j \equiv B_0(z_j), \quad [\mathbf{b}_{N+1}]_j \equiv B_1(z_j). \quad (68)$$

This leads to the coupled linear discrete system

$$\mathcal{A}\mathcal{X} = \mathcal{C} \quad (69)$$

where the $(2m) \times (2m)$ block matrix ($m = 2N + 3$)

$$\mathcal{A} = \left[\begin{array}{c|c} \mathcal{B}_b & 2\kappa^2\mathcal{D}_m\left(\frac{1}{(\phi')^{3/2}}\right)\mathcal{E}_b \\ \hline -2\kappa^2\mathcal{D}_m\left(\frac{1}{(\phi')^{3/2}}\right)\mathcal{E}_b & \mathcal{B}_b \end{array} \right] \quad (70)$$

and the $(2m) \times 1$ column vectors

$$\mathcal{X} = \left[\begin{array}{c} \mathbf{c} \\ \hline \mathbf{d} \end{array} \right], \quad \mathcal{C} = \left[\begin{array}{c} \mathcal{D}_m\left(\frac{1}{(\phi')^{3/2}}\right)\mathbf{F}_1 \\ \hline \mathcal{D}_m\left(\frac{1}{(\phi')^{3/2}}\right)\mathbf{F}_2 \end{array} \right]. \quad (71)$$

It follows that the solutions for $\mathbf{u}_a$ and $\mathbf{v}_a$ at the nodal points are the elements of this column vector $\mathcal{X}$ that are computed from the $(2m) \times (2m)$ block evaluator matrix

$$\left[\begin{array}{c} \mathbf{u}_a \\ \hline \mathbf{v}_a \end{array} \right] = \left[\begin{array}{c|c} \mathcal{E}_b & \mathbf{O} \\ \hline \mathbf{O} & \mathcal{E}_b \end{array} \right] \left[\begin{array}{c} \mathbf{c} \\ \hline \mathbf{d} \end{array} \right], \quad (72)$$

where $\mathbf{O}$ is an $m \times m$ zero matrix. These solutions are related to the approximate nondimensional current components evaluated at the nodal points as

$$\begin{aligned} U_a(z_j) &= u_a(z_j) + \kappa(1 + \sigma - z_j) \cos(\chi), \\ V_a(z_j) &= v_a(z_j) + \kappa(1 + \sigma - z_j) \sin(\chi). \end{aligned}$$

4 Numerical Testing: Constant Eddy Viscosity

We illustrate the accuracy of the Sinc-Galerkin method when applied to the constant eddy viscosity model formulated as a system of real-valued coupled ordinary differential equations. Since the governing equations and variables were nondimensionalized, the only operative constants in (20)-(25) are κ , σ , and χ . In relating these parameters to the “constants of nature”, we adopt the following nominal values: $f = 0.0001 \text{ s}^{-1}$ (appropriate to temperate northern latitudes), sea water density $\rho = 1 \times 10^3 \text{ kg m}^{-3}$, and air density $\rho_{air} = 1.25 \text{ kg m}^{-3}$.

Surface wind stress is assumed to be related to the square of the wind speed W_w (in m s^{-1}) by

$$\tau_w = C_D \rho_{air} W_w^2, \quad (73)$$

where the dimensionless parameter $C_D \approx 0.0012$ for $W_w < 12 \text{ m s}^{-1}$, thereafter increasing linearly to about 0.0025 at gale force winds ($W_w \approx 30 \text{ m s}^{-1}$) [5]. In keeping with [6] and [7], the linear slip bottom stress coefficient k_f is assigned a value of 0.002 m s^{-1} for comparison with other work and to dramatize the change in current speed over the water column. In practice, a value of k_f lower by an order of magnitude may be preferred. Field evidence suggests that the near-surface value of the vertical eddy viscosity is related to the wind speed. Carter [8] has suggested that, if the wind is not fetch-limited and the sea state is fully developed, then $A_v^*(0)$ in units of $\text{m}^2 \text{ s}^{-1}$ is given by

$$A_v^*(0) \approx 0.304 \times 10^{-4} W_w^3. \quad (74)$$

With the parameters and relationships above, and keeping in mind our desire to compare results with those in [9], we choose our constant eddy viscosity to be

$$A_v^*(z^*) \equiv 0.02 \text{ m}^2 \text{ s}^{-1} \quad (75)$$

with $\tau_w = \sqrt{2}/10 = 0.1414 \text{ N m}^{-2}$. Since $D_E = \sqrt{2A_v^*(0)/f} = 20 \text{ m}$, we then have $\kappa = D_0/D_E = D_0/20$. In keeping with [9], we will use $D_0 = 100 \text{ m}$ and hence $\kappa = 5$, which it will be throughout.

The numerical results are compared to the exact solution $W^*(z^*) = U_0[U(z) + iV(z)]$ where $U(z)$ is given by

$$U(z) = \mathcal{R}(W_c(z)) \cos(\chi) - \mathcal{I}(W_c(z)) \sin(\chi) \quad (76)$$

and $V(z)$ is given by

$$V(z) = \mathcal{R}(W_c(z)) \sin(\chi) + \mathcal{I}(W_c(z)) \cos(\chi). \quad (77)$$

Here $\mathcal{R}(W_c(z))$ and $\mathcal{I}(W_c(z))$ denote the real and imaginary parts of $W_c(z)$,

respectively, and

$$W_c(z) = \frac{\kappa(1-i)\sigma \cosh(\kappa(1-i)(1-z)) + \sinh(\kappa(1-i)(1-z))}{(1-i)[\cosh(\kappa(1-i)) + \kappa(1-i)\sigma \sinh(\kappa(1-i))]}.$$

The results of the Sinc-Galerkin approximations $U_a(z_j)$ and $V_a(z_j)$ were compared with the exact solutions for $U(z_j)$ and $V(z_j)$ at the sinc grid points $\mathcal{S}$ with $h = \pi/\sqrt{2N}$ given by

$$\mathcal{S} = \{z_j = \phi^{-1}(jh) = e^{jh}/(e^{jh} + 1) : j = -N-1, \dots, N+1\}. \quad (78)$$

These results were then multiplied by the natural velocity scale U_0 to give a dimensional representation of the velocities.

All numerical simulations were run on a SUN BLADE 1000 with MATLAB Version 6.1. To illustrate the performance of the method, the maximum absolute errors are reported as

$$\begin{aligned} \|U_{\mathcal{S}}\| &= \max_{-N-1 \leq j \leq N+1} \{U_0|U_a(z_j) - U(z_j)|\}, \\ \|V_{\mathcal{S}}\| &= \max_{-N-1 \leq j \leq N+1} \{U_0|V_a(z_j) - V(z_j)|\}, \end{aligned}$$

and

$$\|E_{\mathcal{S}}\| = \max\{\|U_{\mathcal{S}}\|, \|V_{\mathcal{S}}\|\}, \quad (79)$$

where the units are m s^{-1} .

Throughout, comparable graphs (of eddy viscosity functions and velocity components) are shown on the same scale. The horizontal projections of the Ekman spirals are also shown on the same scale. This way visual comparisons of these various quantities are readily made.

Example 1 (Linear stress condition at the seabed) For this example we choose $\chi = 45^\circ$ and for the linear stress condition at the seabed we have $\sigma = A_v^*(D_0)/(k_f D_0) = 0.1$. We find the approximate solution $U_a(z)$ and $V_a(z)$, respectively, using the coupled discrete system of size $2m \times 2m$ ($m = 2N + 3$), given in (69) where $U_0 U_a(z) = U_a^*(z^*)$ and $U_0 V_a(z) = V_a^*(z^*)$. The errors are given in Table 1 and illustrate the classic exponential convergence typical of Sinc-Galerkin methods.

Figure 5 graphically depicts the numerical convergence of the Sinc-Galerkin method to the exact solution with the linear stress bottom boundary condition, as N is repeatedly doubled in size. The horizontal projection of the Ekman spiral for $N = 64$ is indistinguishable from the true solution shown as the solid line.

N	$2m$	h	$\ U_S\ $ (m s ⁻¹)	$\ V_S\ $ (m s ⁻¹)	$\ E_S\ $ (m s ⁻¹)
4	22	1.111	1.080e-03	7.478e-04	1.080e-03
8	38	0.785	2.496e-04	1.293e-04	2.496e-04
16	70	0.555	2.762e-05	1.302e-05	2.762e-05
32	134	0.393	8.985e-07	4.240e-07	8.985e-07
64	262	0.278	5.775e-09	2.770e-09	5.775e-09
128	518	0.196	4.053e-12	1.937e-12	4.053e-12

Table 1

Errors for Example 1 (constant eddy viscosity) on the sinc grid $\mathcal{S}$ with the linear stress bottom condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

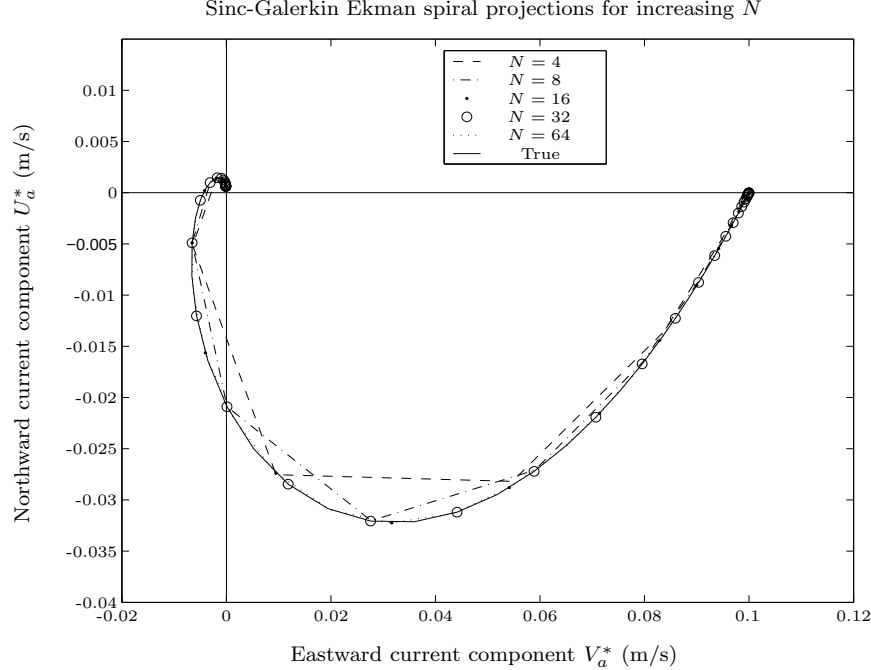


Fig. 5. Sinc-Galerkin Ekman spiral projections for Example 1 with increasing N for constant eddy viscosity with the linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

Example 2 (No-slip condition at the seabed) We set $\sigma = 0$ and $\chi = 45^\circ$ and solve for approximate solutions $U_a(z)$ and $V_a(z)$, respectively, by using the coupled discrete system in (69). The reported errors are shown in Table 2 and are very similar to those for Example 1. The accuracy is no different for the Neumann condition in this example than it is for the mixed condition in Example 1. The horizontal projection of the Ekman spirals is shown in Figure 6. This figure portrays the exponential convergence of the method.

N	$2m$	h	$\ U_S\ $ (m s ⁻¹)	$\ V_S\ $ (m s ⁻¹)	$\ E_S\ $ (m s ⁻¹)
4	22	1.111	1.071e-03	7.500e-04	1.071e-03
8	38	0.785	2.483e-04	1.297e-04	2.483e-04
16	70	0.555	2.752e-05	1.306e-05	2.752e-05
32	134	0.393	8.957e-07	4.255e-07	8.957e-07
64	262	0.278	5.756e-09	2.781e-09	5.756e-09
128	518	0.196	4.073e-12	1.975e-12	4.073e-12

Table 2

Errors for Example 2 (constant eddy viscosity) on the sinc grid $\mathcal{S}$ with the zero-velocity bottom condition for $\sigma = 0$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

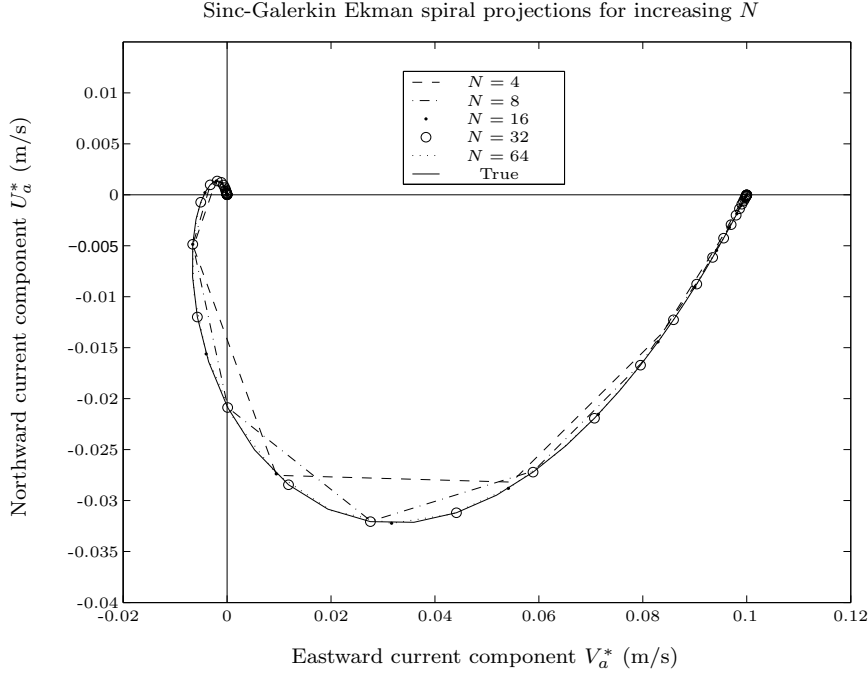


Fig. 6. Sinc-Galerkin Ekman spiral projections for Example 2 with increasing N for the case of constant eddy viscosity with no-slip bottom boundary condition for $\sigma = 0$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

5 Numerical Testing: Variable Eddy Viscosity

The following examples show the results for the system of real-valued coupled ordinary differential equations with either a decreasing or quadratic eddy viscosity function given by (80) or (81). An eddy viscosity which decreases quadratically from the value of $A_v^*(0) = 0.02$ m²s⁻¹ to the minimum value of

$A_v^*(D_0) = 0.00125 \text{ m}^2\text{s}^{-1}$ is given by

$$A_v^*(z^*) = 0.02[1 - (.0075)z^*]^2, \quad 0 < z^* < D_0 = 100 \text{ m}. \quad (80)$$

A graph of this $A_v^*(z^*)$ is shown in Figure 7 where it is contrasted with the constant eddy viscosity of $A_v^*(z^*) \equiv 0.02 \text{ m}^2\text{s}^{-1}$. A quadratic eddy viscosity

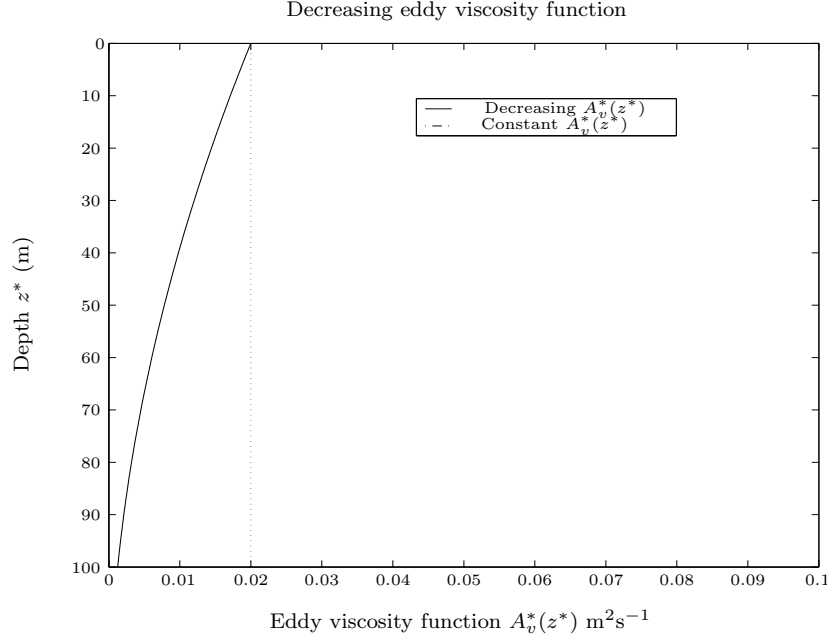


Fig. 7. Eddy viscosity functions $A_v^*(z^*) = 0.02[1 - (.0075)z^*]^2 \text{ m}^2\text{s}^{-1}$ and $A_v^*(z^*) \equiv 0.02 \text{ m}^2\text{s}^{-1}$

function that ranges from $A_v^*(0) = 0.02\text{m}^2\text{s}^{-1}$ back to $A_v^*(D_0) = 0.02 \text{ m}^2\text{s}^{-1}$ is given by

$$A_v^*(z^*) = 0.02[1 + (.12)z^*(1 - (.01)z^*)], \quad 0 < z^* < D_0 = 100 \text{ m}, \quad (81)$$

which is shown in Figure 8. The model with the decreasing and quadratic eddy viscosity functions are solved by the coupled discrete system in (69) of size $2m \times 2m$ ($m = 2N + 3$). Also the results from the coupled system scheme are compared to the results in [1] which used a complex-valued ordinary differential equation.

Example 3 For this example in a sea of depth $D_0 = 100 \text{ m}$, the parameters are chosen to be $\sigma = 0.1$, $\chi = 45^\circ$, and (since $D_E = 20 \text{ m}$) $\kappa = 5$. The decreasing eddy viscosity function $A_v^*(z^*)$ is given by (80). We find the approximate solutions $U_a(z)$ and $V_a(z)$ by using the coupled discrete system in (69). The results comparing the constant eddy viscosity and the decreasing eddy viscosity are shown in Figure 9 and Figure 10 for $N = 32$. Figure 9 contrasts the Ekman spiral projections while Figure 10 depicts the velocity components $U_a^*(z^*)$ and $V_a^*(z^*)$ from the surface to the sea bottom. The convergence of

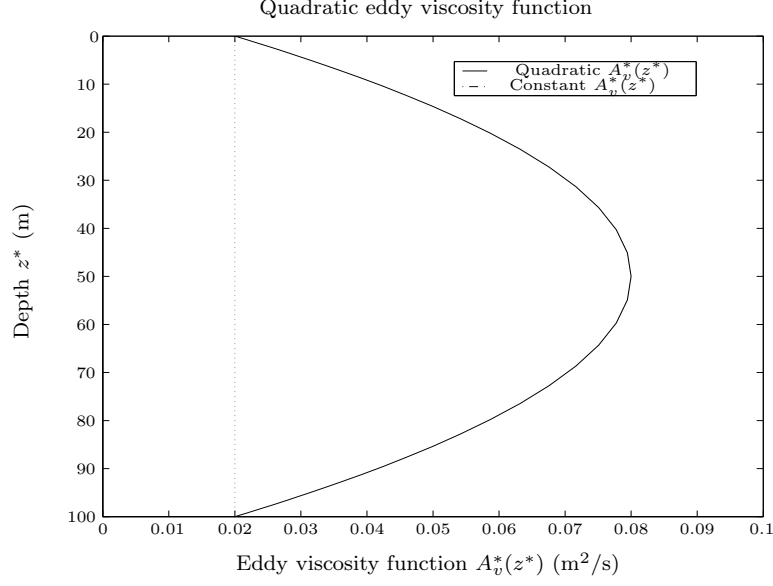


Fig. 8. Eddy viscosity functions $A_v^*(z^*) = 0.02[1 + (.12)z^*(1 - (.01)z^*)]$ m^2s^{-1} and $A_v^*(z^*) \equiv 0.02 \text{ m}^2\text{s}^{-1}$

the Sinc-Galerkin Ekman spiral projections for the decreasing eddy viscosity function in (81) is illustrated in Figure 11.

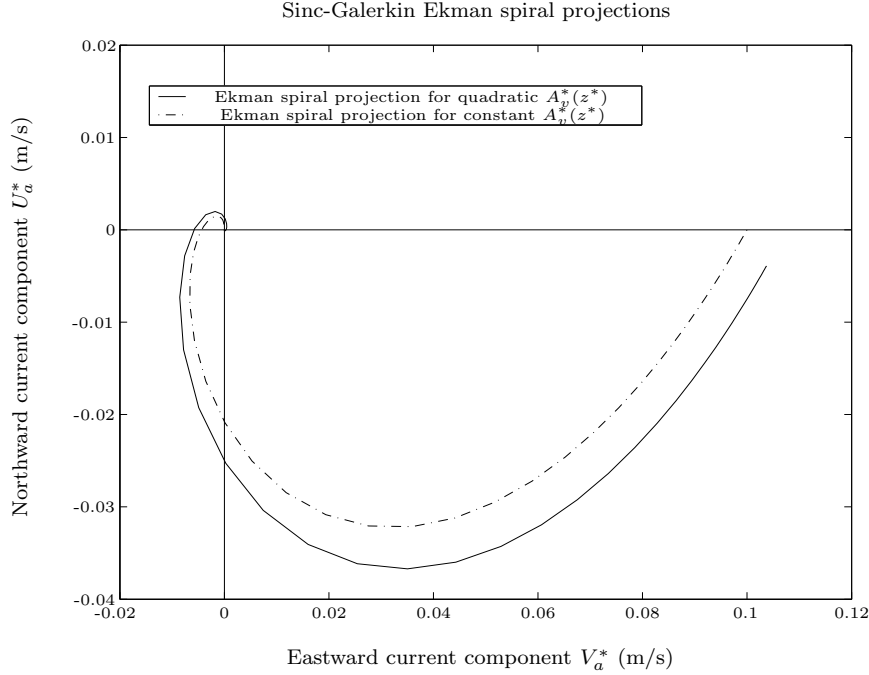


Fig. 9. Sinc-Galerkin Ekman spiral projections for Example 3 with both constant and decreasing eddy viscosity functions and linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100 \text{ m}$, $D_E = 20 \text{ m}$

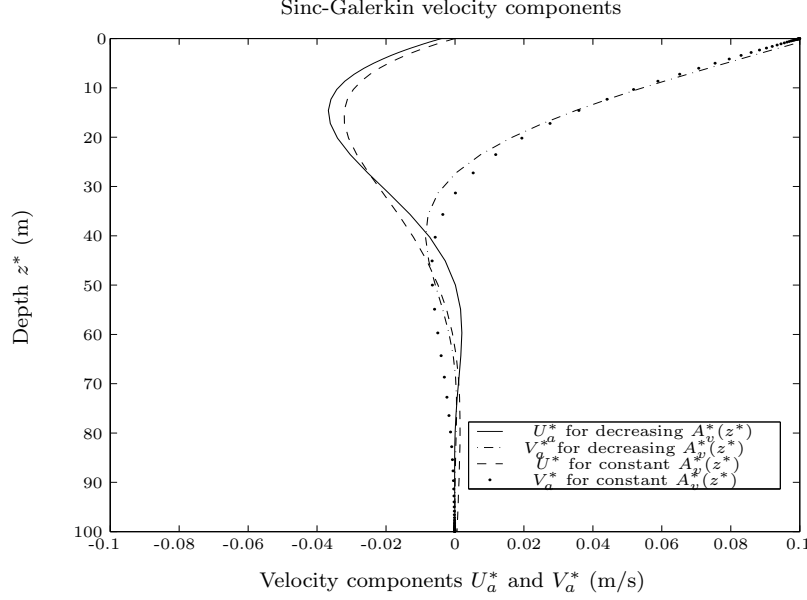


Fig. 10. Sinc-Galerkin northward and eastward calculated velocity profiles for Example 3 with constant and decreasing eddy viscosity functions and linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

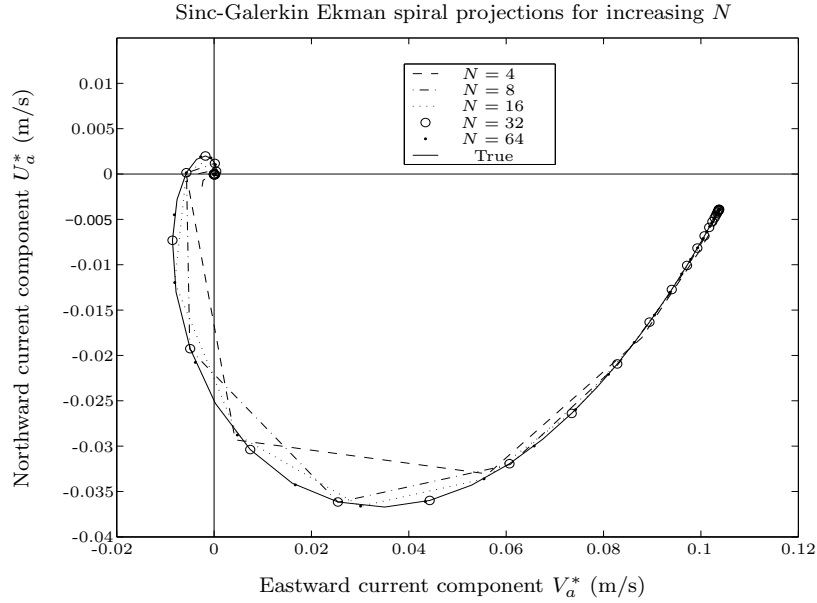


Fig. 11. Sinc-Galerkin Ekman spiral projections for Example 3 with increasing N for the case of the decreasing eddy viscosity function and linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

Example 4 With the quadratic eddy viscosity function in (81), the parameters are again chosen to be $\sigma = 0.1$, $\chi = 45^\circ$, and $\kappa = 5$. The approximate solutions $U_a(z)$ and $V_a(z)$ are illustrated by graphs comparing the constant eddy viscosity with the quadratic eddy viscosity in Figure 12 and Figure 13 for $N = 32$. The convergence of the Sinc-Galerkin Ekman spiral projections

for the quadratic eddy viscosity function in (81) is illustrated in Figure 14.

The approximations are calculated from the $2m \times 2m$ coupled discrete system in (69).

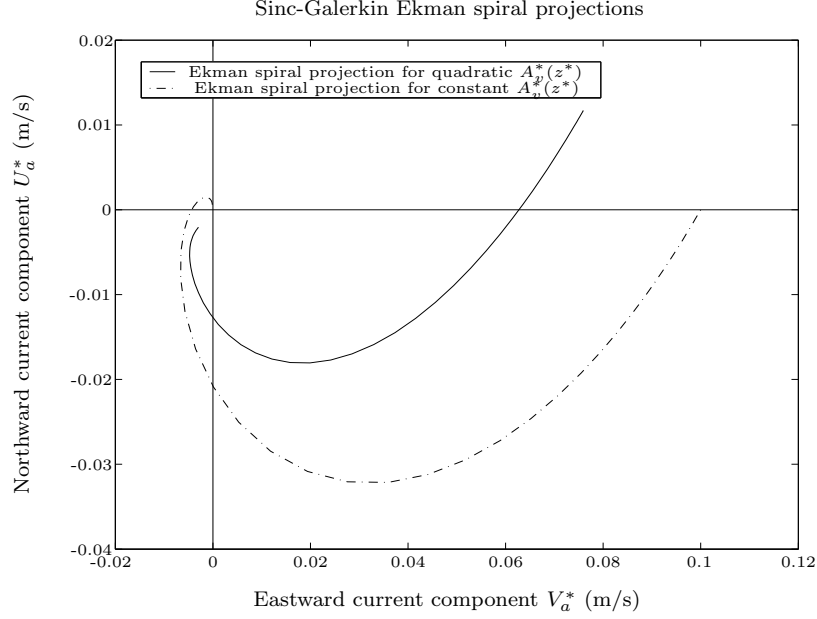


Fig. 12. Sinc-Galerkin Ekman spiral projections for Example 4 with both constant and quadratic eddy viscosity functions and linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

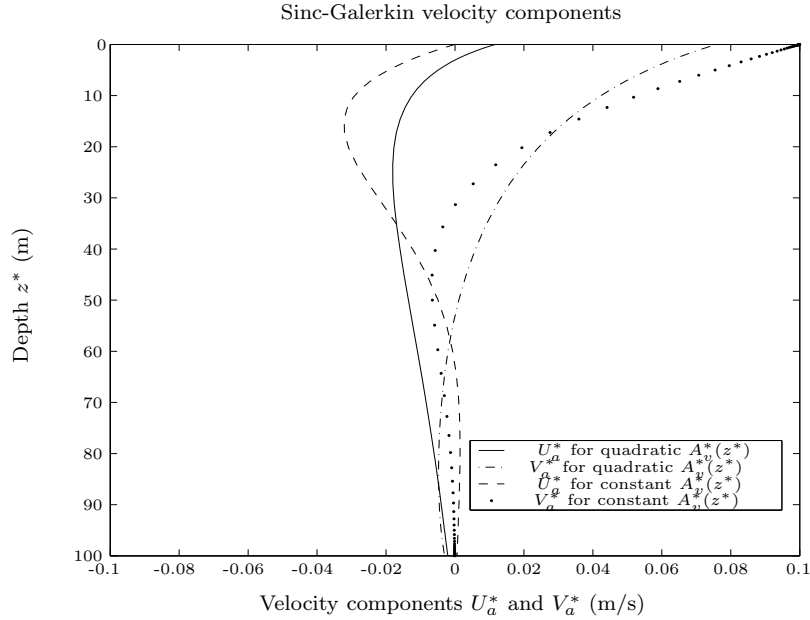


Fig. 13. Sinc-Galerkin northward and eastward calculated velocity profiles for Example 4 with constant and quadratic eddy viscosity functions and linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

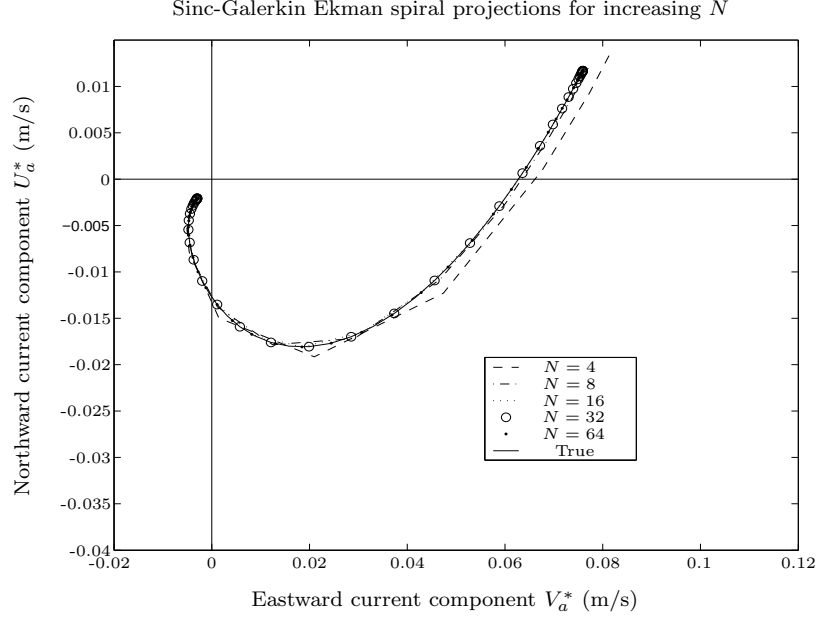


Fig. 14. Sinc-Galerkin Ekman spiral projections for Example 4 with increasing N for the case of the quadratic eddy viscosity function and linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

Next, we want to compare the approximate solutions $U_a(z)$ and $V_a(z)$ for the complex velocity system in [1] and the new coupled system introduced here. To illustrate the comparative performance of both methods, the maximum absolute errors between their respective numerical approximations are reported as

$$\begin{aligned} \|U_C\| &= \max_{-N-1 \leq j \leq N+1} \{U_0 |U_1(z_j) - U_2(z_j)|\}, \\ \|V_C\| &= \max_{-N-1 \leq j \leq N+1} \{U_0 |V_1(z_j) - V_2(z_j)|\}, \\ \|E_C\| &= \max \{\|U_C\|, \|V_C\|\}, \end{aligned}$$

where $U_1(z)$ and $V_1(z)$ are computed by the complex velocity system in [1] and $U_2(z)$ and $V_2(z)$ are computed by the coupled system in (69).

Example 5 This example compares the solution introduced here to that computed by the complex velocity system in [1]. The example used is that described in Example 1. All comparisons show a similarity greater than the accuracy of the method reported in Table 1

N	m	$2m$	$\ U_C\ $	$\ V_C\ $	$\ E_C\ $
4	11	22	1.005e-15	8.163e-16	1.005e-15
8	19	38	9.419e-16	1.193e-15	1.193e-15
16	35	70	6.279e-16	6.217e-15	6.217e-15
32	67	134	1.130e-15	2.009e-15	2.009e-15
64	131	262	5.651e-15	1.155e-14	1.155e-14
128	259	518	1.463e-14	1.758e-14	1.758e-14

Table 3

Comparison between the approximate solutions for the complex velocity system and the coupled system by using the same sinc grid size for Example 1 for the case of constant eddy viscosity with linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

Example 6 This example compares the solution introduced here to the complex velocity system in [1]. The comparison used Example 4. Again the results are extremely similar. Note that the method used here requires none of the complex arithmetic necessary in the work of [1].

N	m	$2m$	$\ U_C\ $	$\ V_C\ $	$\ E_C\ $
4	11	22	1.884e-16	9.419e-16	9.419e-16
8	19	38	3.768e-16	1.193e-15	1.193e-15
16	35	70	2.198e-16	3.579e-15	3.579e-15
32	67	134	1.381e-15	2.261e-15	2.261e-15
64	131	262	4.969e-15	1.068e-14	1.068e-14
128	259	518	4.458e-14	2.003e-14	4.458e-14

Table 4

Comparison between the approximate solutions for the complex velocity system and the coupled system on the same sinc grid for Example 4 for the decreasing eddy viscosity $A_v^*(z^*) = .02(1 + .12z^*(1 - .01z^*))$ with linear stress bottom boundary condition for $\sigma = 0.1$, $\chi = 45^\circ$, $\kappa = 5$, $D_0 = 100$ m, $D_E = 20$ m

References

- [1] D. F. Winter, J. Lund, and K. L. Bowers. Wind-driven currents in a sea with a variable eddy viscosity calculated by a Sinc function Galerkin technique. *Internat. J. Numer. Methods Fluids*, 33:1041-1073, 2000.
- [2] J. Brown, A. Colling, D. Park, J. Phillips, D. Rothery, and J. Wright. *Ocean Circulation*, The Open University, Keynes, 1989.
- [3] J. Lund and K. L. Bowers. *Sinc Methods for Quadrature and Differential Equations*, SIAM: Philadelphia, 1992.
- [4] F. Stenger. *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, New York, 1993.
- [5] W. G. Large and S. Pond. Open ocean momentum flux measurements in moderate to strong winds. *J. Phys. Oceanogr*, 11:324-336, 1981.
- [6] A. M. Davies and A. Owen. Three-dimensional numerical sea model using Galerkin method with a polynomial basis set. *Appl. Math. Modeling*, 3:421-428, 1979.
- [7] N. S. Heaps. On the numerical solution of the three-dimensional hydrodynamical equations for tides and storm surges. *Mem. Soc. Sci. Liege., Ser. 6*, 1:143-180, 1971.
- [8] D. J. T. Carter. Estimation of wave spectra from wave height and period. *Inst. Oceanogr. Sci. Rep*, 135, 1982.
- [9] C. E. Naimie. *A Turbulent Boundary Layer Model for the Linearized Shallow Water Equations*, *NUBBLE USER'S MANUAL (Release 1.1)*. Technical Report NML-96-1, Dartmouth College, July 31, 1996.