Rethink Debye Surface Currents at low ω PhL 8.16.14

Reviewing these notes on 9.10.14

I am reading through Section D.9 and taking notes here on how things are different.

(a) The notion of Debye Surface Currents

I think this section is still completely correct. At $\omega \to 0$ you would just have $\delta \to \infty$ and everything is still valid. Fact 1, the Corollary, and Fact 2 are still true as $\omega \to 0$.

(b) The role of Debye Surface Currents in the boundary condition

I think things are OK down through this equation

$$-j\omega\, n(\theta) = -J_r(a,\theta) - jk\,[\sigma E_z(a,\theta)\,\lambda_D]$$

$$= -\sigma E_r(a,\theta) - jk\,[\sigma E_z(a,\theta)\,\lambda_D]$$

$$= -\sigma E_r(a,\theta)\left[1 - jk\lambda_D \frac{E_z(a,\theta)}{E_r(a,\theta)}\right] \quad . \qquad \text{(D.9.7)}$$

where I continue to dispense with E_θ effects. At this point, my discussion goes into ansatz mode. I assume the second term is small, and with this assumption, the fields I obtain USING the traditional CPBC result in

$$\frac{E_z(a,\theta)}{E_r(a,\theta)} \sim \left|\frac{\beta}{k}\right|$$

which should really be β' in the ratio, but I was always thinking $k << \beta$ so $\beta = \beta'$. Let that lie for now. I then conclude the ansatz with a self-consistency argument to show that indeed, the second term is small in (D.9.7). My end up condition is $(\delta/\lambda_d) >> 1$ which is of course true also as $\omega \to 0$.

But perhaps this is just a self-fulfilling prophecy. IF I assume the second term is small, then I get the App D solutions, and then the second term is indeed small. Look at the logic here"

Step 1: Derive fact that $-j\omega\, n(\theta) = -\sigma E_r(a,\theta)\left[1 - jk\lambda_D \frac{E_z(a,\theta)}{E_r(a,\theta)}\right]$

Step 2: Assume 2nd term can be neglected to get $-j\omega\, n(\theta) = -\sigma E_r(a,\theta)$.

Step 3: Run through Appendix D with this as the CPBC and thereby obtain solutions (D.2.33)

Step 4: Observe that for these (D.2.33) solutions, one has $\frac{E_z(a,\theta)}{E_r(a,\theta)} \sim \left|\frac{\beta}{k}\right|$

Step 5: Using this field ratio, verify that the "second term" in Step 1 can be neglected.

I have then proven the following: "If the 2nd term can be neglected, then it can be neglected" !

[I agree, the argument seems circular, I need to do something to improve this ********]

Ouch. What I really need is some OTHER criterion by which this second term can be neglected.

Something to keep in mind in the Debye layer

$$\mathbf{J} = \sigma\mathbf{E} - D \text{ grad } \rho \ . \qquad \dim(D) = m^2/\text{sec} \qquad \text{(E.1)}$$

Now, go back to earlier statement

$$-j\omega\, n(\theta) = - J_r(a-\lambda_D,\theta) - jk\,[\sigma E_z(a,\theta)\,\lambda_D]$$

$$= - J_r(a-\lambda_D,\theta) - jk\,\lambda_d\, J_z(a,\theta)$$

$$= - J_r(a-\lambda_D,\theta) - jk\,\lambda_d\, J_z^D(a,\theta)$$

$$= - J_r(a-\lambda_D,\theta) - j\,k\, K_z^D(a,\theta)$$

This is the equation I really need to ponder! When is the inside radial current just below the surface much larger than the Debye current? I need some kind of handle or estimate for J_r which does NOT assume the CPBC.

Question. What do I know about divE ? [perhaps a digression not useful]

[I am trying to find a way to compare the size of the two terms in n(θ) above.]

Below the Debye layer there is no free charge, so there I claim that div **E** = 0. To the extent that E_θ is zero in this region (still relatively close to the surface), div **E** = 0 tells me that

$$\partial_r\,(r\,E_r(a-\lambda_D,\theta)) - jka\,E_z(a-\lambda_D,\theta) \approx 0 \qquad r \approx a \qquad \text{below Debye layer}$$

But in the Debye layer I claim that

$$\rho(r) = \rho(0)\, e^{-(a-r)/\lambda_D}$$

Because λ_d is so small, you can regard this as a 1D problem with the above result in terms of r. Then I guess we have this result then:

In the Debye layer just below the surface, we have [from divE = ρ/ε_0]

$$\partial_r\,(r\,E_r(a,\theta)) - jka\,E_z(a,\theta) \approx r\,\rho(0)\,e^{-(a-r)/\lambda_D}/\varepsilon_0 \approx a\,\rho(0)/\varepsilon_0 = a\,e\,n_e/\varepsilon_0 \quad [\ n_e = \text{net}\ ]$$

The RHS is in general a huge number! Stealing a result from below

λ_D e n_e/ε_0 = 10,000 volt/m

a e n_e/ε_0 = (a/λ_d) λ_D e n_e/ε_0 = (a/λ_d) 10,000 volt/m = perhaps 10^{12} volt/m

This seems ridiculous.

Dimension check: LHS = E = volt/m RHS = m * Cou * m^{-3} * m/far = volt/m

So we have shown that

∂_r (r $E_r(a,\theta)$) - jka $E_z(a,\theta) \approx$ a e n_e/ε_0

Now define

f ≡ - $J_r(a,\theta)$ -jk [σ $E_z(a,\theta)$ λ_D] [the two terms of interest]

which is the RHS of the "true" boundary condition, and I want to see when the second term may be neglected. Write

f ≡ - $J_r(a,\theta)$ -jk [σ $E_z(a,\theta)$ λ_D]

= - $J_r(a,\theta)$ -jk$\lambda_D\sigma$ $E_z(a,\theta)$

= - $J_r(a,\theta)$ + $\sigma(\lambda_D/a)$ [-jka$E_z(a,\theta)$]

= - $J_r(a,\theta)$ + $\sigma(\lambda_D/a)$ [e an_e/ε_0 - ∂_r (r $E_r(a,\theta)$)]

=σ { - $E_r(a,\theta)$ + (λ_D/a) [e a n_e/ε_0 - ∂_r (r $E_r(a,\theta)$)] }

so then

f/σ = - $E_r(a,\theta)$ + (λ_D/a) [e a n_e/ε_0 - ∂_r (r $E_r(a,\theta)$)]

and again I want to know when the second term can be neglected. I have managed so far to eliminate E_z from the question, but now I need to know SOMETHING about E_r, but without the App D solutions, I really don't know a thing about E_r ! I can see that if $E_r(a,\theta) \equiv 0$, then you can NOT neglect the second term. Looking at the Helm equation (D.1.17) is not too helpful because it is 2nd order in ∂_r. Write again

f/σ = - $E_r(a,\theta)$ + (λ_D/a) [a e n_e/ε_0] - (λ_D/a) ∂_r (r $E_r(a,\theta)$)

[nothing seems useful in the above really]

Plan A. Perhaps I can argue that $(\lambda_D/a) << 1$ so the last term can be neglected. Then we have

$f/\sigma = - E_r(a,\theta) + \lambda_D\, e\, n_e/\varepsilon_0$

How large is this second term? What are its dimensions?

$\dim(\lambda_D\, en_e/\varepsilon_0) = m * Cou * m^{-3} * m/farad = volt/m$

Here are some numbers for copper

$\lambda_D = 5.55 \times 10^{-11}$ m

$e = - 1.6 \times 10^{-19}$ Coul

$n_e = 10^{22}$ m^{-3} [net n_e inside the Debye layer]

$\varepsilon_0 = 8.8541877 \times 10^{-12}$ farad/m

Maple gives for $\lambda_D\, en_e/\varepsilon_0$:

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Ld := 5.55e-11;
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$$Ld := .555\ 10^{-10}$$

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e := - 1.6e-19;
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$$e := -.16\ 10^{-18}$$

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ne := 1e22;
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$$ne := .1\ 10^{23}$$

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e0 := 8.85e-12;
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$$e0 := .885\ 10^{-11}$$

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Ld*e*ne/e0;
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$$-10033.89831$$

So we then have

$f/\sigma = - E_r(a,\theta) + 10{,}000$ volts / m

An unexpected result. In this plan, the second term would always dominate, and you would never have the CPBC as presented in App D. [so something is not right here]

Plan B: Regroup. In the above I have come up with four distinct equations:

<u>Equation #1:</u>

$-j\omega\, n(\theta) = - J_r(a- \lambda_D,\theta) - jk\, [\sigma\, E_z(a,\theta)\, \lambda_D]$

$= - J_r(a- \lambda_D,\theta) - jk\, \lambda_d\, J_z(a,\theta)$

$= - J_r(a- \lambda_D,\theta) \ -jk \lambda_d J_z^D(a,\theta)$

$= - J_r(a- \lambda_D,\theta) \ -j\, k\, K_z^D(a,\theta)$ // involves J_r *below* Debye layer OK

Equation #2:

$\partial_r (r E_r(a- \lambda_D,\theta)) - jka E_z(a- \lambda_D,\theta) \approx 0$ $r \approx a$ E_r below Debye layer

Equation #3:

$\partial_r (r E_r(a,\theta)) - jka E_z(a,\theta) \approx a\, e\, n_e/\varepsilon_0 \sim 10^{12}$ E_r in the Debye layer

Equation #4:

$E_z(a- \lambda_D,\theta) = E_z(a,\theta)$

Since I am interested in Equation #1 as a modified CPBC, equation #3 is not very useful so perhaps just chuck that from the list of useful equations. I can rewrite #1 and #2 and #4 this way :

$-j\omega\, n(\theta) = - \sigma E_r(a- \lambda_D,\theta) \ -jk [\sigma E_z(a,\theta) \lambda_D]$ #1 + Ohms Law outside Debye layer

$\partial_r (r E_r(a- \lambda_D,\theta)) - jka E_z(a,\theta) \approx 0$ $r \approx a$ #2 and #4 [both OK]

At this point, one thing I *could* do (though probably not useful) is eliminate $E_z(a,\theta)$ to get

$-j\omega\, n(\theta) = - \sigma E_r(a- \lambda_D,\theta) \ - jka E_z(a,\theta) [\sigma (\lambda_D/a)]$

$= - \sigma E_r(a- \lambda_D,\theta) \ - \partial_r (r E_r(a- \lambda_D,\theta)) \ [\sigma (\lambda_D/a)]$

or

$-j\omega\, n(\theta)/\sigma = - E_r(a- \lambda_D,\theta) \ - (\lambda_D/a) \partial_r (rE_r(a- \lambda_D,\theta))$

or

$j\omega\, n(\theta)/\sigma = E_r(a- \lambda_D,\theta) \ + (\lambda_D/a) \partial_r (rE_r(a- \lambda_D,\theta))$

or

$j\omega\, n(\theta)/\sigma = E_r(a-\lambda_D,\theta) \ + (\lambda_D/a) E_r(a- \lambda_D,\theta) + (\lambda_D/a)\, a\ \partial_r E_r(a- \lambda_D,\theta)$

Now we know that $(\lambda_D/a) << 1$ for a practical transmission line a value, so this says

$j\omega\, n(\theta)/\sigma = E_r(a-\lambda_D,\theta) + \lambda_d\ \partial_r E_r(a- \lambda_D,\theta)$ [seems OK]

This is the first time in this doc I have stated this claimed equation. Only the E_r field below the Debye layer is involved here! I want to know the relative size of the two terms shown.

At high ω, I might guess this to be true below the Debye layer and toward wire center,

$$E_r(r,\theta) = E_r(a-\lambda_D,\theta)\sqrt{\frac{a}{r}}\; e^{(r-a)/\delta}$$

and then roughly we get

$$\partial_r E_r(a-\lambda_D,\theta) \approx (1/\delta)\, E_r(a-\lambda_D,\theta)$$

Then one gets

$$j\omega\, n(\theta)/\sigma = E_r(a-\lambda_D,\theta) + (\lambda_d/\delta)\, E_r(a-\lambda_D,\theta)$$

$$= [\,1 + (\lambda_d/\delta)\,]\, E_r(a-\lambda_D,\theta)$$

Even at 100 GHz we know that $(\lambda_d/\delta) \approx 1/4000$ and the first term dominates. Below this, the first term dominates even more, and this is how I have used Appendix D. So maybe this is a better argument than the one I now present in Section D (9) (b). [taking note of that]

This really says nothing about what happens at low ω [agreed] . If you just say $\delta \to \infty$ in the above formula as a representation of low ω, you get the same result as for high ω. Let's go back to

$$j\omega\, n(\theta)/\sigma = E_r(a-\lambda_D,\theta) + \lambda_d\, \partial_r E_r(a-\lambda_D,\theta)$$

radial pump Debye term

Another form

$$j\omega\, n(\theta) = J_r(a-\lambda_D,\theta) - jk\,[J_z(a,\theta)\,\lambda_D] \qquad \#1 + \text{Ohms Law outside Debye layer}$$

[so, to this point I have learned nothing new about these two terms]

Concern: Have I ever said i(z) = dq(z)/dt ? I hope not. In my notation, i(z) is the total current in a conductor, whereas q(z) is the charge per unit length all of which lies on the surface. So $\partial_t q(z)$ is only the current feeding the surface charge, not the total current! I will collect some evidence:

1. Below Fig 3.6a I say this:

$$i(z) = q(z)\, v_d, \qquad (4.11.19a)$$

where I am quoting from a future location in lines doc. Going there I see first

$$\partial_z\, i(z) = -j\omega q(z)\,. \qquad (4.11.19)$$

which seems to say that $\partial_z i(z,t) = -\partial_t q(z,t)$ assuming no dielectric current out box sides. This is supposed to come from

$$\text{div}\,\mathbf{J} = -j\omega\rho \quad \Leftrightarrow \quad -j\omega[\int_V \rho\, dV] = \int_S \mathbf{J} \bullet d\mathbf{S}\,. \qquad (1.1.35)$$

Why doesn't capacitor current appear here? Because symbol **J** is conduction current only! So Ok, if the total wire current i(z) is decreasing in z, it has to be going somewhere, and the only place is q(z). I think then that (4.11.19) is OK.

Now look at my larger context,

$$\partial_z\, i(z) = -j\omega q(z)\,. \qquad (4.11.19)$$

Jumping the gun again, if we again use $\partial_z \rightarrow -jk$ with k = (ω/v), we arrive at the intuitive relation

$$i(z) = q(z)\, v \qquad (4.11.19a)$$

which just says the charge per unit length is [in effect, see D.9 (c)] traveling down the line at phase velocity v. In the lossless case $v = v_d$ (dielectric speed of light), whereas more generally v is complex.

Now saying $k = \omega/v_\varphi$ can be regarded as a definition of the complex phase velocity v_φ. Then with that definition, we can say $\partial_z i(z) = -j\omega q(z) \Rightarrow -jk\, i(z) = -j\omega q(z) \Rightarrow k\, i(z) = \omega q(z) \Rightarrow i(z) = (\omega/k) q(z)$ and finally that $i(z) = v_\varphi\, q(z)$. This is just a definition. The more interesting claim is just that

$$i(z) = (\omega/k) q(z)$$

which is a relationship between the TOTAL current in the conductor to the surface charge. I don't think I have said anything wrong yet, and I have not said i(z) = dq(z)/dt so far.

[so this little "concern" turned up nothing in lines doc that is wrong]

Plan C. Let's go back to an earlier stage:

$$-j\omega\, n(\theta) = -J_r(a-\lambda_D,\theta) - jk\,[\sigma E_z(a,\theta)\,\lambda_D]$$

where I think the second term is the Debye current contribution. I want a solution where this second term "carries the whole load". Something like this for very low ω

$$-j\omega\, n(\theta) = -jk\,[\sigma E_z(a,\theta)\,\lambda_D]$$

Since n(θ) is a function of θ, how are you going to avoid having $E_z(a,\theta)$ depend on θ? Well. I was hoping that $J_z^D = \sigma E_z$ would depend on θ only in the Debye layer, and then below that layer I wanted to see $J_z = \sigma E_z$ being a constant in θ! But then how do I get E_z being continuous at the "Debye boundary"? Maybe my model of a hard boundary is no good.

[at least a new idea just presented: maybe you can have J_z be uniform inside the round wire, while at the same time the Debye current on the surface is non-uniform.]

(next day Aug 17, 2014)

Plan D. Have been working on scratch this AM. Here is a possible logic thread. At high ω, strong skin effect, current constrained to sheath, Z_0 is low and some constant value. Then i(z) = V(z)/Z_0 = some finite value. Perhaps V = 10 volts and Z_0 = 75 ohms so i(z) = 1/7 amp. The sheath is δ thick, so for symmetric round wire you could say that 2πa δ J_z = i(z), which says J_z(sheath) = i(z)/[2πaδ]. For some large ω there is some small δ and there is then some J_z(sheath) value. You cannot have J_z(sheath) be 10 times this value! The Debye surface current is swamped by the δ sheath current as I show somewhere, so we can ignore the Debye layer. Perhaps v_φ = c, speed of light. Now with all this background, we ask if n(θ) can just move along the Debye layer and no J_r is necessary to make n(θ) "go". Then $J_z = n_e e v_\varphi = n_e e c$ in order to make that work. [Why do I put phase velocity into the J = nev formula, that seems wrong. It should be a drift velocity. I think therefore Plan D is no good.] What is this value? $J_z = n_e e c = 10^{22} * 1.6 \times 10^{-19} * 3\times10^{8} = 5 \times 10^{11}$ amps/m^2. The required E_z field to do this is then $E_z = J_z/\sigma = 10^{12}/10^{8} = 10^{4}$ volts/m. But our limit is J_z(sheath) = i(z)/[2πaδ] which for a = Belden = 10^{-3} m and δ = 0.21 x 10^{-6} at 100 GHz so then

$$J_z(\text{sheath}) = i(z)/[2\pi a\delta] = (1/7) / [\, 6 \times 10^{-3} \times 0.21 \times 10^{-6} \,] = 10^9/8 \sim 10^8 \text{ amp/m}^2$$

Since J_z(sheath) = 10^8 amp/m^2 and $E_z = J_z/\sigma = 10^8/10^8 = 1$ volt/m, our attempt to make drift velocity be "c" would require $J_z = 10^{12}$ amp/m^2 and $E_z = 10^4$ volt/m, so that is why you cannot "feed n(θ)" by the Debye mechanism at 100 GHz.

So why could you maybe "feed n(θ)" by Debye at low ω? In this case, perhaps $J_z = i(z)/(\pi a^2)$ and there is some corresponding E_z in the wire. Now $J_z = n_e e v_\varphi$ is small because v_φ is very small. In this case, we have $Z_0 = \sqrt{R/2C}\ 1/\sqrt{\omega}\ [\,1\text{-}j\,] = \sqrt{R/C}\ 1/\sqrt{\omega}\ e^{-j\pi/4}$ from App Q. Then $i(z) = V(z)/Z_0 \sim V(z)\sqrt{\omega}\sqrt{C/R}\ e^{+j\pi/4}$ and this is getting very small. So then $J_z = i(z)/(\pi a^2) = (1/\pi a^2)\, V(z)\sqrt{\omega}\ \sqrt{C/R}\ e^{+j\pi/4}$ which is getting small as $\sqrt{\omega}$. The required Debye solution would need $J_z = n_e e v_\varphi$ with $v_\varphi = \sqrt{\frac{2}{RC}}\ \omega^{1/2}$ [same error as above I think]according to my last doc, so we then have $J_z = n_e e v_\varphi = n_e e \sqrt{\omega}/\sqrt{RC/2}$. If we were to equate these values

$J_z = n_e e v_\varphi = n_e e \sqrt{\omega}/\sqrt{RC/2}$ required Debye value of J_z^D

$J_z = i(z)/(\pi a^2) = (1/\pi a^2)\, V(z)\sqrt{\omega}\ \sqrt{C/R}\ e^{+j\pi/4}$ value determined by uniform J_z assumption

These have the same ω dependence, which is encouraging. But to be equal we would need

$$n_e e \sqrt{\omega}/\sqrt{RC/2} = (1/\pi a^2)\, V(z)\sqrt{\omega}\ \sqrt{C/R}\ e^{+j\pi/4}$$

or

$$n_e e/\sqrt{RC/2} = (1/\pi a^2)\, V(z)\ \sqrt{C/R}\ e^{+j\pi/4}$$

or

$$n_e e = (1/\pi a^2)\, V(z)\ \sqrt{C/R}\ e^{+j\pi/4}\sqrt{RC/2} = (1/\pi a^2)\, V(z)\, C\, e^{+j\pi/4}/\sqrt{2} = (1/\pi a^2)\, q(z)\, e^{+j\pi/4}/\sqrt{2}$$

LHS = Coul/m^3 RHS = m^{-2} * volt * far/m = Coul/m^3 OK after 8 fixes

So what does it mean to require that $n_e e = (1/\pi a^2)\, q(z)\, e^{+j\pi/4}/\sqrt{2}$?

$q(z) = (\pi a^2)\, n_e e \sqrt{2}\; e^{-j\pi/4}$ — $n_e e$ = total 3D free charge density in the Debye layer; $n(\theta) = n_e e\, \lambda_D$

$q(z) = (2\pi a)\, a\, n_e e\, (1/\sqrt{2})\, e^{-j\pi/4}$

$q(z) = (2\pi a)\, (a/\lambda_d)\, n(\theta)\, (1/\sqrt{2})\, e^{-j\pi/4}$

$q(z) = q(z)\, (a/\lambda_d)\, (1/\sqrt{2})\, e^{-j\pi/4}$ — hmmmm.. problem here

Well, I am playing in the right city at least, but things just remain fuzzy all the time, now noon.

[so in Plan D I wonder if at low ω the Debye layer could carry all the current, but I reach no conclusion]

Comments (9.10.14)

I read all the above stuff today, it is a lot of failing around. Since n_e is always small compared to the large number of carrier electrons, maybe the surface charge has no effect at all on the total J_z current. If E_z is the same in the Debye later as below the Debye layer, J_z will be about the same in both regions. This would be true at any ω. OK, but this does not answer the question of how the z directed Debye current contributes to n(θ) as compared to the radial J_r current. So everything is inclusive.