Problem with the curl B Maxwell Equation and low ω limit of theory PhL 10.4.14

I had a paradox here where it seemed that when I use the low-ω E and low-ω B fields, the Maxwell curl B equation was not being satisfied. But then I realized that in the limit of small x which those fields represent, it really was being satisfied for small x, an interesting little puzzle of the day.

Comment: Only today did I first think about checking the Maxwell curl B equation *after* taking the low-ω limits of the round wire E and B fields. As I show below, this curl B equation is violated. I will first show why this is the case, then try to find the cause of the problem. I do know that before the low-ω limit is taken, this curl B equation is OK because I verify it with Maple in "B field verify 9_20". At that point everything is given in terms of Bessel functions.

I have spent a lot of time looking for "something wrong" in my derivation of the low-ω limit of the round wire theory, and finally I think I have found something wrong. I first latched on to this problem in "Infinite B fields directly from (D.4.13).doc", but I will move that work to this doc now.

Review of verifying the curl B equation before the limit is taken.

I obtain the B fields from the curl E equation and then I show that the combined E and B fields satisfy the curl B equation, all in B field verify 9_20.mws.

So, *how* did this work out right for the full E and B fields?? I did that in D.5 and "B field verify 9_20". In that case I had k, β^2 and β'^2 all floating around. Looking at that mws file, you see that, in verifying the equation curl **B** = j (β^2/ω) **E,** the curl **B** components are functions of β' and k. And the **E** components are functions of β' and k as well. On the right side which says j (β^2/ω) **E** I replace β^2 with $\beta'^2 - k^2$. Thus, in verifying curl **B** = j (β^2/ω) **E** I have each side being a function of β' and k, so at least it *stands a chance* of working our right, and as Maple shows, it does work out right.

Problem verifying after the low-x limit is taken

I first take the low x limits of the E fields. In doing this I use these low-x limits of the f_m type functions,

$f_m = (r/a)^m (m+1) (2/\beta'a)$ $\quad f_0 = 4/(a\beta')$

$g_m = (r/a)^{m+1} + (r/a)^{m-1}$ $\quad g_0 = 2 (r/a)$

$h_m = (r/a)^{m+1} - (r/a)^{m-1}$ $\quad h_0 = 0$ $\quad x = \beta'r$ $\quad x_a = \beta'a$

In doing these limits, I assume a priori that $x_a = \beta'a \ll 1$, and that in turn forces $x = \beta'r \ll 1$. For the moment, I defer the claim that this small x_a limit aligns with the small ω limit and/or the small k limit. Let's just think of this as a small-x limit !

Notice that β' only appears in two places above, and there are no k appearances. When these f_m etc are used in the E field expressions, I get

$$E_z(r,m) = (1/2)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(r/a)^{|m|}\,(|m|+1) \qquad (7.4.1)$$

$$E_r(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$E_z(r,0) = CV\, R_{dc}\, (\omega/k)$
$E_r(r,0) = (j/2)\, CV\, R_{dc}\, (\omega r)$
$E_\theta(r,0) = 0$ // low ω E fields

A key point is that *only k appears*, there are no β' appearances (and of course no β appearances).

At this point there are two different methods of obtaining the corresponding B fields. In the first method, I work directly with the above expressions and compute curl **E** of curl **E** = -jω**B**. This is my approach in Chapter 7, and here is what I find:

$B_z(r,m) = (j/2)\, \eta_m CVR_{dc}\, (r/a)^m\, (m+1)$
$B_r(r,m) = (1/4)\, \eta_m CVR_{dc}\, (r/a)^{m-1}\, (1/ak)[\, -2m(m+1) + k^2(a^2-r^2)]$
$B_\theta(r,m) = (j/4)\, \eta_m CVR_{dc}\, (r/a)^{m-1}\, (1/ak)\, [\, -2m(m+1) + k^2(r^2+a^2)]$ $\quad m > 0$

$B_z(r,0) = 0$
$B_r(r,0) = 0$
$B_\theta(r,0) = (j/2)\, CV\, R_{dc}\, kr$ $\quad m = 0$

Not surprisingly, since these B fields are derived from the E fields, only k appears. In the second method of obtaining the B fields, I work directly from the "full" B field expressions which are these

$B_z(r,m) = \quad (j/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (k\, \beta'\, e_m\,)$
$B_r(r,m) = -\, (1/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (\ r^{-1}m\ \beta'\, f_m + k^2\ h_m\ )$
$B_\theta(r,m) = \quad (j/4)\, (a/\omega)\, \eta_m\, I\, R_{dc}\, (\, k^2\, g_m - \beta'^2 f_m\, [\, (m/x) - J_{m+1}(x)/J_m(x)]\,)$

and I take the small x limit just as I did for the E fields. I show in "infinite B fields directly..." that doing this direct reduction from the full B fields gives the same low-x B fields (at least for m > 0) as obtained by the first method.

Now here is a simple statement of the paradoxical problem. I have E and B fields which in the low-x limit depend only on k. The curl operation can only introduce factors of k due to the $\partial_z \rightarrow -jk$ fact which of course follows from the famous e^{-jkz} ansatz. Therefore, curlB and E are each functions only of k, and not of β or β'. But then I have to verify this Maxwell curl B equation:

$$\text{curl}\ \mathbf{B} = \mu\, \mathbf{J} + \mu\, j\omega\varepsilon \mathbf{E} \quad = \mu(\sigma + j\omega\varepsilon)\, \mathbf{E} \ = \mu(j\omega)(\, \varepsilon - j\sigma/\omega)\, \mathbf{E} = j\omega\, \mu\xi\, \mathbf{E} \qquad \text{(D.5.1)}$$
$$= j\, (\beta^2/\omega)\, \mathbf{E} \quad . \qquad \text{// see (1.5.1c)}$$

which I write simply as

$$\text{curl}\ \mathbf{B} = j\, (\beta^2/\omega)\, \mathbf{E}\ .$$

Since curl **B** on the left and **E** on the right are functions only of k, it is impossible that this equation be satisfied! So that is our little conundrum addressed in this doc!

Statement: Assuming the e^{-jkz} ansatz, I obtain full E and B fields in terms of Bessel functions, and I show that these full fields satisfy all four Maxwell equations including the curl B equation. My problem

is that when I take the small x limit of the E and B fields, it seems that the curl B equation is no longer satisfied. The fact that things work with the full fields tells me that the e^{-jkz} ansatz is *not* the problem.

Not enough terms in the small x limit?

Somehow this seems a likely culprit. Especially when you combine this with the derivatives which appear in the curl operator. I addressed this issue at one time just for the E fields and concluded that extra terms made no difference and could be ignored, but I did not do that for the B fields. My previous work could have been wrong of course! Let's put this idea on hold for a moment.

Treating β^2 for small x ?

We know that

$$\beta'^2 = \beta^2 - k^2$$

so that

$$\beta^2 = \beta'^2 + k^2$$

Now one argument might be that in our small-x limit, we are assuming small β', so in that case

$$\beta^2 = k^2$$

in our limit. Then the curl B equation becomes this:

$$\text{curl } \mathbf{B} = j\,(k^2/\omega)\,\mathbf{E}$$

and this equation *does* have a chance of being valid. Let's check it. I did this in "B for small omega" and it all works just fine.

My paradox has gone away. The point is that you only expect curl $\mathbf{B} = j\,(\beta^2/\omega)\,\mathbf{E}$ to be valid for small x, not in general, and for small x we have $\beta \approx k$.

Does this conflict with my use of k(ω) ?

In order to make things work for the curl B equation, I had to say that

$$\beta^2 = \beta'^2 + k^2 \approx k^2$$ dropping terms of order β'^2 which is small

In the low-x limit we *must* have β' be small. We do not need to have k or β be small. Here are two ways that we can have β' be small

1) β and k need not be small, but we must have $\beta \approx k$ in order for β' to be small

2) β and k are both small, so in that case $\beta'^2 = \beta^2 - k^2$ will be small. In this case, we do not need to have $\beta \approx k$ in order that β' be small.

In this second case, consider

$$\text{curl } \mathbf{B} = j\,(\beta^2/\omega)\,\mathbf{E} \;\; = \; j\,(\beta'^2/\omega)\,\mathbf{E} \;\; + j\,(k^2/\omega)\,\mathbf{E}$$

We can certainly neglect the first term since we are in low-x. But in this case, the second term is also negligible. I don't think there are any problems here.