Redo Sec 4 with averaging everywhere PhL 9.14.14

I think this is the first doc where I attempt to maintain $(\partial_x A_{x12}(\mathbf{x}) + \partial_y A_{y12}(\mathbf{x}))$ in the Chapter 4 analysis leading to the transmission line equations. I made some mistakes below, such as T(z) missing from one of the transmission line pair. I am also trying to support the general notion of averaging better. All this stuff ended up in Appendix S where I think it is done OK.

The doc ends with a lot of scratch work, and one item starts to show that "superposition" does not seem to fix that J_z asymmetry problem.

Reviewed 10.8.14.

I think the very first thing would be a new version of (4.4.1),

$$V(\mathbf{x}_1,\mathbf{x}_2) \equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$$
$$= \frac{1}{4\pi\xi_d}\, q(z) \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_{12}} \right\}$$
$$- \frac{1}{4\pi\xi_d}\, q(z) \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_{22}} \right\} \quad (4.4.1)$$

Continuing on, we would then get to

$$V(\mathbf{x}_1,\mathbf{x}_2) = q(z) \frac{1}{4\pi\xi_d} \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \right\}$$

$$s_{21}^2 = (x_2-x_1')^2 + (y_2-y_1')^2 \qquad s_{22}^2 = (x_2-x_2')^2 + (y_2-y_2')^2 \quad (4.4.6)$$
$$s_{11}^2 = (x_1-x_1')^2 + (y_1-y_1')^2 \qquad s_{12}^2 = (x_1-x_2')^2 + (y_1-y_2')^2 \ .$$

Next would come

$$\frac{1}{C'(\mathbf{x}_1,\mathbf{x}_2)} = \frac{V(\mathbf{x}_1,\mathbf{x}_2)}{q(z)}$$
$$= \frac{1}{4\pi\xi_d} \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \right\}$$
$$= \frac{1}{4\pi\xi_d}\, K(\mathbf{x}_1,\mathbf{x}_2) \text{ and } \quad V(\mathbf{x}_1,\mathbf{x}_2) = q(z) \frac{1}{4\pi\xi_d}\, K(\mathbf{x}_1,\mathbf{x}_2) \quad (4.4.7)$$

Now I guess I can decompose C' into its real and imaginary parts and give them names as shown,

$$C'(\mathbf{x}_1,\mathbf{x}_2) = C(\mathbf{x}_1,\mathbf{x}_2) + G(\mathbf{x}_1,\mathbf{x}_2)/(j\omega) \quad (4.4.9)$$

And next of course

$C(\mathbf{x}_1,\mathbf{x}_2) = 4\pi\varepsilon_d/K(\mathbf{x}_1,\mathbf{x}_2)$ capacitance per unit length of the transmission line

$G(\mathbf{x}_1,\mathbf{x}_2) = 4\pi\sigma_d/K(\mathbf{x}_1,\mathbf{x}_2)$ conductance per unit length of the transmission line

so

$$C(\mathbf{x}_1,\mathbf{x}_2)\,/G(\mathbf{x}_1,\mathbf{x}_2) \;=\varepsilon_d/\sigma_d \;. \tag{4.4.10}$$

I don't have to interpret anything, things like $C(\mathbf{x}_1,\mathbf{x}_2)$ are just functions with dimensions you know.

$$L_e(\mathbf{x}_1,\mathbf{x}_2) = (\mu_d/4\pi)K(\mathbf{x}_1,\mathbf{x}_2) \;. \tag{4.4.11}$$

$$Z_0(\mathbf{x}_1,\mathbf{x}_2) \;=\; \sqrt{\frac{L_e}{C}} \;= (1/4\pi)\, K(\mathbf{x}_1,\mathbf{x}_2)\,\sqrt{\mu_d/\varepsilon_d} \tag{4.4.14}$$

OK, and so ends Section 4.4. I then do the two examples, and theory resumes in 4.7. But I think we can skip down to Section 4.10 where we talk about W between the two conductors.

$$W(\mathbf{x}_1,\mathbf{x}_2) \equiv A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)$$

$$= \frac{\mu_d}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \,\{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_{12}} \,\}$$

$$- \frac{\mu_d}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \,\{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_{22}} \,\} \tag{4.10.1}$$

and then later

$$W(\mathbf{x}_1,\mathbf{x}_2) = i(z)\,\frac{\mu_d}{4\pi}\,\{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1')\, \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2')\, \ln(s_{22}^2/s_{12}^2) \,\} \tag{4.10.4}$$

$$W(\mathbf{x}_1,\mathbf{x}_2) = L_e(\mathbf{x}_1,\mathbf{x}_2)\; i(z) \;. \tag{4.10.7}$$

$$L_e(\mathbf{x}_1,\mathbf{x}_2) \;= \frac{W(\mathbf{x}_1,\mathbf{x}_2)}{i(z)} \;=\; \frac{\mu_d}{4\pi}\; K_L(\mathbf{x}_1,\mathbf{x}_2) \tag{4.10.8}$$

where K_L is the following dimensionless real number,

$$K_L(\mathbf{x}_1,\mathbf{x}_2) \equiv \;\int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1')\, \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2')\, \ln(s_{22}^2/s_{12}^2) \;. \tag{4.10.9}$$

$$K(\mathbf{x}_1,\mathbf{x}_2) \;\equiv\; \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1')\, \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2')\, \ln(s_{22}^2/s_{12}^2) \;. \tag{4.4.8}$$

Thus ends Section 4.10 and we start into Section 4.11.

$$\mathbf{E} = -\,\text{grad}\,\varphi - \partial_t \mathbf{A} \tag{1.3.1}$$

$$\text{div}\,\mathbf{A} = -\,\mu_d\varepsilon_d\,\partial_t\varphi - \mu\sigma\varphi \;. \qquad \text{// the King gauge} \tag{1.3.18}$$

Both the above equations are exact. But then my next step is this

$$E_z(\mathbf{x}) = -\partial_z\varphi(\mathbf{x}) - j\omega A_z(\mathbf{x})$$
$$\partial_z A_z(\mathbf{x}) + (\partial_x A_x + \partial_y A_y) = -j\,(\beta_d^2/\omega)\,\varphi(\mathbf{x}) \quad . \qquad (4.11.3)$$

The first above is still exact, the second is NOT exact and I will have more to say about it later on (next section of this doc). I will nevertheless just carry on with this inexactness being allowed. [But I am now going back and adding the missing stuff in blue.]

Next comes

$$E_z(\mathbf{x}) = -\partial_z\varphi_{12}(\mathbf{x}) - j\omega A_{z12}(\mathbf{x}) \qquad (4.11.4a)$$
$$\partial_z A_{z12}(\mathbf{x}) + (\partial_x A_{x12}(\mathbf{x}) + \partial_y A_{y12}(\mathbf{x})) = -j\,(\beta_d^2/\omega)\varphi_{12}(\mathbf{x}) \quad . \qquad (4.11.4b)$$

These are implied by the previous pair, so no new assumptions. Then

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)] - j\omega[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] \qquad (4.11.5a)$$
$$\partial_z[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] = -j\,(\beta^2/\omega)[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)] \quad . \qquad (4.11.5b)$$

$$\partial_z[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] + (\partial_x A_{x12}(\mathbf{x}_1) - \partial_x A_{x12}(\mathbf{x}_2)\,] + (\partial_y A_{y12}(\mathbf{x}_1) - \partial_y A_{y12}(\mathbf{x}_2)\,]$$
$$= -j\,(\beta^2/\omega)[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)]$$

Again, there are no new assumptions, I just did a subtraction. I then define

$$V(\mathbf{x}_1,\mathbf{x}_2) \equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2) \qquad \textit{(4.4.1)}$$
$$W(\mathbf{x}_1,\mathbf{x}_2) \equiv A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2) \qquad \textit{(4.10.1)}$$

and this leads to

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z V(\mathbf{x}_1,\mathbf{x}_2) - j\omega W(\mathbf{x}_1,\mathbf{x}_2) \qquad (4.11.6a)$$
$$\partial_z W(\mathbf{x}_1,\mathbf{x}_2) = -j\,(\beta_d^2/\omega)\,V(\mathbf{x}_1,\mathbf{x}_2) \quad . \qquad (4.11.6b)$$

$$\partial_z W(\mathbf{x}_1,\mathbf{x}_2) = (\partial_x A_{x12}(\mathbf{x}_1) - \partial_x A_{x12}(\mathbf{x}_2)\,] + (\partial_y A_{y12}(\mathbf{x}_1) - \partial_y A_{y12}(\mathbf{x}_2)\,] - j\,(\beta_d^2/\omega)\,V(\mathbf{x}_1,\mathbf{x}_2)$$

Suppose for shorthand I write this as

$$\partial_z W(\mathbf{x}_1,\mathbf{x}_2) = T(\mathbf{x}_1,\mathbf{x}_2) - j\,(\beta_d^2/\omega)\,V(\mathbf{x}_1,\mathbf{x}_2) \qquad T = \text{"Transverse derivatives"}$$

If I double average this equation, I get something like this

$$\partial_z W(z) = T(z) - j\,(\beta_d^2/\omega)\,V(z)$$

I then average the first equation of the previous pair to get

$$[\langle E_z(\mathbf{x}_1)\rangle_{c1} - \langle E_z(\mathbf{x}_2)\rangle_{c2}]$$
$$= -\partial_z[\langle\varphi_{12}(\mathbf{x}_1)\rangle_{c1} - \langle\varphi_{12}(\mathbf{x}_2)\rangle_{c2}] - j\omega\,[\langle A_{z12}(\mathbf{x}_1)\rangle_{c1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{c2}\,] \quad .$$

We now *redefine* V and W to be the averages appearing in these equations, along with E_{z1} and E_{z2} :

$$E_{z1}(z) \equiv \langle E_z(\mathbf{x}_1)\rangle_{C1} = (1/P_1)\int_{C1} ds_1\, E_z(\mathbf{x}_1)$$

$$E_{z2}(z) \equiv \langle E_z(\mathbf{x}_2)\rangle_{C2} = (1/P_2)\int_{C2} ds_2\, E_z(\mathbf{x}_2)$$

$$\begin{aligned} V(z) &\equiv \langle \varphi_{12}(\mathbf{x}_1)\rangle_{C1} - \langle \varphi_{12}(\mathbf{x}_2)\rangle_{C2} = \langle V(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2} \\ W(z) &\equiv \langle A_{z12}(\mathbf{x}_1)\rangle_{C1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{C2} = \langle W(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2} \end{aligned} \qquad (4.11.7)$$

with this result

$$[E_{z1}(z) - E_{z2}(z)] = -\partial_z V(z) - j\omega W(z) \ . \qquad (4.11.8)$$

I think the above equation with its averages is exact!

Meanwhile, the surface impedances on C_1 and C_2 are defined by (see C.2.1) ,

$$\begin{aligned} E_{z1}(\mathbf{x}_1) &= Z_{s1}(\mathbf{x}_1)\, i_1(z) \\ E_{z2}(\mathbf{x}_2) &= Z_{s2}(\mathbf{x}_2)\, i_2(z) \end{aligned} \qquad \textit{(C.2.1)}$$

which we average in the same way to obtain

$$\begin{aligned} E_{z1}(z) &= Z_{s1}\, i_1(z) \qquad & Z_{s1} &\equiv (1/P_1)\int_{C1} ds_1\, Z_{s1}(\mathbf{x}_1) \\ E_{z2}(z) &= Z_{s2}\, i_2(z) \qquad & Z_{s2} &\equiv (1/P_2)\int_{C2} ds_2\, Z_{s2}(\mathbf{x}_2) \ . \end{aligned} \qquad (4.11.9)$$

There will be some location on C_1 where $E_{z1}(\mathbf{x}_1)$ and thus $Z_{s1}(\mathbf{x}_1)$ will be maximal (for example on the walls of the gap in Fig 4.12). Referring to this value as $Z_{s1,max}$ we can define

$$\begin{aligned} p_1 &\equiv (Z_{s1}/Z_{s1,max})P_1 \\ p_2 &\equiv (Z_{s2}/Z_{s2,max})P_2 \end{aligned} \qquad (4.11.10)$$

Then using $i(z) = i_1(z) = -i_2(z)$ and (4.11.9), rewrite (4.11.8) and (4.11.6b) as

$$\begin{aligned} [Z_{s1} + Z_{s2}]\, i(z) &= -\partial_z V(z) - j\omega W(z) \\ \partial_z W(z) &= -j\,(\beta_d^2/\omega)\, V(z) + T(z) \end{aligned} \qquad (4.11.11)$$

Again, the first equation here is exact in terms of my averages, but the second is NOT exact for the reasons mentioned above going back to (4.11.3). I now back up to

$$W(\mathbf{x}_1,\mathbf{x}_2) = L_e(\mathbf{x}_1,\mathbf{x}_2)\, i(z) \ . \qquad (4.10.7)$$

which do a double average of the functions to get

$$W(z) = L_e\, i(z)$$

and then this is also exact, given the averages.

Then install this to get

$$(Z_{s1} + Z_{s2})\, i(z) = -\,\partial_z V(z) - j\omega\, L_e\, i(z)$$
$$L_e\, \partial_z\, i(z) = -\,j\,(\beta_d^2/\omega)\, V(z) + T(z)$$

which we then rearrange as

$$\partial_z V(z) = -\,[\,Z_{s1} + Z_{s1} + j\omega L_e]\, i(z)$$
$$\partial_z\, i(z) = -\,[\,j\beta_d^2/(\omega L_e)]\, V(z) + T(z)/L_e \qquad (4.11.14a)$$

Again, the second equation is not exact (but I made it exact in blue).

These are the classical **transmission line equations**. They are usually written in this form: [$\partial/\partial z = d/dz$]

$$\frac{dV(z)}{dz} = -\,\mathbf{z}\, i(z) \qquad \frac{di(z)}{dz} = -\,\mathbf{y}\, V(z) + T(z)/L_e$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C\,. \qquad (4.11.14b)$$

$$Z_{s1} \equiv (1/P_1) \int_{C1} ds_1\, Z_{s1}(\mathbf{x}_1)$$

$$Z_{s2} \equiv (1/P_2) \int_{C2} ds_2\, Z_{s2}(\mathbf{x}_2)$$

$$V(z) \equiv \langle \varphi_{12}(\mathbf{x}_1) \rangle_{C1} - \langle \varphi_{12}(\mathbf{x}_2) \rangle_{C2} = \langle V(\mathbf{x}_1,\mathbf{x}_2) \rangle_{C1,C2}$$
$$W(z) \equiv \langle A_{z12}(\mathbf{x}_1) \rangle_{C1} - \langle A_{z12}(\mathbf{x}_2) \rangle_{C2} = \langle W(\mathbf{x}_1,\mathbf{x}_2) \rangle_{C1,C2} \qquad (4.11.7)$$

$$L_e \equiv \langle L_e(\mathbf{x}_1,\mathbf{x}_2) \rangle_{C1,C2}$$

Thus, we end up with R,L,C,G which are functions of the above averages. I don't have any great interpretations for the averages, but they are just averages of the functions shown. So we have

$$\mathbf{z} = R + j\omega L = Z_{s1} + Z_{s2} + j\omega L_e \qquad (4.11.17)$$

$$\mathbf{y} = G + j\omega C = j\beta_d^2/(\omega L_e) \qquad (4.11.18)$$

where the Z's and L_e are the averaged values.

<u>But what about the $K = K_L$ business?</u> Despite the averaging done above, I think there are still some loose ends that I have to deal with. Go back to:

$$\frac{1}{C'} = \frac{V(z)}{q(z)} = \frac{1}{4\pi\xi_d} K \qquad (4.4.7)$$

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu_d}{4\pi} K_L \qquad (4.10.8)$$

Here we have to regard this as saying

$$\left\langle \frac{1}{C'(x_1,x_2)} \right\rangle_{c1,c2} = 1/q(z) * \langle V(x_1,x_2)\rangle_{c1,c2} = \frac{1}{4\pi\xi_d} \langle K(x_1,x_2)\rangle_{c1,c2}$$

and then I define

$$1/C' \equiv \left\langle \frac{1}{C'(x_1,x_2)} \right\rangle_{c1,c2} \qquad \text{so} \qquad 1/C' = V(z)/q(z) = \frac{1}{4\pi\xi_d} K$$

Question: Is this true:

$$\left\langle \frac{1}{C'(x_1,x_2)} \right\rangle_{c1,c2} = 1 / \left[\langle C'(x_1,x_2)\rangle_{c1,c2} \right]$$

The answer is no. For example, $\int dx\ 1/x \neq 1/\int dx\ x$ over some interval. So be careful! I think it is still OK to define C' as I have done it 2 lines above.

In the same way everything here is an average

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu_d}{4\pi} K_L \qquad \text{// this average seems clear}$$

There is more to do. Consider some next equations:

$$\partial_z\, i(z) = -[j\omega C']\, V(z) + (1/L_e)T(z) \ . \quad [\text{ I omitted this! }] \qquad (4.11.20)$$

$$\partial_z\, i(z) = -[j\beta_d^2/(\omega L_e)]\, V(z) + T(z)/L_e \quad [\text{ updated}] \qquad (4.11.14a)$$

Rewrite them both in this manner which is more suitable for talking about averaging,

$$-V(z) = \partial_z\, i(z) \frac{1}{j\omega} \frac{1}{C'}$$

$$-V(z) = \partial_z\, i(z) \frac{\omega}{j\beta_d^2} L_e$$

Now set these equal to find that

$$\frac{1}{j\omega}\frac{1}{C'} = \frac{\omega}{j\beta_d^2} L_e$$

We could then average this equation to get the same result with 1/C' and L_e being the averages discussed above. THEN we can say (using the averaged values),

$$L_e C' = \beta_d^2/\omega^2 = \mu_d \xi_d \quad . \quad \text{// see (1.5.1a) regarding } \beta_d^2 \tag{4.11.21}$$

$$\mathbf{y} = G + j\omega C = j\beta_d^2/(\omega L_e) = j\,(\beta_d^2/\omega^2)\,(\omega/L_e) = j\,(L_e C')\,(\omega/L_e) = j\omega C' \,. \tag{4.11.22}$$

We NOW have

$$\mathbf{z} = R + j\omega L = Z_{s1} + Z_{s2} + j\omega L_e \tag{4.11.17}$$

$$\mathbf{y} = G + j\omega C = j\omega C' \tag{4.11.18}$$

where everything the Z's and L_e and C' are the averages above.

$$\mathbf{y} = G + j\omega C = j\omega C' = j\omega(\xi_d/\varepsilon_d)C = j\omega\,(1 - j\sigma_d/\varepsilon_d\omega)C = j\omega C + (\sigma_d/\varepsilon_d)C \tag{4.11.24}$$

Once we have C' by doing the average of $1/C'(\mathbf{x}_1,\mathbf{x}_2)$, we can use C' as a constant. For example,

$$C'/C = q_c/q_s = (\xi_d/\varepsilon_d) \quad . \tag{4.11.23}$$

$$(1/C) = (\xi_d/\varepsilon_d)\,(1/C')$$

$$(1/C(x_1,x_2)) = (\xi_d/\varepsilon_d)\,(1/C'(x_1,x_2))$$

Now apply the average and we get

$$(1/C) = (\xi_d/\varepsilon_d)\,(1/C')$$

$$C'/C = q_c/q_s = (\xi_d/\varepsilon_d)$$

where C' and C are well defined quantities. More equations that are still true in averages.

$$L_e C' = \mu_d \xi_d \quad \text{// I just did this above, both are averages} \tag{4.11.27}$$

$$L_e C = \mu_d \varepsilon_d = 1/v_d^2 \quad \text{// just using result above} \tag{4.11.28}$$

Recall now that in terms of averages,

$$L_e = \frac{\mu_d}{4\pi} K_L \,. \tag{4.11.29}$$

Now go back to this result, in terms of averages,

$$L_e C' = \beta_d^2/\omega^2 = \mu_d \xi_d$$

write as

$$L_e = \mu_d \xi_d \ \frac{1}{C'}$$

but this says

$$\frac{\mu_d}{4\pi} K_L = \mu_d \xi_d \frac{1}{4\pi \xi_d} K$$

and finally we have our $K_L = K$ in terms of averages.

Conclusion: I think my averaging idea is in fact exact and unambiguous, and in itself, it puts no frequency range limitation on the resulting transmission line equations.

Possible Break

But now we go back to the assumption made earlier. In (4.11.3) I write

$$\text{div } \mathbf{A} = -j(\beta_d^2/\omega)\varphi \ . \qquad \text{// the King gauge, see (1.5.1a) and (1.5.5) re } \beta_d^2 \qquad (4.11.2)$$

$$\partial_z A_z(\mathbf{x}) = -j(\beta_d^2/\omega)\varphi(\mathbf{x}) \ . \qquad (4.11.3)$$

so I am assuming we can ignore the transverse A_t component *derivatives*. In new App M I claim this is justified for the *non*-derivatives all the way down to DC for the A_t but I say nothing about derivatives. Now look back at this:

$$\Rightarrow \ \varphi/v_d - A_z = E_z/(j\omega) \qquad (3.7.6)$$

Inside a good conductor E_z is small to begin with, and in the limit $\delta \to 0$ ($\omega \to \infty$) the right side of (3.7.6) is small due to this fact and due to the $1/\omega$ factor. Therefore,

$$A_z \approx \varphi/v_d \qquad \text{small or extreme skin effect} \qquad (3.7.7)$$

Now I would guess that (3.7.7) is NOT valid as $\omega \to 0$ since in that case $E_z/(j\omega)$ becomes very large despite the fact that E_z is very small! So this might be a key fact. This is my Step 1 result.

Note for manana: There is something afoot here. Before I derive the TL equations (with averaging), I do get to the steps shown above as (4.11.2) and then (4.22.3). This is then the source of the second equation needed to get the TL equations! Thus, I really am assuming that ω is larger than some amount!!! I think I have finally found something! I will pursue this tomorrow.

It is now manana. I have three arguments for the same conclusion:

Argument A. We have to go back to Chapter 3 and review this logic sequence:

1. If ω is large, then $A_z \approx \varphi/v_d$ as shown in (3.7.7), using the strict $E_z = -\partial_z\varphi - j\omega A_z$ relation.

2. If $|\partial_x A_x+\partial_y A_y| << |\partial_z A_z|$, then $A_z \approx \varphi/v_d$ as shown below (3.7.8), this using the King gauge.

From these two facts I conclude that

If ω is small, then this is NOT true: $|\partial_x A_x+\partial_y A_y| << |\partial_z A_z|$,

Argument B. I reach this same conclusion roughly in doc " App M on the subject of derivatives of A_x and A_z" by considering the Helmholtz integrals.

Argument C. Write the above condition replacing $\partial_z \rightarrow -jk$:

$|\partial_x A_x+\partial_y A_y| << |kA_z|$,

Large ω means large k and that makes the above be true.
Small ω means small k and that makes it be NOT true.

In a cruder sense, we might write

$(1/D)\,|A_x| << (1/\lambda)\,|A_z|$?

Then this would be true of λ is not too large relative to D, which means moderate ω.
But for low ω, λ is very large and then this is NOT true. In the extreme transmission line limit, λ is very large, ω is very low, and then this is NOT true.

OK, let's now back up to the derivation of the TL equations.

OK, I have included the omitted transverse terms in the above math and I end up with this

$$\frac{dV(z)}{dz} = -\mathbf{z}\, i(z) \qquad \frac{di(z)}{dz} = -\mathbf{y}\, V(z) + T(z)/L_e$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C\,. \qquad (4.11.14b)$$

where blue shows the now-extra term. Diff to get

$$\partial_z^2 V(z) = -\mathbf{z}\,\partial_z\, i(z) \qquad \partial^2 i(z) = -\mathbf{y}\,\partial_z V(z) + (1/L_e)\partial_z T(z)$$

We then get

$$\partial_z^2 V(z) = -\mathbf{z}\,[-\mathbf{y}\, V(z) + T(z)/L_e]$$

$$\partial^2 i(z) = - \mathbf{y}\,[-\mathbf{z}\, i(z)\,] + (1/L_e)\partial_z T(z)$$

Then the transmission line 2nd order equations become

$$[\partial_z^2 - \mathbf{zy}\,]\, V(z) = (1/L_e)T(z)$$

$$[\partial_z^2 - \mathbf{zy}\,]\, i(z) = (1/L_e)\partial_z T(z)$$

Without the blue right sides, these equations have the solution e^{-jkz} where $k^2 = -\ \mathbf{zy}$.

With the blue right side, the solutions do NOT have this simple exponential form!!! Thus, the ansatz of Appendix D is not realized. In this case, the solutions have to be integrals over k.

This is then a REASON why I get anomalous results as $\omega \rightarrow 0$.

OK, what implications does this have?

1. For "large ω", the ansatz form e^{-jkz} is justified and $k^2 = -\ \mathbf{zy}$. Thus, in the solutions of Appendix D, we set $k = -j\sqrt{\mathbf{zy}}$.

2. For "small ω", since the TL equations are no longer valid, the ansatz form e^{-jkz} is *not* justified. In this case, the general form of an Appendix D partial wave solution is this

$$E_z(r,m,z) = (1/4)\,\eta_m\, I\, R_{dc} \int dk\ C(k)\, f_m(\sqrt{\beta^2 - k^2}\, r)\, e^{-jkz}$$

$$E_r(r,m,z) = (j/4)\,\eta_m\, I\, R_{dc} \int dk\ C(k)\, g_m(\sqrt{\beta^2 - k^2}\, r)\, e^{-jkz}$$

$$E_\theta(r,m.z) = (1/4)\,\eta_m\, I\, R_{dc} \int dk\ C(k)\, h_m(\sqrt{\beta^2 - k^2}\, r)\, e^{-jkz}$$

where a coefficient function C(k) determines the distribution of k in the solution. For large ω then we must have

$$C(k) \rightarrow \delta(k - [-j\sqrt{\mathbf{zy}}\,])$$

For small ω, C(k) must be determined by solving the full problem including the matching boundary conditions on all fields (or potentials) at the two conductor surfaces, a problem we have not attempted.

Note: We know the solutions for a given value of k, namely.

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

This is the solution for some value $k = k_1$. The solution has the same form for $k = k_2$. I guess I could do this linear combination to form a third solution:

$$E_z(r,m)^{(3)} = A_m E_z(r,m; k_1) + B_m E_z(r,m; k_2)$$
$$E_r(r,m)^{(3)} = A_m E_r(r,m; k_1) + B_m E_r(r,m; k_2)$$
$$E_\theta(r,m)^{(3)} = A_m E_\theta(r,m; k_1) + B_m E_\theta(r,m; k_2)$$

which really says

$$E_i(r,m)^{(3)} = A_m E_i(r,m; k_1) + B_m E_i(r,m; k_2)$$

If I now look at the 4 equations of (D.1.20), what happens? We know that the z equation says

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \partial_z^2 + \beta^2\ ]\ e^{j(\omega t - k1z)} E_z(r,\theta;k_1) = 0\ . \qquad (1)$$

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \partial_z^2 + \beta^2\ ]\ e^{j(\omega t - k2z)} E_z(r,\theta;k_2) = 0\ . \qquad (1)$$

Then I suppose

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \partial_z^2 + \beta^2\ ]\ E_z(r,\theta)^{(3)}$$

=

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \partial_z^2 + \beta^2\ ]\ \{\ A_m E_i(r,m; k_1) + B_m E_i(r,m; k_2)$$

3. Thus, for small ω, the fact that our $k = -j\sqrt{\mathbf{zy}}$ Appendix D solution set has the two anomalies already noted can be explained as being due to the fact that the z dependence is not given by e^{-jkz} for low ω, but rather must be a linear combination as shown with C(k) which must be determined.

It is a pretty complicated explanation, but I have been looking for an explanation for a long time and have tried everything I could think of.

So let's now go back to the critical step in Chapter 3. There I started with this

$$\varphi/v_d - A_z = E_z/(j\omega) \qquad (3.7.6)$$

I argued that for large ω, this says

$A_z \approx \varphi/v_d$ small or extreme skin effect (3.7.7)

I then show that this same result obtains from the King gauge condition ,

$(\partial_x A_x+\partial_y A_y) - j\beta_d A_z = -j\,(\beta_d^2/\omega)\varphi$. (3.7.8)

if we neglect the transverse terms. I conclude that

Fact: The transverse derivative terms can be neglected if $|\ \varphi/v_d - A_z\ | >> |E_z/(j\omega)|$. This is all inside a conductor where E_z is small.

You might crudely say

$V/v_d >> E_z/\omega$

Do I have an estimate for E_z ? It is J_z/σ so condition is

$V/v_d >> J_z/(\sigma\omega)$

In a ballpark sense, maybe J_z = I/area so we then have

$V/v_d >> I/(\sigma\omega * \text{area})$

$\omega Z_0 >> v_d / (\sigma\pi a^2)$

$\omega >> [v_d / (\sigma\pi a^2)]\ (1/Z_0)$

Here I am already thinking in the low ω range with this area statement.

$\dim(\omega) = \text{m/sec} * \text{m}^{-2} * \text{ohm-m} * 1/\text{ohm} = 1/\text{sec} = \text{OK}$

Now at low ω maybe write

Fact 8: The small ω limit for $Z_0(\omega)$, assuming $\omega_d = 0$, is given by (Q.8.6)

$$\text{Re}(Z_0) \approx \sqrt{R_{dc}/(2C)}\ 1/\sqrt{\omega} + (1/2)\,[L_{dc}/\sqrt{2R_{dc}C}\,]\,\sqrt{\omega} + O(\omega^{3/2})$$
$$\text{Im}(Z_0) \approx -\sqrt{R_{dc}/(2C)}\ 1/\sqrt{\omega} + (1/2)\,[L_{dc}/\sqrt{2R_{dc}C}\,]\,\sqrt{\omega} + O(\omega^{3/2})$$

So the general idea is

$Z_0 = \sqrt{R_{dc}/(2C)}\ (1/\sqrt{\omega}\,)(1-j)$

$|Z_0| = \sqrt{R_{dc}/(C)}\ \ (1/\sqrt{\omega}\ )$

Then our inequality is

$\omega >> [v_d / (\sigma\pi a^2)]\ \ * \sqrt{\omega}\ \ / \sqrt{R_{dc}/(C)}$

or

$\sqrt{\omega}\ \ >> [v_d / (\sigma\pi a^2)]\ \ / \sqrt{R_{dc}/(C)}$

But for a twinlead each of radius a, perhaps $R_{dc} = 2\ \sigma/(\pi a^2)$ and we then have

$\omega >> [v_d / (\sigma\pi a^2)]^2\ (C/R_{dc})$

Now we can estimate $R_{dc} = 2 * 1/(\sigma\pi a^2)$ so $1/R_{dc} = (1/2)\ \sigma\pi a^2$ and then we get

$\omega >> [v_d^2 / (\sigma^2\pi^2 a^4)]\ \ (C)\ (1/2)\ \sigma\pi a^2$

or

$\omega >> [v_d^2 / (\sigma\pi a^2)]\ \ (C)\ (1/2)$

or

$\omega >> [Cv_d^2 / (2\pi\sigma a^2)]$

Another dimension check

dim(RHS) = far/m * m^2/sec^2 * m^{-2} * ohm-m = far * $1/sec^2$ * ohm = far-ohm/sec^2 = sec^{-1} OK

Now we know that $v_d^2 = 1/(\mu_d\varepsilon_d)$ so we have

$\omega >> [C/ (2\pi\ \mu_d\varepsilon_d\sigma a^2)]$

We also know that

$C = \ 4\pi\varepsilon_d/K$

so then we have

$\omega >> [4\pi\varepsilon_d / (2\pi\ K\ \mu_d\varepsilon_d\sigma a^2)]\ \ =\ \ 2 / (K\ \mu_d\sigma a^2)]$

or

$\omega\mu_d\sigma\ \ >> (2/Ka^2)$

Now $\delta \equiv \sqrt{2/\omega\mu\sigma}$, and if we assume the same μ we have

$\delta^2 = 2/(\omega\mu\sigma)$ => $\omega\mu_d\sigma = 2/\delta^2$

and then

$2/\delta^2 >> (2/Ka^2)$

$\delta^2 << Ka^2$

$(\delta/a) << \sqrt{K}$

This is how ALL inequalities end up, something about the ratio δ/a being small.

Plan A. Let's write a new section 4.11 (g) which discusses averaging being OK at any ω, but the TL equations are not valid for low ω . Perhaps that is a good place to insert this new stuff.

Or should I just rewrite ALL of Chapter 4 in the full $\mathbf{x_1}$, $\mathbf{x_2}$ notation? That will really mess things up a lot. Maybe I have a section reviewing the averaging idea, and then another section on the low ω implication.

9/16/14.

So exactly how does this explanation remove my two paradoxes? We now have

$$E_z(r,m,z) = (1/4)\,\eta_m\, I\, R_{dc} \int dk\; C(k)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}$$

$$E_r(r,m,z) = (j/4)\,\eta_m\, I\, R_{dc} \int dk\; C(k)\, g_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}$$

$$E_\theta(r,m.z) = (1/4)\,\eta_m\, I\, R_{dc} \int dk\; C(k)\, h_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}$$

It would then seem that

$$E_z(r,m,z) / E_z(r,0,z) = \eta_m \int dk\; C(k)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz} \Big/ \int dk\; C(k)\, f_0(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}$$

or

$$\frac{E_z(r,m,z)}{E_z(r,0,z)} = \eta_m \frac{\int dk\; C(k)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}}{\int dk\; C(k)\, f_0(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}}$$

There could be a wide spectrum of k values in C(k), so I cannot take any particular limit of f_m. Is it possible that the upper integral on the right side could be 0 ? In fact, the low ω requirement really is

$$\int dk\; C(k)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz} = 0 \qquad \text{for all m, all r, all z, as } \omega \to 0$$

Now C(k) can have either sign, but how can this possibly be true as a function of z ?

$F(m,\beta,r,z) = 0$ as $\omega \rightarrow 0$??