App M on the subject of derivatives of Ax and Az PhL 9.15.14

Here I do some calculations in an attempt to see what Appendix M says about the size of $\partial_x A_x$ versus $\partial_z A_z$ in terms of the Helmholtz Integrals. My first results are

$$\partial_z A_z(\mathbf{x},\omega) = -\frac{1}{4\pi}\,(1+j\beta_d)\,\Sigma_i \int \mu_i\; J_{zi}(x',y',\omega)\, I_1(x',y')\; dx'\, dy'$$

$$\partial_x A_x(\mathbf{x},\omega) = -\frac{1}{4\pi}\,(1+j\beta_d)\,\Sigma_i \int \mu_i\; J_{xi}(x',y',\omega)\, I_2(x',y')\,(x-x')\, dx'\, dy'$$

where

$$I_1 \;=\; -2\,j\, e^{-jkz} \int_0^\infty dz''\, [z''\sin(kz'')]\,(1+j\beta_d R)\, e^{-j\beta_d R}/R^3 \quad = \text{even in } (x-x')$$

$$I_2 = \;\; 2\, e^{-jkz} \int_0^\infty dz''\; [\cos(kz'')]\;(1+j\beta_d R)\, e^{-j\beta_d R}/R^3 \qquad = \text{even in } (x-x')$$

I continue on, but these integrals are a bit messy though they can be done. I would associate large ω with large k since k = ω/v_d, so large number of waves per meter at large ω. Then large ω chops up the I_2 integral, making I_2 small, and then $\partial_x A_x(\mathbf{x},\omega)$ becomes small, which is the desired result. In contrast, the I_1 integral seems small at small k and at large k, since large k just chops up zero.

As ω→ 0 and k→ 0, the I_2 integral grows larger, causing $\partial_x A_x(\mathbf{x},\omega)$ to increase in size. So this supports my general conjecture.

I just added this conjecture as Observation 3 at the end of Appendix M just now !

Does App M say something about these transverse derivatives by implication? That appendix shows that the current ratio J_r/J_z is small, and then it uses $\mathbf{A_t}(\mathbf{x}) = -\frac{1}{4\pi}\;\Sigma_i\,\mu_i \int \mathbf{J}_{t,i}(x',y')\,\ln(s^2)\,dx'\,dy'$ to argue that A_r/A_z is therefore small. You could perhaps consider

$$\partial_x A_x \;= -\frac{1}{4\pi}\;\Sigma_i\,\mu_i \int J_{x,i}(x',y')\,\partial_x \ln(s^2)\,dx'\,dy'$$

$$\partial_z A_z \;= -\frac{1}{4\pi}\;\Sigma_i\,\mu_i \int J_{z,i}(x',y')\,\partial_z \ln(s^2)\,dx'\,dy' \;\; = \;0$$

The second line gives 0 only after we have done the approximate process leading from (M.2) to (M.3), and this suggests the strange result that $\partial_x A_x >> \partial_z A_z$. So maybe I have to back up and write

$$A_z(\mathbf{x},\omega) = \frac{1}{4\pi}\;\Sigma_i \int \mu_i\; J_{zi}(x',y',\omega)\, e^{-jkz'}\,\frac{e^{-j\beta_d R}}{R}\,dV' \qquad R = |\,\mathbf{x}-\mathbf{x}'\,| \qquad \textit{(1.5.9)}$$

$$\partial_z A_z(\mathbf{x},\omega) = \frac{1}{4\pi} \sum_i \int \mu_i \; J_{zi}(x',y',\omega)\, e^{-jkz'}\, \partial_z\left[\frac{e^{-j\beta_d R}}{R}\right] dV' \qquad \text{// using } e^{-jkz} \text{ ansats}$$

What do I know about this ∂_z object in there? I don't think $\partial_z\left[\frac{e^{-j\beta_d R}}{R}\right]$ has ever appeared before in lines doc! I would have to compute this out,

$$\partial_z\left[\frac{e^{-j\beta_d R}}{R}\right] = \{\, R\,[\partial_z\, e^{-j\beta_d R}] - e^{-j\beta_d R}\partial_z R\,\}\,/R^2$$

But $\partial_z\, e^{-j\beta_d R} = e^{-j\beta_d R}\,[-j\beta_d\, \partial_z R]$ so then

$$\partial_z\left[\frac{e^{-j\beta_d R}}{R}\right] = \{\, R\, e^{-j\beta_d R}(-j\beta_d)\;\partial_z R - e^{-j\beta_d R}\partial_z R\,\}\,/R^2$$

$$= e^{-j\beta_d R}\,(\partial_z R)\,\{\, -j\beta_d R - 1\,\}/R^2 \qquad \text{// Maple verified}$$

Now

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$\partial_z R^2 = 2(z-z')$$
$$\partial_z R^2 = 2R(\partial_z R) \qquad \Rightarrow \qquad (\partial_z R) = (z-z')/R$$

Therefore

$$(\partial_z R) = 2(z-z')/(2R) = (z-z')/R \quad = \text{very simple}$$

and then

$$\partial_z\left[\frac{e^{-j\beta_d R}}{R}\right] = -\, e^{-j\beta_d R}\,(z-z')\,(j\beta_d R + 1)/R^3$$

So we are then left with

$$\partial_z A_z(\mathbf{x},\omega) = \frac{1}{4\pi} \sum_i \int \mu_i \; J_{zi}(x',y',\omega)\, e^{-jkz'}\, \partial_z\left[\frac{e^{-j\beta_d R}}{R}\right] dV'$$

$$= -\frac{1}{4\pi} \sum_i \int \mu_i \; J_{zi}(x',y',\omega)\, e^{-jkz'} e^{-j\beta_d R}\,(1+j\beta_d R)\,(z-z')\,/R^3 \; dV'$$

The same processing would yield,

$$\partial_x A_x(\mathbf{x},\omega) = \frac{1}{4\pi}\ \Sigma_i \int \mu_i\ J_{zi}(x',y',\omega)\ e^{-jkz'}\ \partial_x[\frac{e^{-j\beta_d R}}{R}\ ]dV'$$

$$= -\frac{1}{4\pi}\ \Sigma_i \int \mu_i\ J_{xi}(x',y',\omega)\ e^{-jkz'} e^{-j\beta_d R}\ (1+j\beta_d)\ (x-x')\ /R^3\ dV'$$

To summarize

$$\partial_z A_z(\mathbf{x},\omega) = -\frac{1}{4\pi}\ \Sigma_i \int \mu_i\ J_{zi}(x',y',\omega)\ e^{-jkz'} e^{-j\beta_d R}\ (1+j\beta_d R)\ (z-z')\ /R^3\ dV'$$

$$\partial_x A_x(\mathbf{x},\omega) = -\frac{1}{4\pi}\ \Sigma_i \int \mu_i\ J_{xi}(x',y',\omega)\ e^{-jkz'} e^{-j\beta_d R}\ (1+j\beta_d R)\ (x-x')\ /R^3\ dV'$$

Now we have two different dz' integrals here which are then integrands in the above,

$$I_1\ \equiv\ \int_{-\infty}^{\infty} dz'\ e^{-jkz'}\ e^{-j\beta_d R}\ (1+j\beta_d R)/R^3 * (z-z')$$

$$I_2 \equiv\ \int_{-\infty}^{\infty} dz'\ e^{-jkz'}\ e^{-j\beta_d R}\ (1+j\beta_d R)/R^3 \qquad R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2\ = s^2 +\ (z-z')^2$$

In terms of these integrals, we can write

$$\partial_z A_z(\mathbf{x},\omega) = -\frac{1}{4\pi}\ \Sigma_i \int \mu_i\ J_{zi}(x',y',\omega)\ I_1(x',y')\ dx'\ dy'$$

$$\partial_x A_x(\mathbf{x},\omega) = -\frac{1}{4\pi}\ \Sigma_i \int \mu_i\ J_{xi}(x',y',\omega)\ I_2(x',y')\ (x-x')\ dx'\ dy'$$

We now study the two integrals.

Rewrite the first using z" = z-z' and dz" = -dz' so z' = z - z"

$$I_1 = \int_{-\infty}^{\infty} dz''\ e^{-jkz}\ e^{+jkz''}\ (1+j\beta_d R)\ e^{-j\beta_d R}/R^3 * (z'') \qquad R^2 = s^2 + z''^2$$

or

$$I_1 =\ e^{-jkz}\ \int_{-\infty}^{\infty} dz''\ z''\ e^{jkz''}\ (1+j\beta_d R)\ e^{-j\beta_d R}/R^3$$

$$= e^{-jkz}\ \int_{-\infty}^{\infty} dz''\ z''\ [\cos(kz'') + j\sin(kz'')]\ e^{-j\beta_d R}/R^3$$

$$= j\, e^{-jkz} \int_{-\infty}^{\infty} dz''\, z'' [j\sin(kz'')] (1+j\beta_d R)\, e^{-j\beta_d R}/R^3$$

$$= -j\, e^{-jkz} \int_{-\infty}^{\infty} dz'' [z'' \sin(kz'')] (1+j\beta_d R)\, e^{-j\beta_d R}/R^3$$

I expect this integral to be dominated by the region near z'' = 0, but there we have z'' sin(kz'') active.

Meanwhile,

$$I_2 = \int_{-\infty}^{\infty} dz'\; e^{-jkz'} (1+j\beta_d R) e^{-j\beta_d R}/R^3 =$$

$$= e^{-jkz} \int_{-\infty}^{\infty} dz''\; e^{+jkz''} (1+j\beta_d R) e^{-j\beta_d R}/R^3$$

$$= e^{-jkz} \int_{-\infty}^{\infty} dz''\; [\cos(kz'') + j\sin(kz'')] (1+j\beta_d R) e^{-j\beta_d R}/R^3$$

$$= e^{-jkz} \int_{-\infty}^{\infty} dz''\; [\cos(kz'')]\; (1+j\beta_d R) e^{-j\beta_d R}/R^3$$

and here we have no such zeroing out factor. So I expect I_2 = large and I_1 = small. To summarize,

$$I_1 = -j\, e^{-jkz} \int_{-\infty}^{\infty} dz'' [z'' \sin(kz'')] (1+j\beta_d R) e^{-j\beta_d R}/R^3$$

$$= -2\, j\, e^{-jkz} \int_{0}^{\infty} dz'' [z'' \sin(kz'')] (1+j\beta_d R)\, e^{-j\beta_d R}/R^3$$

$$I_2 = e^{-jkz} \int_{-\infty}^{\infty} dz''\; [\cos(kz'')]\; (1+j\beta_d R)\, e^{-j\beta_d R}/R^3$$

$$= 2\, e^{-jkz} \int_{0}^{\infty} dz''\; [\cos(kz'')]\; (1+j\beta_d R)\, e^{-j\beta_d R}/R^3$$

I wonder if these integrals can be done?

Skip this, it is all wrong since I failed to have the $(1+j\beta_d R)$ factor when I did them!

Write

$$J_1 \equiv \int_{0}^{\infty} dx\, [x \sin(kx)]\, e^{-jbR}/R^3 \qquad R^2 = s^2 + x^2$$

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Maple can do both as some hypergeometric series things. Wow. So maybe then it is worth trying to look these up in GR7. Perhaps change from x to R as integration variable

$$R^2 = s^2 + x^2 \qquad RdR = xdx \qquad x = \sqrt{R^2-s^2}$$

$$J_1 = \int_0^\infty RdR \sin(k\sqrt{R^2-s^2}\,)\, e^{-jbR}/R^3$$

$$= \int_0^\infty dR \sin(k\sqrt{R^2-s^2}\,)\, e^{-jbR}/R^2$$

I cannot find anything like this in any of the forms. BUT, maybe I can do a contour integration. Write

$$K_2 \equiv \int_{-\infty}^\infty dx\, [x \sin(kx)]\, e^{-jbR}/R^3$$

This thing has zeros on the real axis which are fine. The only singularities are where R vanishes and these unfortunately are branch points at R = ± s. OK I give up. How about method of **stationary phase**? Consider

$$T_1 \equiv \int_0^\infty dt\, e^{jkt}\ [t\, e^{-j\beta_d R}/R^3] \qquad R = s^2 + t^2 \qquad \text{let } x = k$$

$$T_1 \equiv \int_0^\infty dt\, e^{jxt}\ f(t) \qquad f(t) = t\, e^{-j\beta_d R}/R^3 \qquad \psi(t) = t \qquad \psi'(t) = 1$$

Look at the boundaries, and note that $f(t)/\psi'(t) = f(t)$. At t = 0 we have f(t) = 0 so it is zero at one of the boundaries and this invalidates my little set of notes. But let's just follow the notes anyway

$$\text{parts} = [\, f(t)/(ix) * e^{ixt}]^{t=\infty}_0 = (1/ix)\, [\, f(t)\, e^{ixt}]^{t=\infty}_0$$

As $t\to\infty$ we have $R\to t\neq\infty$ so then $f(t) = t\, e^{-jbt}/t^3 \sim 1/t^2 = 0$.
As $t\to-\infty$ same thing. So the parts then just vanish and you get no answer.

Suppose we assume that most of the integral comes from some central region. Then try

$$R_1 \equiv \int_{-a}^{a} dt\, e^{jkt}\ [t\, e^{-j\beta_d R}/R^3] \qquad R^2 = s^2 + t^2 \qquad \text{let } x = k$$

$$R_1 \equiv \int_{-a}^{a} dt\, e^{jxt}\ f(t) \qquad f(t) = t\, e^{-j\beta_d R}/R^3 \qquad \psi(t) = t \qquad \psi'(t) = 1$$

Now do the parts and we have

$$\text{parts}_1 = [\, f(t)/(ix) * e^{ixt}]^{t=a}_{-a} = (1/ix)\, [\, f(a)e^{iax} - f(-a)e^{-iax}\,]$$

$= (1/ix)\,[a\, e^{-j\beta_d R}/R^3\, e^{iax} - (-a)\, e^{-j\beta_d R}/R^3\, e^{-iax}]$ $\qquad R^2 = s^2 + a^2$

$= (1/ix)\, a\, e^{-j\beta_d R}/R^3\, [e^{iax} + e^{-iax}]$

$= (1/ix)\, a\, e^{-j\beta_d R}/R^3\, [2\cos(ax)]$

$= (1/ix)\cos(ax)\, 2a\, e^{-j\beta_d R}/R^3$ $\qquad R^2 = s^2 + a^2$

$= (1/ik)\cos(ak)\, 2a\, e^{-j\beta_d R}/R^3$ $\qquad R^2 = s^2 + a^2$

Well, this just says integral is small as k gets large which I already imagine. The other integral is missing the t factor so it might be

$\text{parts}_2 = [\, f(t)/(ix) * e^{ixt}]^{t=a}_{-a} = (1/ix)\,[\, f(a)e^{iax} - f(-a)e^{-iax}\,]$

$= (1/ix)\; e^{-j\beta_d R}/R^3\, [2j\sin(ax)]$

$= (1/ik)\; 2j\sin(ak)\, e^{-j\beta_d R}/R^3$

Then one might argue that

$\text{parts}_1 = (1/ik)\cos(ak)\, 2a\, e^{-j\beta_d R}/R^3$
$\text{parts}_2 = (1/ik)\sin(ak)\; 2j\, e^{-j\beta_d R}/R^3 * D$ $\qquad D$ = transverse dimension

Then somehow $\text{parts}_2/\text{parts}_1 \approx D/a$ = small if I take a large and then $I_2 << I_1$ which is the desired result. So somehow I have managed to get my desired result.

Conclusion: It is not impossible to imagine *based on the above* that $|\partial_x A_x| >> |\partial_z A_z|$ at low ω but just the opposite at large ω.

Should I put anything into App M on this issue?

$$J_1 = \int_0^\infty [(R/\sqrt{R^2-s^2})dR]\,\sqrt{R^2-s^2}\;\sin(k\sqrt{R^2-s^2})\; e^{-jbR}/R^3$$

For the usual reasons (domination by small z" region) I expect $I_2 >> I_1$. Now going back

$$\partial_z A_z(\mathbf{x},\omega) = -\frac{1}{4\pi}\,(1+j\beta_d)\,\Sigma_i \int \mu_i\; J_{zi}(x',y',\omega)\, I_1(x',y')\, dx'\, dy'$$

$$\partial_x A_x(\mathbf{x},\omega) = -\frac{1}{4\pi}\,(1+j\beta_d)\,\Sigma_i \int \mu_i\; J_{xi}(x',y',\omega)\,(x-x')\, I_2(x',y')\, dx'\, dy'$$

Now just based on this idea that $I_2 >> I_1$ you might conclude that $\partial_x A_x(\mathbf{x},\omega) >> \partial_z A_z(\mathbf{x},\omega)$ which is a rather amazing conclusion, and this confirms the impression noted above for the 2D analysis.

Let's write out the above in more detail

$$\partial_z A_z(\mathbf{x},\omega) = -\frac{1}{4\pi}(1+j\beta_d)\,\Sigma_i \int \mu_i\; J_{zi}(x',y',\omega)\,\{\; -j\, e^{-jkz} \int_{-\infty}^{\infty} dz''\,[z''\sin(kz'')]\, e^{-j\beta_d R}/R^3\}\, dx'\, dy'$$

$$\partial_x A_x(\mathbf{x},\omega) = -\frac{1}{4\pi}(1+j\beta_d)\,\Sigma_i \int \mu_i\; J_{xi}(x',y',\omega)\,\{\, e^{-jkz} \int_{-\infty}^{\infty} dz''\;[\cos(kz'')]\; e^{-j\beta_d R}/R^3\}\,(x-x')\, dx'\, dy'$$