Pondering Low ω Issues PhL 9.11.14

Reviewing this on 10.8.14, it is a very clear exposition of the low-ω problem. I give a list of four items which quickly result in the low-ω problem. One of those items is the e^{-jkz} ansatz, and I now think that is the culprit and have written Ch 7 to argue that point. The best part of this doc is the start through the end of the Thought Experiment.

I put the most recent group of low-ω papers into a separate folder. I started these, then went off for a month doing EV visit and App Q rewrite and App M repair, and then I came back and reviewed those papers. Here I will try to extract the essence of the low-ω problems.

Background

1. In Appendix D, I make the famous ansatz e^{-jkz} for all E field components, where k is a completely free parameter, determined by nothing at all. I take the E field Helmholtz equation with its β^2 and solve it for the E fields along with div E = 0. I get a unique solution for all E_i up to two constants per partial wave. I then apply my two boundary conditions, and these constants are then all nailed down. Here is the solution:

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ (D.2.33)

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$ $g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$ $h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

where k is still a free parameter both explicitly and as part of β', and of course $\beta^2 = -j\omega\mu\sigma$.

The Asym Problem

Now right at this point, I can study the "J_z asym problem". From above I see that

$$\frac{J_z(r,m)}{J_z(r,0)} = \frac{\eta_m f_m}{f_0} = \eta_m \frac{\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]}{2\frac{J_0(x)}{J_1(x_a)}}$$

Now *regardless* of the value selected for k (and thus, regardless of the value of β'), I can see that this ratio does not vanish for a given $m \neq 0$, unless $\eta_m = 0$. If I take $\omega \to 0$, then regardless of what k approaches in this limit (be it large, small, or in between), the above ratio is still not zero! For a given limiting form of k as $\omega \to 0$, the above ratio is some complicated function of β'a and β'r and m. If I know

that k→0 in some manner, I could simplify that above form perhaps, but that does not matter! The point is that as ω→0, all the partial waves which have $\eta_m > 0$ are still present, and therefore J_z is asymmetric ! [agreed!]

Fact 1: The J_z asym problem as ω→0 is present independent of any assumption about the form of k.

Thus, for example, I cannot blame J_z asymmetry on the transmission line equations which give rise to a particular form for k. [agreed!]

Just for the record, if it happens that $k^2 \rightarrow A\omega^s$ as ω→ 0, with s > 0, then we have

$$\beta'^2 = \beta^2 - k^2 = -j\omega\mu\sigma - A\omega^s$$

In this case, in the limit ω→0 we have β'→0 so we can then use the small-x limits of the Bessel functions, and in that case we find that

$$\begin{aligned} E_z(r,m) &= (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\, (|m|+1) \\ E_r(r,m) &= (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \\ E_\theta(r,m) &= (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}] \end{aligned} \qquad \text{(D.11.7)}$$

$$\begin{aligned} E_z(r,0) &= I\, R_{dc} \\ E_r(r,0) &= (j/2)\, I\, R_{dc}\, (ak)\, (r/a) \\ E_\theta(r,0) &= 0 \end{aligned} \qquad \text{// low ω E fields}$$

and then

$$\frac{J_z(r,m)}{J_z(r,0)} = \eta_m \frac{(1/2)\,(r/a)^{|m|}\,(|m|+1)}{1} = (\eta_m/2)\,(r/a)^{|m|}\,(|m|+1)$$

and now it is very simple to see that this ratio is not going to 0! But it is important to keep Fact 1 above in mind: it does not matter WHAT you assume about the small ω limit of k, you get asymmetry. [agreed]

Corollary 1: The J_z asym problem is really contained within this set of assumptions: [agreed]

(a) the e^{-jkz} ansatz for each E_i component (for some unspecified k(ω) form as ω→0)
(b) the BC that $E_\theta = 0$ at the round wire surface
(c) the BC that $J_r(r=a-\varepsilon,\theta) = j\omega\, n(\theta)$ which is then $E_r(r=a,m) = (j\omega/\sigma)\, N_m$
(d) the fact that as ω→ 0, the surface charge moments N_m do not vanish

Thought Experiment #1

Imagine two semi-infinite solid copper cylinders at some very small ω in the very ultra-low range. Each cylinder is a foot in diameter, they are spaced by 1/10", you are staring at them. There is a voltage driver applying $Ve^{j\omega t}$ to the available end of the cylinders at z = 0. The dielectric is a vacuum, let's say. There

is some $C > 0$ and some $R_{dc} > 0$. Perhaps ω is such that the period is something like 1 day, so you have plenty of time to measure things at your leisure. Since the cylinders are infinitely long, at DC they will have a total resistance of ∞ (since R_{dc} per length $\neq 0$), so for that reason we expect current I to be very small for ω very close to DC. Our driver then has no trouble maintaining the voltage V at the driver end. We could simulate the line with 100 m of line with proper termination, but let's just keep it semi-infinite for now.

We now make some observations. At the driving end we have some voltage V, the line has some C, so there will be some $n(\theta)$ on each cylinder. Just the presence of $V \neq 0$ causes $n(\theta)$ to exist since there is some $C > 0$, this cannot be avoided. Moreover, we can imagine the E field pattern in the vacuum between the two cylinders, and we know that this causes $n(\theta)$ to have a dramatic peak at $\theta = \pi$, so for sure $n(\theta)$ is not a constant. Thus, in general we will have $\eta_m \neq 0$ for many partial waves. Thus, $N_m \neq 0$ and we then have item (d) in the above list for sure.

Next, since things are almost static, and charges can move very fast, it really does seem that $E_\theta = 0$ on the surface will be true. Recall that $E_\theta = -(\text{grad}\ \varphi)_\theta - j\omega A_\theta$. Even if there is some small A_θ, we would think $j\omega A_\theta$ would be extremely small, so then $E_\theta = -(1/a)\partial_\theta \varphi$ and we really expect both $E_\theta = 0$ on the cylinder surface, and φ = constant around the surface. This φ will be the same as V. So it certainly seems that item (b) in the above list will be met by our physical example.

Similarly, in the limit we expect $J_r \rightarrow 0$ and then $J_r(r=a-\varepsilon,\theta) = j\omega\, n(\theta)$ which is item (c) is also met.

To summarize, in our thought experiment physical apparatus, items (b), (c), (d) are met.
[and a good thought experiment it is!]

What about item (a) ? We presume that only J_z exists, so item A is claiming that

$$J_z(z) = J_z(0)\, e^{-jkz}$$

Close to the DC limit, we shall assume that both C and R_{dc} are playing a role and we ignore any possible inductance effect L, and we have already said G = 0. At this very low ω, it certainly seems that the network model would apply to my physical apparatus, and I know that in that case the $k(\omega)$ formula applies, and I know that the low ω limit of that formula says

$$k \equiv -j\sqrt{(R+j\omega L)(G+j\omega C)} = -j\sqrt{(R_{dc})(j\omega C)} = -j\, j^{1/2}\, \omega^{1/2}\sqrt{R_{dc}C}$$

$$= (j^{1/2}/j)\, \omega^{1/2}\sqrt{R_{dc}C} = j^{-1/2}\, \omega^{1/2}\sqrt{R_{dc}C} = (1-j)/\sqrt{2} * \omega^{1/2}\sqrt{R_{dc}C}$$

$$= (1-j)\, \omega^{1/2}\sqrt{R_{dc}C/2}$$

and this agrees with (Q.4.9). Then item (a) claims that

$$J_z(z) = J_z(0)\exp(-j\,\omega^{1/2}\sqrt{R_{dc}C/2}\, z)\exp(-\omega^{1/2}\sqrt{R_{dc}C/2}\, z)$$

As we then take $\omega \rightarrow 0$, this says, for any finite value of z,

$$J_z(z) = J_z(0)\ (1 - j\,\omega^{1/2}\sqrt{R_{dc}C/2}\, z)(1 - \omega^{1/2}\sqrt{R_{dc}C/2}\, z)$$

For any finite value of z, as ω → 0 we then get

$$J_z(z) \rightarrow J_z(0)$$

I know that $J_z(0) = 0$ in this limit, but still VERY close to the limit, $J_z(z) = J_z(0)$ is what I expect, since the current can no longer flow across C, so it just stays constant along z.

So item (a) seems *consistent with* my thought experiment physical situation. [but the experiment does not prove that the ansatz is correct. All I did here was show that if k → 0, then $e^{-jkz} \rightarrow 1$. I don't think this little section has anything to say about the validity of e^{-jkz} other than it *can* explain decay on the line.]

So with minimum detail, I have just shown that my theory contains a low frequency paradox: Regardless of what k does as ω→0, we get a non-physical result that J_z is asymmetric as ω→ 0.

A Theory Check. Go back to item (c). Note that this says $J_r(a,\theta) \rightarrow$ constant * ω as ω→ 0. Similarly, the m space version says that $J_r(a,m) \rightarrow j\omega N_m$ = constant * ω. Is this consistent with the low β' solutions above? If we assume that k → 0 in some manner as ω→0, then we get to use

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\, (ak)\, (r/a)$$

Both of these are proportional to I k. We know that

$$I = 2\pi\omega\, (a/k)\, N_0 \quad \Rightarrow \quad I\,k = 2\pi\omega\, a\, N_0 = \text{constant} * \omega$$

This tells us that both $E_r(r,m)$ and $E_r(r,0) \rightarrow 0$ as constant * ω, so things are in fact consistent.

$$E_r(r,m) = (j/4)\,\eta_m\, 2\pi\omega\, a\, N_0\, R_{dc}\, (a)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_r(r,0) = (j/2)\, 2\pi\omega\, a\, N_0\, R_{dc}\, (a)\, (r/a)$$

So fine, these go to zero as ω→ 0, I have learned nothing from this check except things are consistent.

The curl E Issue : Infinite B fields

Let's start over on this new subject. Making only assumption (a) above, we know that

$$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1} jm E_z + jk E_\theta$$

(1) Let us now assume that $k \rightarrow \omega^s$ with some s > 0 as ω→ 0. Then k → 0 and then we can use the small x expressions above for the E fields to get

$$[\text{curl } \mathbf{E}(r,m,z)]_r = j\,(1/4)\,\eta_m\, R_{dc}\, \{\, 2\, I\, r^{-1}\, m\, (r/a)^{|m|}\, (|m|+1) + a\, k^2 I\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\, \}$$

$= j\,(1/4)\,\eta_m\,R_{dc}\,\{\,2\,2\pi\omega\,(a/k)\,N_0\;\; r^{-1}\;\; m\,(r/a)^{|m|}\,(|m|+1)\;\; + a\;k\;\; 2\pi\omega\;a\;N_0\,(\,[(r/a)^{|m|+1} -\;\; (r/a)^{|m|-1}]\,\}$

where we used I k = 2πω a N_0 in each term. So this form is then

$$[\text{curl } \mathbf{E}(r,m,z)]_r \;= m\;A\;(1/k)\;\omega + B\;k\;\omega$$

$$\rightarrow\; m\;A\;\omega^{1-s} + B\;\omega^{1+s}$$

and so

$$B_r = +(j/\omega)\,[\text{curl } \mathbf{E}]_r \rightarrow\;\; j\;m\;A\;\omega^{-s} + j\;B\;\omega^{+s}$$

For any s > 0, the <u>first term blows up</u> and we have $B_r \rightarrow \infty$! For example, s = 1/2 causes blow up.

(2) Let us assume instead that $k \rightarrow k_0$, some constant, as $\omega \rightarrow 0$. If we assume that k_0 is small enough that the resulting β' is small and x is small, then we get

$$[\text{curl } \mathbf{E}(r,m,z)]_r \;= m\;A\;(1/k)\;\omega + B\;k\;\omega \;\rightarrow m\;A\;(1/k_0)\;\omega + B\;k_0\;\omega$$

$$= [\;m\;A\;(1/k_0) + B\;k_0]\;\omega$$

and

$$B_r = +(j/\omega)\,[\text{curl } \mathbf{E}]_r \rightarrow\; j\;[\;m\;A\;(1/k_0) + B\;k_0]$$

In this case, we at least get a *finite* B_r result! More specifically we get

$$B_r(r,m)$$

$$j\,(1/4)\,\eta_m\,R_{dc}\,\{\,2\,2\pi\,(a/k_0)\,N_0\;\; r^{-1}\;\; m\,(r/a)^{|m|}\,(|m|+1)\;\; + a\;k_0\;\; 2\pi\;a\;N_0\,(\,[(r/a)^{|m|+1} -\;\; (r/a)^{|m|-1}]\,\}$$

This function of r cannot vanish for all $r \le a$. For r = a we would get

$$B_r(a,m) = j\,(1/4)\,\eta_m\,R_{dc}\,\{\,2\,2\pi\,(a/k_0)\,N_0\;\; a^{-1}\;\; m\,(|m|+1)\;\; + a\;k_0\;\; 2\pi\;a\;N_0\,(\,[1^{|m|+1} -\;\; 1^{|m|-1}]\,\}$$

$$= j\,(1/4)\,\eta_m\,R_{dc}\,\{\,2\,2\pi\,(a/k_0)\,N_0\;\; a^{-1}\;\; m\,(|m|+1)$$

So in this scenario we get some finite radial B_r field existing at DC. The limiting field is a function of both r and θ.

Fact 1: If we assume (a), (b), (c) and (d) as shown in Corollary 1, AND if we assume that $k \rightarrow 0$, we get a B_r which is infinite. That is impossible, so at least one of our assumptions is invalid. On the other hand, if $k \rightarrow k_0$ which is very small, then $B_r(a,m)$ = constant, and we have some radial B_r field that is a function of r and θ.

Our G = 0 model for k(ω) gives $k \rightarrow \omega^{1/2}$ so we have the problem of infinite B_r. For G > 0 we get Rek→0 and Imk→ k_0 which I think is very small, being $\sqrt{R_{dc}G_{dc}}$, so at least we have a finite B_r. [but I think it is infinite here as well, see Chap 7]

In this second case, what happens to the other B fields? I did this in the curl E doc, and in that case we find that all B fields are finite.

Perhaps the most obvious and glaring paradox is that for G = 0, my theory gives $B_r = \infty$.

Plan A. Maybe the e^{-jkz} ansatz (which is really the separation of variables ansatz into two groups which are z and then r,θ) is OK except near ω = 0. Perhaps I could attempt to solve $(\nabla^2 + \beta^2)\ \mathbf{E}(\mathbf{x},\omega) = 0$ without making this separation assumption, but for β^2 being very small allowing some kind of perturbation theory. Then maybe I could show that the solution of that problem does NOT separate, and then I could really blame my problems on the e^{-jkz} ansatz with some actual evidence. Maybe I could obtain a value of ω below which the separation assumption is no longer valid! That would be great!

[Here I just leave ∂_z in place, but this leads to same old same old.]

Let's take a few steps down this road to see what happens. I can write the Helmholtz equation components from Appendix D as follows: [here we have $E_i(r,\theta,z)$ or $E_i(r,m,z)$ with no z separation assumed]

The z equation:

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \partial_z^2 + \beta^2\ ]\ E_z = 0$$

The r equation:

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2 + \partial_z^2 - (1/r^2) + \beta^2]\ E_r - (2/r^2)\,\partial_\theta E_\theta = 0$$

The θ equation:

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2 + \partial_z^2 - (1/r^2) + \beta^2]\ E_\theta + (2/r^2)\,\partial_\theta E_r = 0$$

The divE = 0 equation:

$$\partial_r\,(r\,E_r) + \partial_\theta E_\theta + r\,\partial_z E_z = 0$$

Now I will still assume the partial wave equation, so I can write this equation set as $\partial_\theta \rightarrow jm$

$$[\partial_r^2 + (1/r)\,\partial_r - (1/r^2)\,m^2 + \partial_z^2 + \beta^2\ ]\ E_z = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)\,m^2 + \partial_z^2 - (1/r^2) + \beta^2]\ E_r - (2/r^2)\,jmE_\theta = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_\theta + (2/r^2)\, jmE_r = 0$$

$$\partial_r (r\, E_r) + jmE_\theta + r\, \partial_z E_z = 0$$

This is a complex problem with three field components and two variables r and z. It is a coupled set of four differential equations. I can still solve the last equation to write

$$jmE_\theta = - \partial_r (r\, E_r) - r\, \partial_z E_z$$

and I can still then insert this into the middle two equations to get this set of only 3 equations:

$$[\partial_r^2 + (1/r)\, \partial_r - (1/r^2)\, m^2 + \partial_z^2 + \beta^2\ ]\, E_z = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)\, m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_r - (2/r^2)\, [- \partial_r (r\, E_r) - r\, \partial_z E_z] = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, (1/jm)[- \partial_r (r\, E_r) - r\, \partial_z E_z] + (2/r^2)\, jmE_r = 0$$

Now we have only two field components to worry about, E_z and E_r and still two variables r and z.

The first equation has only E_z. It is a partial differential equation

$$[\partial_r^2 + (1/r)\, \partial_r - (1/r^2)\, m^2 + \partial_z^2 + \beta^2\ ]\, E_z\, (r,z) = 0$$

I can back up and write this as a scalar Helmholtz equation

$$(\nabla^2 + \beta^2)\, E_z\, (r,z) = 0$$

So Moon and Spencer have anything useful to say about this situation? Page 15 is their stuff on this. They are of course writing separated solutions where $\kappa^2 = \beta^2$. They of course have three coordinates and not just two, so maybe I will back out my partial wave equations and rewrite the above as

$$[\partial_r^2 + (1/r)\, \partial_r + (1/r^2)\, \partial_\theta^2 + \partial_z^2 + \beta^2\ ]\, E_z\, (r,z) = 0$$

Since things have to be single valued in θ for the round wire situation, their p = my m = integer, so here is their separated solution

$$[J_m(iqr), Y_m(iqr)],\ [\sin(m\theta), \cos(m\theta)],\ [\sin(\sqrt{\beta^2 + q^2}\, z),\ \cos(\sqrt{\beta^2 + q^2}\, z)]$$

where q is a separation constant. Suppose I define $k \equiv \sqrt{\beta^2 + q^2}$ so $k^2 = \beta^2 + q^2$. Then

$$\beta^2 = k^2 - q^2 \qquad \text{and} \qquad q^2 = k^2 - \beta^2 = - \beta'^2 \qquad \beta'^2 = \beta^2 - k^2$$

so then I identify $q = -j\beta'$ and replace q with β' to get

$iqr = jqr = j(-j\beta')r = \beta' r$

$[J_m(\beta' r), Y_m(\beta' r)]$, $[\sin(m\theta), \cos(m\theta)]$, $[\sin(kz), \cos(kz)]$

$[J_m(\beta' r), Y_m(\beta' r)]$, $[\sin(m\theta), \cos(m\theta)]$, $[e^{jkz}, e^{-jkz}]$

So this is nothing new, except of course k is a continuous separation constant. So I guess this says that the most general solution might have the form

$$E_z(r,z) = \Sigma_m \, e^{jm\theta} \int dk \; J_m(\beta' r) \, [\, A_{km} \, e^{jkz} + B_{km} \, e^{-jkz}]$$

Maybe the physical solution has to be this linear combination and not just an atomic form. So my "ansatz" then is saying that the physical solution is assumed to be an isolated atomic form. Hold on this.

Plan B. Look at the integral form of the curl E equation

$$\oint_C \mathbf{E} \bullet d\mathbf{s} = -j\omega[\int_S \mathbf{B} \bullet d\mathbf{S}]$$

If $\mathbf{E} \rightarrow \sqrt{\omega}$ as $\omega \rightarrow 0$, then we have $\oint_C \mathbf{E} \bullet d\mathbf{s} \rightarrow \sqrt{\omega}$ and we must then have $\int_S \mathbf{B} \bullet d\mathbf{S} \rightarrow 1/\sqrt{\omega}$, and this says the captured flux goes infinite. [fine, but what help is that?]

Plan C. Consider our two Appendix Q limits for k

Fact 3: The small ω limit for k(ω), assuming $\omega_d > 0$, is given by (Q.4.6)

$$\mathrm{Re}(k) \approx (\omega/2)\,(R_{dc} + \omega_d L_{dc})\sqrt{C/(\omega_d R_{dc})} + O(\omega^2) \qquad \omega < \omega_d = (\sigma_d/\varepsilon_d)$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc} C \omega_d}\;[\,1 + (\tan_L/2)\,(\omega/\omega_d)] + O(\omega^2) \qquad G_{dc} = C\omega_d$$

Fact 4: The small ω limit for k(ω), assuming $\omega_d = 0$, is given by (Q.4.9)

$$\mathrm{Re}(k) \approx +\sqrt{R_{dc}C/2}\;\sqrt{\omega}\,(\,1 - \tan_L/2) + O(\omega^{3/2})$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C/2}\;\sqrt{\omega}\,(\,1 + \tan_L/2) + O(\omega^{3/2})$$

What causes the general nature of the limit to change when we have $\omega_d = 0$ versus $\omega_d \rightarrow 0$? Maybe think of a function of two variables ω and ω_d. So it is $k(\omega,\omega_d)$. We find that

$$\lim_{\omega \rightarrow 0} k(\omega,\omega_d) = (\omega/2)\,(R_{dc} + \omega_d L_{dc})\sqrt{C/(\omega_d R_{dc})} - j\sqrt{R_{dc} C \omega_d}$$

$\lim_{\omega_d \to 0} k(\omega,\omega_d) = \sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (1 - j)$ [just a related question]

Plan D. Funny cut structure? [I am grasping at straws here]

$$k(\omega) = -j\sqrt{(R+j\omega L)(G+j\omega C)} = -j\sqrt{(R_{dc}+j\omega L_{dc})(C(\omega_d + \tan_L \omega)+j\omega C)}$$

$$= -j\sqrt{C}\sqrt{(R_{dc}+j\omega L_{dc})(1(\omega_d + \tan_L \omega)+j\omega 1)}$$

$$= -j\sqrt{C}\sqrt{(R_{dc}+j\omega L_{dc})(\omega_d + \tan_L \omega + j\omega)}$$

$$= -j\sqrt{C}\sqrt{(R_{dc}+j\omega L_{dc})(\omega_d + \omega(j+\tan_L))}$$

Assume that our interest lies in ω low enough that L_{dc} does not matter, so then we have

$$k(\omega) = -j\sqrt{R_{dc}L}\ \sqrt{\omega_d + \omega(j+t)}$$

Now let's ignore the constant leading factor and focus on the rest

$$f(\omega) = \sqrt{\omega_d + \omega(j+t)}$$

This is the essence of the function we are interested in. We are already assuming small ω, but have not taken any limit yet. Change from ω_d and ω to x and y, so we have

$f(x,y) = \sqrt{x + y(j+t)}$ where y is already small but no limit yet.

How should you treat both x and y small at the same time?

$f(x,y) = f(0,0) + x\,\partial_x f + y\,\partial_y f + xy\,\partial_{xy} f + \ldots$ // from my "proof of Taylor" doc

So

$\partial_x f = (1/f)$ $\partial_y f = (1/f)(j+t) = (j+t)\,f^{-1}$

$\partial_{xy} f = \partial_x [(j+t)\,f^{-1}] = (j+t)(-1)\,f^{-2}\,(1/f) = -(j+t)(1/f^3)$

So then the double variable expansion for small variables is:

$f(x,y) = \sqrt{x + y(j+t)} = 0 + x\,(1/f) + y\,(j+t)\,f^{-1} - xy\,(j+t)(1/f^3) + \ldots$

But this is meaningless since f(0,0) = 0 is 1/f = ∞.

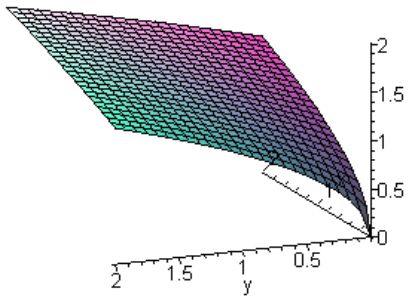
Consider simpler problem

$f(x,y) = \sqrt{x+y}$ $x \geq 0$ $y \geq 0$

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f := sqrt(x+y);
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$$f := \sqrt{x+y}$$

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plot3d(f,x=0..2,y=0..2,axes=normal);
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This surface comes into the origin with infinite slope in both directions, so you can't really model it in a polynomial way near the origin! If either variable is a constant > 0, you can approach with the other to 0 and get something smooth.

Start over. First, assume that ω is small:

$$f(\omega_d,\omega) = \sqrt{\omega_d + \omega(j + t)}$$

$$= \sqrt{\omega_d}\ \sqrt{1+ (j+t)(\omega/\omega_d)}\ \approx \sqrt{\omega_d}\ [1 + (j+t)\ (\omega/\omega_d)]$$

$$\approx\ \sqrt{\omega_d}\ [\ 1 + j\ (\omega/\omega_d)\ ]$$

$$\approx \sqrt{\omega_d}\quad = \text{constant} \qquad \text{valid only for } \omega < \omega_d$$

$$\text{Ref} \approx \sqrt{\omega_d} \qquad \text{Imf} \approx \sqrt{\omega_d}\ (\omega/\omega_d)$$

Now do the other way assuming ω_d is small:

$$f(\omega_d,\omega) = \sqrt{\omega_d + \omega(j + t)}\ \ = \sqrt{\omega(j+t)}\ \sqrt{1 + \omega_d/[\omega(j+t)]}$$

$$\approx\ \sqrt{\omega(j+t)}\ [\ 1 + \omega_d/[\omega(j+t)]\ ]$$

$$\approx\ \sqrt{\omega(j+t)}\ [\ 1 + (\omega_d/\omega)(1/(j+t)]\ ]$$

$$\approx\ \sqrt{\omega(j+t)}\ [\ 1 + (\omega_d/\omega)(1/(j)]\ ]$$

$\approx \sqrt{\omega(j+t)}\ [\ 1 - j\ (\omega_d/\omega)\ ]$

$\approx \sqrt{j\omega}\ [\ 1 - j\ (\omega_d/\omega)\ ]$

$\approx \sqrt{j\omega}$ = constant * $\sqrt{\omega}$ valid only for $\omega_d < \omega$

$= j^{1/2}\sqrt{\omega} = (1+j)\sqrt{\omega}/\sqrt{2}$

Ref $\approx \sqrt{\omega}/\sqrt{2}$ Imf $\approx \sqrt{\omega}/\sqrt{2}$