Infinite B fields directly from (D.4.13) PhL 9.30.14

Here I do this twice. The second pass (see red below) is more current, but both worked.

Start with (D.4.13) :

Summary of E and B fields inside a round wire (D.4.13)

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $\quad x = \beta' r$ $\quad x_a = \beta' a$ $\quad \beta'^2 = \beta^2 - k^2$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$

$B_z(r,m) = (j/4)\,(a/\omega)\,\eta_m\, I\, R_{dc}\, (k\,\beta'\, e_m)$

$B_r(r,m) = -(1/4)\,(a/\omega)\,\eta_m\, I\, R_{dc}\, (\, r^{-1}m\;\beta'\, f_m + k^2\; h_m\,)$

$B_\theta(r,m) = (j/4)\,(a/\omega)\,\eta_m\, I\, R_{dc}\, (\, k^2\, g_m - \beta'^2 f_m\, [\,(m/x) - J_{m+1}(x)/J_m(x)]\,)$

$$e_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad R_{dc} = \frac{1}{\sigma\pi a^2}$$

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

I argue that both β and k are very small so β'a is very small and I can then use the small argument limits of all these items. I already do the low arg limits below (D.11.5). Here is a new section for e_m

$$e_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m/m!}{(x_a/2)^{m+1}/(m+1)!} + \frac{(x/2)^m/m!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= [\,(m+1)\,(x/x_a)^m\,(2/x_a) + (1/m)\,(x/x_a)^m(x_a/2)\,] = (x/x_a)^m\,[\,(m+1)\,(2/x_a) + (1/m)\,(x_a/2)\,]$$

$$\approx (x/x_a)^m\,(m+1)\,(2/x_a) \qquad \text{// as } x_a \to 0 \qquad \text{// only first term matters}$$

$$e_0 = \left[\frac{J_0(x)}{J_1(x_a)} + \frac{J_0(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_0(x)}{J_1(x_a)} - \frac{J_0(x)}{J_1(x_a)}\right] = 0$$

So here is a full summary

$e_m = (r/a)^m\,(m+1)\,(2/\beta' a)$	$e_0 = 0$
$f_m = (r/a)^m\,(m+1)\,(2/\beta' a)$	$f_0 = 4/(a\beta')$
$g_m = (r/a)^{m+1} + (r/a)^{m-1}$	$g_0 = 2\,(r/a)$
$h_m = (r/a)^{m+1} - (r/a)^{m-1}$	$h_0 = 0$

But now I also need a limit for this quantity

$$(m/x) - J_{m+1}(x)/J_m(x) \quad = (m/x) - \frac{J_{m+1}(x)}{J_m(x)} \quad = (m/x) - \frac{(x/2)^{m+1} / (m+1)!}{(x/2)^m / m!}$$

$$= (m/x) - \frac{(x/2) / (m+1)!}{1 / m!} = (m/x) - \frac{(x/2)}{(m+1)} \quad \approx \quad m/x \quad = m/(\beta' r) \qquad \text{2nd term does not matter}$$

For m = 0 let's make sure:

$$(0/x) - J_1(x)/J_0(x) \quad = (0/x) - \frac{J_1(x)}{J_0(x)} \quad = (0/x) - \frac{(x/2)^1 / (1)!}{1} \quad = - x/2 \quad = -(\beta' r)/2$$

I can now insert these expressions into the above E and B fields for m > 0 only:

Summary of E and B fields inside a round wire m > 0 (D.4.13)

$$E_z(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\, (a\beta')\, (r/a)^m\, (m+1)\, (2/\beta' a)$$
$$E_r(r,m) = (j/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [r/a)^{m+1} - (r/a)^{m-1}]$$

$$B_z(r,m) = (j/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (k\, \beta'\, (r/a)^m\, (m+1)\, (2/\beta' a))$$
$$B_r(r,m) = -(1/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (\, r^{-1} m\, \beta'\, (r/a)^m\, (m+1)\, (2/\beta' a) + k^2\, [(r/a)^{m+1} - (r/a)^{m-1}]\,)$$
$$B_\theta(r,m) = (j/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (\, k^2\, [(r/a)^{m+1} + (r/a)^{m-1}] - \beta'^2 (r/a)^m\, (m+1)\, (2/\beta' a)\, (m/\beta' r)\,)$$

and here I simplify as needed

Summary of E and B fields inside a round wire m > 0 (D.4.13)

$$E_z(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\, (r/a)^m\, (m+1)\, (2)$$
$$E_r(r,m) = (j/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [r/a)^{m+1} - (r/a)^{m-1}]$$

$$B_z(r,m) = (j/4)\, (k/\omega)\, \eta_m\, I\, R_{dc}\, ((r/a)^m\, (m+1)\, (2))$$
$$B_r(r,m) = -(1/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (\, r^{-1} m\, (r/a)^m\, (m+1)\, (2/a) + k^2\, [(r/a)^{m+1} - (r/a)^{m-1}]\,)$$
$$B_\theta(r,m) = (j/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (\, k^2\, [(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^m\, m(m+1)\, (2/ra)\,)$$

Things don't simplify very much as you can see. But next I will install (D.2.31c)

$$I = (\omega/k)\, CV \qquad \text{// right here I am assuming G = 0 !!}$$

to get

Summary of E and B fields inside a round wire m > 0 (D.4.13)

$E_z(r,m) = (1/4)\,\eta_m\,(\omega/k)\,CV\,R_{dc}\,(r/a)^m\,(m+1)\,(2)$
$E_r(r,m) = (j/4)\,\eta_m\,(\omega/k)\,CV\,R_{dc}\,(ak)\,[(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = (1/4)\,\eta_m\,(\omega/k)\,CV\,R_{dc}\,(ak)\,[r/a)^{m+1} - (r/a)^{m-1}]$

$B_z(r,m) = (j/4)\,(k/\omega)\,\eta_m\,(\omega/k)\,CV\,R_{dc}\,((r/a)^m\,(m+1)\,(2))$
$B_r(r,m) = -(1/4)\,(a/\omega)\,\eta_m\,(\omega/k)\,CV\,R_{dc}\,(r^{-1}m\,(r/a)^m\,(m+1)\,(2/a) + k^2\,[(r/a)^{m+1} - (r/a)^{m-1}])$
$B_\theta(r,m) = (j/4)\,(a/\omega)\,\eta_m\,(\omega/k)\,CV\,R_{dc}\,[\,k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^m\,m(m+1)\,(2/ra)\,]$

and then I will try to simplify the above

Summary of E and B fields inside a round wire $m > 0$ (D.4.13)

$E_z(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(r/a)^m\,(m+1)\,(2)$
$E_r(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[r/a)^{m+1} - (r/a)^{m-1}]$

$B_z(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^m\,(m+1)\,(2)$
$B_r(r,m) = -(1/4)\,\eta_m\,CV\,R_{dc}\,((2/kr)\,m\,(r/a)^m\,(m+1) + ak\,[(r/a)^{m+1} - (r/a)^{m-1}])$
$B_\theta(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,[\,(ak)\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^m\,m(m+1)\,(2/rk)\,]$

$\dim(CV\,R_{dc}) = \text{far/m} * \text{volts} * \text{ohms/m} = \text{sec/m}^2 * \text{volts} = \text{volt-sec/m}^2 = \text{tesla}$

Now finally I reorder the terms in B_θ so

$E_z(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(r/a)^m\,(m+1)\,(2)$
$E_r(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[r/a)^{m+1} - (r/a)^{m-1}]$

$B_z(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^m\,(m+1)\,(2)$
$B_r(r,m) = -(1/4)\,\eta_m\,CV\,R_{dc}[\,(ak)\,[(r/a)^{m+1} - (r/a)^{m-1}] + (2/kr)\,(r/a)^m\,m(m+1)\,]$
$B_\theta(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,[(ak)\,[(r/a)^{m+1} + (r/a)^{m-1}] - (2/rk)\,(r/a)^m\,m(m+1)\,]$

Now the last two B fields are very similar in appearance. Now use

$(r/a)^m/r = (r/a)^m\,(a/r)(1/a) = (r/a)^{m-1}(1/a)$

and the last two fields then become

$$B_r(r,m) = -(1/4)\,\eta_m\ CV\ R_{dc}[\ (ak)\ [(r/a)^{m+1} - (r/a)^{m-1}] + (2/ak)\ m(m+1)\ (r/a)^{m-1}]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ [(ak)\ [(r/a)^{m+1} + (r/a)^{m-1}] - (2/ak)\ m(m+1)\ (r/a)^{m-1}]$$

Now replace

$$(r/a)^{m+1} = (r/a)^{m-1}\ (r^2/a^2)$$

$$B_r(r,m) = -(1/4)\,\eta_m\ CV\ R_{dc}[\ (ak)\ [(r/a)^{m-1}\ (r^2/a^2) - (r/a)^{m-1}] + (2/ak)\ m(m+1)\ (r/a)^{m-1}]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ [(ak)\ [(r/a)^{m-1}\ (r^2/a^2) + (r/a)^{m-1}] - (2/ak)\ m(m+1)\ (r/a)^{m-1}]$$

Now factor out $(r/a)^{m-1}$ to get

$$B_r(r,m) = -(1/4)\,\eta_m\ CV\ R_{dc}(r/a)^{m-1}\ [\ (ak)\ [(r^2/a^2) - 1] + (2/ak)\ m(m+1)]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ (r/a)^{m-1}\ [(ak)\ [(r^2/a^2) + 1] - (2/ak)\ m(m+1)]$$

Next, do a little adjusting

$$B_r(r,m) = -(1/4)\,\eta_m\ CV\ R_{dc}(r/a)^{m-1}\ [\ (k/a)\ [r^2-a^2] + (2/ak)\ m(m+1)]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ (r/a)^{m-1}\ [(k/a)\ [r^2+a^2] - (2/ak)\ m(m+1)]$$

Then factor out (1/ak)

$$B_r(r,m) = -(1/4)\,\eta_m\ CV\ R_{dc}(r/a)^{m-1}\ (1/ak)\ [k^2\ [r^2-a^2] + (2)\ m(m+1)]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ (r/a)^{m-1}\ (1/ak)\ [k^2\ [r^2+a^2] - (2)m(m+1)]$$

and final sign fiddle

$$B_r(r,m) = (1/4)\,\eta_m\ CV\ R_{dc}(r/a)^{m-1}\ (1/ak)\ [-k^2\ [r^2-a^2] - 2m(m+1)]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ (r/a)^{m-1}\ (1/ak)\ [k^2\ [r^2+a^2] - 2m(m+1)]$$

or

$$B_r(r,m) = (1/4)\,\eta_m\ CV\ R_{dc}(r/a)^{m-1}\ (1/ak)\ [-2m(m+1) - k^2\ (r^2-a^2)\ ]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ (r/a)^{m-1}\ (1/ak)\ [-2m(m+1) + k^2\ (r^2+a^2)]$$

or

$$B_r(r,m) = (1/4)\,\eta_m\ CV\ R_{dc}(r/a)^{m-1}\ (1/ak)\ [-2m(m+1) + k^2\ (a^2-r^2)\ ]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\ CV\ R_{dc}\ (r/a)^{m-1}\ (1/ak)\ [-2m(m+1) + k^2\ (a^2+r^2)\ ]$$

Gather up all the B field results (only for $m > 0$)

$$E_z(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\ \ (r/a)^m\,(m+1)\ (2)$$

$$E_r(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}(\omega a)\,[r/a)^{m+1} - (r/a)^{m-1}]$$

$$B_z(r,m) = (j/2)\,\eta_m\,CV\,R_{dc}\,(r/a)^m\,(m+1)$$
$$B_r(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^{m-1}\,(1/ak)\,[-2m(m+1) + k^2\,(a^2-r^2)\,]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^{m-1}\,(1/ak)\,[-2m(m+1) + k^2\,(a^2+r^2)\,]$$

This agrees with the m > 0 portion of (7.4.2). You see the problem now that if k → 0, the last two B fields go infinite for m > 0. This is the "infinite B field problem".

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Let's do a quick check on the computation of I

$$I = \int_0^{2\pi} d\theta \int_0^a r\,dr\,J_z(r,\theta) = \int_0^{2\pi} d\theta \int_0^a r\,dr\,\{\sigma \sum_{m=-\infty}^{\infty} E_z(r,m)\,e^{jm\theta}\} \quad // (D.1.3a)$$

$$= \sigma \sum_{m=-\infty}^{\infty} \int_0^a r\,dr\,E_z(r,m) \int_0^{2\pi} d\theta\,e^{jm\theta} = 2\pi\sigma \int_0^a r\,dr\,E_z(r,0)$$

$$= 2\pi\sigma \int_0^a r\,dr\,\{-j(\beta'/k)\frac{K_0}{2} J_0(x)\} \quad // (D.2.21) \text{ for } E_z(r,0)$$

$$= -j(\beta'/k)\,2\pi\sigma\frac{K_0}{2}\int_0^a r\,dr\,J_0(x) \quad // x = \beta' r \text{ so } xdx = \beta'^2\,rdr$$

$$= -j(\beta'k)^{-1}\,2\pi\sigma\frac{K_0}{2}[\int_0^{x_a} dx\,x\,J_0(x)] = -j(\beta'k)^{-1}\,2\pi\sigma\frac{K_0}{2}[\,x_a\,J_1(x_a)\,] \quad // \text{GR7 5.52.1}$$

$$= -j(\beta'k)^{-1}\,2\pi\sigma\,\{(j\omega/\sigma)\,N_0\,/\,J_1(x_a)\}\,[\,x_a\,J_1(x_a)\,] \quad // (D.2.28) \text{ for } \frac{K_0}{2}$$

$$= (\beta'k)^{-1}\,2\pi\omega\,N_0\,x_a = (\beta'k)^{-1}\,2\pi\omega\,N_0\,\beta' a$$

$$= 2\pi\omega\,(a/k)\,N_0$$

so that

$$I = 2\pi\omega\,(a/k)\,N_0 \quad (D.2.31a)$$

$$N_0 = (k/2\pi\omega a)\,I\,. \quad // I \text{ is called } i(z=0) \text{ in (4.9.2) so } I = i(0) \quad (D.2.31b)$$

From (D.1.8) we know that $N_0 = \langle n(\theta)\rangle = (1/2\pi a)\,q$ and since q = CV we get the alternate form,

$$I = 2\pi a\,(\omega/k)\,N_0 = 2\pi a\,(\omega/k)\,(1/2\pi a)\,q = (\omega/k)\,CV\,. \quad (D.2.31c)$$

I cannot find anything wrong with this computation of I. I have checked it many times and just did it again.

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10.4.14. Now restate our results from above

$E_z(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\;(r/a)^m\,(m+1)$ (2)
$E_r(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\;[(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\;[r/a)^{m+1} - (r/a)^{m-1}]$ $\quad m > 0$ and $G = 0$

$B_z(r,m) = (j/2)\,\eta_m\,CV\,R_{dc}\;(r/a)^m\,(m+1)$
$B_r(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\;(r/a)^{m-1}\,(1/ak)\,[-2m(m+1) + k^2\;(a^2-r^2)\;]$
$B_\theta(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^{m-1}\,(1/ak)\,[-2m(m+1) + k^2\;(a^2+r^2)\;]$

Are these equations consistent with the Maxwell curl B equation? You would think so since I showed this for the general solutions, but let's just check to make sure. Set the coefficients to 1 for simplicity since constant won't affect the result, so

$E_z(r,m) = (1/2)\,(\omega/k)\;(r/a)^m\,(m+1)$
$E_r(r,m) = (j/4)\,(\omega a)\;[(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = (1/4)\,(\omega a)\;[r/a)^{m+1} - (r/a)^{m-1}]$ $\quad m > 0$ and $G = 0$ and const = 1

$B_z(r,m) = (j/2)\;(r/a)^m\,(m+1)$
$B_r(r,m) = (1/4)\,(r/a)^{m-1}\,(1/ak)\,[-2m(m+1) + k^2\;(a^2-r^2)\;]$
$B_\theta(r,m) = (j/4)\,(r/a)^{m-1}\,(1/ak)\,[-2m(m+1) + k^2\;(a^2+r^2)\;]$

Maybe check ALL of the Maxwell equations at this point! I am suspicious. In the Maple file "B for small omega.mws" (App D) I directly enter the above E_i expressions and that file *computes* the B expressions. you see above, using

$$\mathbf{B} = (j/\omega)\,\text{curl}\,\mathbf{E}$$

Right now I will add more code to that mws file in an attempt to verify all of Maxwell's equations. I can steal some of this code from "B field verify 9_20.mws". I am doing this right now only for m > 0. I am able to verify the first three Maxwell equations OK. The fourth one is this:

$$\text{curl}\,\mathbf{B} = \mu\,\mathbf{J} + \mu\,j\omega\varepsilon\mathbf{E} = \mu(\sigma + j\omega\varepsilon)\,\mathbf{E} = \mu(j\omega)(\,\varepsilon - j\sigma/\omega)\,\mathbf{E} = j\omega\,\mu\xi\,\mathbf{E} \qquad (D.5.1)$$
$$= j\,(\beta^2/\omega)\,\mathbf{E} \quad . \qquad \text{// see (1.5.1c)}$$

which I take simply to be

$$\text{curl}\,\mathbf{B} = j\,(\beta^2/\omega)\,\mathbf{E}.$$

However, in my small-x limit in which everything is being done here, we have $\beta' << 1$ so to speak, and that means we have $\beta = k$ so you can replace the above with

$$\text{curl}\,\mathbf{B} = j\,(k^2/\omega)\,\mathbf{E}$$

In "B for small omega m GT 0.mws" and "B for small omega m EQ 0.mws" I show that my low-x fields for E and B do in fact satisfy all four Maxwell equations, including the curl **B** one above. I thought there was a problem here at first and wrote " Problem with the curl B Maxwell Equation", but in that doc I realized that there is in fact no problem.

10.8.14. Let's try this all again now that I have a fancier statement of the B fields in D.9.

Summary of E and B fields inside a round wire (D.9.39)

$E_z(r,m) = (1/4)\,\eta_m\,\mathcal{B}\,(\omega/k)\,(a\beta')\,f_m$ $\quad x = \beta' r$ $\quad x_a = \beta' a$ $\quad \beta'^2 = \beta^2 - k^2$

$E_r(r,m) = (j/4)\,\eta_m\,\mathcal{B}\,(\omega a)\,g_m$

$E_\theta(r,m) = (1/4)\,\eta_m\,\mathcal{B}\,(\omega a)\,h_m$ $\quad \mathcal{B} \equiv (\xi_d/\varepsilon_d)\,CV\,R_{dc}$

$B_z(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(\beta'\,e_m)$ $\quad (\xi_d/\varepsilon_d) = 1 + (G/j\omega C)$

$B_r(r,m) = -(1/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r^{-1}m\,\beta'\,f_m + k^2\,h_m)$

$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(k^2\,g_m - \beta'^2 f_m\,[(m/x) - J_{m+1}(x)/J_m(x)])$ $\quad \eta_m = N_m/N_0$

$$e_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad R_{dc} = \frac{1}{\sigma\pi a^2}$$

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad G \geq 0$$

I know that for small x and x_a (and small β')

$e_m = (r/a)^m\,(m+1)\,(2/\beta' a)$ $\quad e_0 = 0$

$f_m = (r/a)^m\,(m+1)\,(2/\beta' a)$ $\quad f_0 = 4/(a\beta')$

$g_m = [(r/a)^{m+1} + (r/a)^{m-1}]$ $\quad g_0 = 2\,(r/a)$

$h_m = [(r/a)^{m+1} - (r/a)^{m-1}]$ $\quad h_0 = 0$

Thus the B field expressions for small x are:

$B_z(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(\beta'\,e_m)$

$B_r(r,m) = -(1/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r^{-1}m\,\beta'\,f_m + k^2\,h_m)$

$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(k^2\,g_m - \beta'^2 f_m\,[(m/x) - J_{m+1}(x)/J_m(x)])$

$B_z(r,0) = (j/4)\,(a)\,\mathcal{B}\,(\beta'\,e_0)$

$B_r(r,0) = -(1/4)\,(a)\,\mathcal{B}\,(1/k)\,(r^{-1}m\,\beta'\,f_0 + k^2\,h_0)$

$B_\theta(r,0) = (j/4)\,(a)\,\mathcal{B}\,(1/k)\,(k^2\,g_0 - \beta'^2 f_0\,[(m/x) - J_{m+1}(x)/J_m(x)])$

Now earlier I showed that

$(m/x) - J_{m+1}(x)/J_m(x) \approx m/(\beta' r)$ $\quad m > 0$

$(m/x) - J_{m+1}(x)/J_m(x) \approx -(\beta' r)/2$ $\quad m = 0$

So install things to get

$$B_z(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\;\left(\beta'(r/a)^m\,(m+1)\,(2/\beta'a)\right)$$
$$B_r(r,m) = -(1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(r^{-1}m\;\beta'\,(r/a)^m\,(m+1)\,(2/\beta'a) + k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - \beta'^2(r/a)^m\,(m+1)\,(2/\beta'a)\,[m/(\beta'r)]\right)$$

$$B_z(r,0) = (j/4)(a)\,\mathcal{B}\;\left(\beta'\,0\right)$$
$$B_r(r,0) = -(1/4)(a)\,\mathcal{B}\,(1/k)\left(r^{-1}\;0\;\beta'\;4/(a\beta') + k^2\;\;0\right)$$
$$B_\theta(r,0) = (j/4)(a)\,\mathcal{B}\,(1/k)\left(k^2\,2\,(r/a) - \beta'^2\,4/(a\beta')\,[-(\beta'r)/2]\right)$$

Do a rewrite and simplification

$$B_z(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\;\left((r/a)^m\,(m+1)\,(2/a)\right)$$
$$B_r(r,m) = -(1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(r^{-1}m\;(r/a)^m\,(m+1)\,(2/a) + k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^m\,(m+1)\,(2/a)\,[m/(r)]\right)$$

$$B_z(r,0) = 0$$
$$B_r(r,0) = 0$$
$$B_\theta(r,0) = (j/4)(a)\,\mathcal{B}\,(1/k)\left(k^2\,2\,(r/a) + \beta'^2\,2r/a\right)$$

Now use this

$$(r/a)^m/r = (r/a)^m\,(a/r)(1/a) = (r/a)^{m-1}(1/a)$$

to get

$$B_r(r,m) = -(1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(r^{-1}m\;(r/a)^m\,(m+1)\,(2/a) + k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^m\,(m+1)\,(2/a)\,[m/(r)]\right)$$

$$B_r(r,m) = -(1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(m\;(r/a)^{m-1}(1/a)\,(m+1)\,(2/a) + k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^{m-1}(1/a)\,(m+1)\,(2/a)\,[m]\right)$$

$$B_r(r,m) = -(1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(m(m+1)\;(r/a)^{m-1}\,(2/a^2) + k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^{m-1}m\,(m+1)\,(2/a^2)\right)$$

$$B_r(r,m) = -(1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(2m(m+1)\;(r/a)^{m-1}\,(1/a^2) + k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(k^2\,[(r/a)^{m+1} + (r/a)^{m-1}] - (r/a)^{m-1}2m\,(m+1)\,(1/a^2)\right)$$

$$B_r(r,m) = (1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(-2m(m+1)\;(r/a)^{m-1}\,(1/a^2)\;\; - k^2\;[(r/a)^{m+1} - (r/a)^{m-1}]\right)$$
$$B_\theta(r,m) = (j/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left(-\;2m(m+1)\,(r/a)^{m-1}\,(1/a^2)\;+ k^2\,[(r/a)^{m+1} + (r/a)^{m-1}]\right)$$

$$B_r(r,m) = (1/4)(a)\,\eta_m\,\mathcal{B}\,(1/k)\left((r/a)^{m-1}\,\{-2m(m+1)\,(1/a^2) + k^2\} - k^2(r/a)^{m+1}\right)$$

$$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,((r/a)^{m-1}\,\{-2m(m+1)\,(1/a^2) + k^2\} + k^2(r/a)^{m+1}\,)$$

$$B_r(r,m) = (1/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\,(\{-2m(m+1)\,(1/a^2) + k^2\} \;- k^2(r/a)^2\,)$$
$$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\,(\,\{-2m(m+1)\,(1/a^2) + k^2\} + k^2(r/a)^2\,)$$

$$B_r(r,m) = (1/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\,(-2m(m+1)\,(1/a^2) + k^2 \;- k^2(r/a)^2\,)$$
$$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\,(-2m(m+1)\,(1/a^2) + k^2 \;+ k^2(r/a)^2\,)$$

$$B_r(r,m) = (1/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\,(1/a^2)\,(-2m(m+1) + k^2a^2 \;- k^2r^2\,)$$
$$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\;\,(1/a^2)\,(-2m(m+1) + k^2a^2 \;+ k^2r^2\,)$$

$$B_r(r,m) = (1/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\,(1/a^2)\,(-2m(m+1) + k^2(a^2 - r^2\,)\,)$$
$$B_\theta(r,m) = (j/4)\,(a)\,\eta_m\,\mathcal{B}\,(1/k)\,(r/a)^{m-1}\;\,(1/a^2)\,(-2m(m+1) + k^2(a^2 + r^2\,)\,)$$

$$B_r(r,m) = (1/4)\,\eta_m\,\mathcal{B}\,(1/ka)\,(r/a)^{m-1}\,(-2m(m+1) + k^2(a^2 - r^2\,)\,)$$
$$B_\theta(r,m) = (j/4)\,\eta_m\,\mathcal{B}\,(1/ka)\,(r/a)^{m-1}\,(-2m(m+1) + k^2(a^2 + r^2\,)\,)$$

The complete results at this point are then

$$B_z(r,m) = (j/2)\;\,\eta_m\,\mathcal{B}\;\,(r/a)^m\,(m+1)$$
$$B_r(r,m) = (1/4)\,\eta_m\,\mathcal{B}\,(1/ka)\,(r/a)^{m-1}\,(-2m(m+1) + k^2(a^2 - r^2\,)\,)$$
$$B_\theta(r,m) = (j/4)\,\eta_m\,\mathcal{B}\,(1/ka)\,(r/a)^{m-1}\,(-2m(m+1) + k^2(a^2 + r^2\,)\,)$$

$$B_z(r,0) = 0$$
$$B_r(r,0) = 0$$
$$B_\theta(r,0) = (j/2)\;\,\mathcal{B}\,(kr)$$

And this agrees exactly with (7.4.2), confirmed!