Redo App D solutions for m = 0 explicitly PhL 9.25.14

Here -- yet once again -- I verify that the "second summary" is valid for m = 0. I was having doubts midway below, but then the old results turned out to be valid, and I added a comment in lines doc pointing out that $a_0 = 0$ and also $(\frac{K_0}{2} + \frac{a_0}{0}) = 0$, so $E_\theta(r,0) \equiv 0$.

The first summary is this

First summary of the E field solutions (D.2.21)

$E_z(r,m) = -j(\beta'/k)\frac{K_m}{2} J_m(x)$ $x = \beta' r$ *(D.1.27)*

$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x)$ $\beta'^2 = -k^2$ $\beta' = jk$ *(D.2.11)*

$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + (\frac{K_m}{2} + \frac{a_m}{m}) J_{m+1}(x)$ *(D.2.15)*

I don't like seeing the a_m/m thing. Where did it come from? It comes from the fact that I use (D.26) to make a replacement, which equation says $jmE_\theta(r,m)$ = stuff. Suppose I directly study these equations in the case m = 0 and see what happens then. So I am now going to do Section D.2 with m = 0 from the start (and for the moment I will forget ω=0 and just leave β' as its original self). I will now write new versions of various equations with m = 0:

(a) The E_z Solution

$[r^2\partial_r^2 + r\,\partial_r + (r^2\,\beta'^2)]\,E_z(r,0) = 0$ (D.2.1)

$[x^2\partial_x^2 + x\,\partial_x + (x^2)]\,E_z(x/\beta',0) = 0\,.$ $x = \beta' r$ (D.2.3)

$E_z(r,0) = C_{z0}\,J_0(\beta' r)$ (D.2.4)

I think Y_0 is logarithmically divergent just like ln r in the 2D thing and can be left out. So nothing is new here so far.

(b) The E_r Solution

$[r^2\partial_r^2 + r\partial_r - (1) + r^2\beta'^2]\,E_r(r,0) = 0$ (D.2.5)

Equation is no longer coupled with E_θ. I think the solution to this equation is just

$E_r(r,0) = C_{r0}\,J_1(\beta' r)$ // Schaum says yes

Again, I would reject $Y_1(\beta' r)$. So this makes for a much briefer E_r solution section. Continuing,

(c) The E_θ Solution

The equation here is

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-k^2)]\, E_\theta(r,m) + 2jmE_r(r,m) = 0 \qquad (D.1.18)$$

which becomes

$$[r^2\partial_r^2 + r\partial_r - (1) + r^2\beta'^2]\, E_\theta(r,0) = 0$$

and this is the *same* as for E_θ, so I write

$$E_\theta(r,0) = C_{\theta 0}\, J_1(\beta' r)$$

(c') The div E = 0 Condition // this section does not exist in App D

But I then have the div E equation which has not been used at all, and it says

$$\partial_r [r\, E_r(r,m)] + jmE_\theta(r,m) + r\,(-jk)E_z(r,m) = 0$$

or for m = 0

$$\partial_r [r\, E_r(r,0)] + r\,(-jk)E_z(r,0) = 0$$

and it then links E_r and E_z together. Inserting my solutions then gives

$$\partial_r [r\, C_{r0}\, J_1(\beta' r)] + r\,(-jk)\, C_{z0}\, J_0(\beta' r) = 0 \qquad x = \beta' r$$

I think I can write this as

$$C_{r0}\, \partial_x [x\, J_1(x)] + C_{z0}\, (x/\beta')(-jk) J_0(x) = 0$$

Now Schaum p 137 24.21 with n = 1 says

$$\partial_x [x\, J_1(x)] = x\, J_0(x)$$

Then my divE = 0 condition becomes

$$C_{r0}\, x\, J_0(x) + C_{z0}\, (x/\beta')(-jk) J_0(x) = 0$$

or

$$x\, J_0(x)\, [C_{r0} - jkC_{z0}/\beta'] = 0$$

or

$$C_{r0} = j(k/\beta')C_{z0}$$

I feel I am treading on new territory here, I don't remember doing this before.

Now recall that

$$C_{zm} = (1/2j)(\beta'/k)\, K_m \,. \qquad \text{(D.2.20)}$$

which is just my *definition* of the K_m constants. I can then of course say

$$C_{z0} = (1/2j)(\beta'/k)\, K_0 \,. \qquad \text{(D.2.20)}$$

Then my div E = 0 result above says

$$C_{r0} = j(k/\beta')C_{z0} = j(k/\beta')(1/2j)(\beta'/k)\, K_0 = (1/2)\, K_0$$

I then end up with

$$C_{z0} = (1/2j)(\beta'/k)\, K_0$$
$$C_{r0} = (1/2)\, K_0$$

where both are related to the single unknown constant K_0. And then

$$E_z(r,0) = C_{z0}\, J_0(\beta'r) = (1/2j)(\beta'/k)\, K_0\, J_0(\beta'r) = -j\,(\beta'/k)\,\frac{K_0}{2}\, J_0(\beta'r)$$

$$E_r(r,0) = C_{r0}\, J_1(\beta'r) = (1/2)\, K_0\, J_1(\beta'r) = \frac{K_0}{2}\, J_1(\beta'r)$$

$$E_\theta(r,0) = C_{\theta 0}\, J_1(\beta'r)$$

I can then restate my "first summary" for the E fields as

First summary of the E field solutions (D.2.21)

For $m \neq 0$:

$E_z(r,m) = -j\,(\beta'/k)\,\frac{K_m}{2}\,J_m(x)$ $\qquad x = \beta' r$ *(D.1.27)*

$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)$ $\qquad \beta'^2 = \beta^2 - k^2$ *(D.2.11)*

$jE_\theta(r,m) = -a_m\,x^{-1}\,J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)$ *(D.2.15)*

For $m = 0$:

$E_z(r,0) = -j\,(\beta'/k)\,\frac{K_0}{2}\,J_0(\beta' r)$

$E_r(r,0) = \frac{K_0}{2}\,J_1(\beta' r)$

$E_\theta(r,0) = C_{\theta 0}\,J_1(\beta' r)$ // but $C_{\theta 0} = 0$ from below !

Observations:

- The E_z and E_r solutions for $m = 0$ can be obtained from the $m \neq 0$ solution set by setting $m = 0$ and assuming that $a_0 = 0$ (as I do assume in App D).

- The E_θ solution can NOT be obtained in this manner! It is independent. I think in App D I would say that $j\,E_\theta(r,0) = \frac{K_0}{2}\,J_1(x)$, but I now see that is WRONG! So perhaps for the first time in a long time I have found a bug in App D. Go back and rereads this section. Done. The reason E_θ is independent is that the divE = 0 condition for m = 0 does not involve E_θ !!!!

NOW let's apply the boundary conditions.

$E_r(r=a,m) = (j\omega/\sigma)\,N_m$ (D.2.26)

$E_\theta(r=a,m) = 0$ (D.2.27)

For $m \neq 0$, things are unchanged, but for $m = 0$ I now get for the second,

$E_\theta(r=a,0) = 0 \quad \Rightarrow \quad C_{\theta 0}\,J_1(\beta' a) = 0 \quad \Rightarrow \quad C_{\theta 0} = 0$

and now $C_{\theta 0}$ is GONE from the problem forever. The other BC says

$E_r(r=a,0) = (j\omega/\sigma)\,N_0 \quad \Rightarrow \quad \frac{K_0}{2}\,J_1(\beta' a) = (j\omega/\sigma)\,N_0$

and this tells us that

$K_0 = 2\ (j\omega/\sigma)\ N_0 / J_1(\beta' a)$ // which is the same as I had before by the way

Now the new version of coefficients box (D.2.28) is this

For $m \neq 0$:

$$a_m = (j\omega/2\sigma)\ 2m\ N_m \frac{1}{J_{m-1}(x_a)} \quad . \qquad \text{(D.2.28)}$$

$$\frac{K_m}{2} = (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right]$$

For $m = 0$:

$$\frac{K_0}{2} = (j\omega/\sigma)\ N_0 \frac{1}{J_1(x_a)}$$

Now I can insert these into the previous box to get:

First for $m \neq 0$:

$$E_z(r,m) = -j\ (\beta'/k) \frac{K_m}{2}\ J_m(x) = -j\ (\beta'/k)\ (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= -j\ (\beta'/k)\ (j\omega/2\sigma)\ N_m f_m = (\beta'/k)\ (\omega/2\sigma)\ N_m f_m$$

$$E_r(r,m) = a_m\ x^{-1}\ J_m(x) + \frac{K_m}{2}\ J_{m+1}(x)$$

$$= (j\omega/2\sigma)\ 2m\ N_m \frac{1}{J_{m-1}(x_a)}\ x^{-1}\ J_m(x) + (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x)$$

$$= (j\omega/2\sigma)\ N_m \left\{ 2m \frac{1}{J_{m-1}(x_a)}\ x^{-1}\ J_m(x) + \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x) \right\}$$

At this point I have to use Schaum 24.17 which says

$$2m\ x^{-1} J_m = J_{m+1} + J_{m-1}$$

to get

1 2 3 4

$$= (j\omega/2\sigma)\, N_m \left\{ \frac{1}{J_{m-1}(x_a)} \left[J_{m+1}(x) + J_{m-1}(x)\right] + \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x) \right\}$$

Terms 1 and 4 cancel giving

$$= (j\omega/2\sigma)\, N_m \left\{ \frac{1}{J_{m-1}(x_a)} \left[J_{m-1}(x)\right] + \left[\frac{1}{J_{m+1}(x_a)}\right] J_{m+1}(x) \right\} = (j\omega/2\sigma)\, N_m\, g_m$$

Now finally,

$$jE_\theta(r,m) = - a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x)$$

$$= - (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}\, x^{-1}\, J_m(x) + (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x)$$

$$= (j\omega/2\sigma)\, N_m \left\{ - 2m \frac{1}{J_{m-1}(x_a)}\, x^{-1}\, J_m(x) + \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x) \right\}$$

$$= (j\omega/2\sigma)\, N_m \left\{ \underset{1}{-} \frac{1}{J_{m-1}(x_a)} \underset{2}{\left[J_{m+1}(x) + J_{m-1}(x)\right]} + \underset{3}{\left[\frac{1}{J_{m+1}(x_a)}\right.} \underset{4}{\left. + \frac{1}{J_{m-1}(x_a)}\right]} J_{m+1}(x) \right\}$$

Now terms 1 and 4 cancel and we get

$$= (j\omega/2\sigma)\, N_m \left\{ - \frac{1}{J_{m-1}(x_a)} \left[J_{m-1}(x)\right] + \left[\frac{1}{J_{m+1}(x_a)}\right] J_{m+1}(x) \right\}$$

$$= (j\omega/2\sigma)\, N_m\, h_m$$

So here is my Second Summary for $m \neq 0$:

$$E_z(r,m) = (\beta'/k)\, (\omega/2\sigma)\, N_m\, f_m$$

$$E_r(r,m) = (j\omega/2\sigma)\, N_m\, g_m$$

$$jE_\theta(r,m) = (j\omega/2\sigma)\, N_m\, h_m$$

Now for $m = 0$, we can just copy the first two above, but we set the last to $\equiv 0$!!!

$$E_z(r,0) = (\beta'/k)\, (\omega/2\sigma)\, N_0\, f_0$$

$$E_r(r,0) = (j\omega/2\sigma)\, N_0\, g_0$$

$E_\theta(r,0) = 0$

Now, if you go look at (D.2.31a) , since $E_z(r,m)$ with m = 0 has not changed, the relation between I and N_0 does not change!! So we can go ahead and write

$N_0 = (k/2\pi\omega a)\, I$

$N_m = \eta_m (k/2\pi\omega a)\, I$

Use these then to re-express the above 6 fields

<u>m ≠ 0:</u>

$E_z(r,m) = (\beta'/k)(\omega/2\sigma)\eta_m(k/2\pi\omega a) I f_m = (\beta' a)(1/2\sigma)\eta_m(1/2\pi a^2) I f_m = (1/4)(\beta' a)\eta_m I R_{dc} f_m$

$E_r(r,m) = (j\omega/2\sigma)\eta_m(k/2\pi\omega a) I g_m = (j/2\sigma)\eta_m(ka/2\pi a^2) I g_m = (j/4)(ka)\eta_m I R_{dc} g_m$

$jE_\theta(r,m) = (j\omega/2\sigma)\eta_m(k/2\pi\omega a) I h_m = (j/2\sigma)\eta_m(k/2\pi a) I h_m = (j/4)(ka)\eta_m I R_{dc} h_m$

Now for m = 0, we can just copy the first two above, but we set the last to ≡ 0 !!!

$E_z(r,0) = (\beta'/k)(\omega/2\sigma)(k/2\pi\omega a) I f_0 = (\beta' a)(1/2\sigma)(1/2\pi a^2) I f_0 = (1/4)(\beta' a) I R_{dc} f_0$

$E_r(r,0) = (j\omega/2\sigma)(k/2\pi\omega a) I g_0 = (j/2\sigma)(ka/2\pi a^2) I g_0 = (j/4)(ka) I R_{dc} g_0$

$E_\theta(r,0) = 0$

Here then is my "new" second summary

<u>m ≠ 0:</u>

$E_z(r,m) = (1/4)(\beta' a)\eta_m I R_{dc} f_m$
$E_r(r,m) = (j/4)(ka)\eta_m I R_{dc} g_m$
$E_\theta(r,m) = (1/4)(ka)\eta_m I R_{dc} h_m$

<u>m = 0</u>:

$E_z(r,0) = (1/4)(\beta' a) I R_{dc} f_0$
$E_r(r,0) = (j/4)(ka) I R_{dc} g_0$
$E_\theta(r,0) = 0$

But let's now compare this to the existing box (D.2.33),

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ (D.2.33)

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$ $g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$ $h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

But we know that

$$f_0 = [\frac{J_0(x)}{J_1(x_a)} - \frac{J_0(x)}{J_{-1}(x_a)}] = 2\frac{J_0(x)}{J_1(x_a)}$$

$$g_0 = [\frac{J_1(x)}{J_1(x_a)} + \frac{J_{-1}(x)}{J_{-1}(x_a)}] = 2\frac{J_1(x)}{J_1(x_a)}$$

$$h_0 = [\frac{J_1(x)}{J_1(x_a)} - \frac{J_{-1}(x)}{J_{-1}(x_a)}] = 0 \qquad \text{D.6.1}$$

Where have I written this down? They are all in D.6.1.

Since $h_0 \equiv 0$, I now see that the "second summary" is unchanged! It already gives $E_\theta(r,0) = 0$ since $h_0 = 0$. So the upshot is that there was some fuzziness with $E_\theta(r,0)$ when written in the general form, but in the end, there is no change to my Second Summary now that I have completely redone the m = 0 case separately. That would explain why these second summary fields solved the Helm equation as I have Maple show below (D.2.35). There I set both $a_0 = 0$ and $a_0/0 = 0$.

I do think something is fishy with (D.2.21) so just now I added a clarifying note regarding m = 0 after that equation to show that $E_\theta(r,0) \equiv 0$ from the git go.