Show that linear combinations solve Appendix D PhL 9.16.14

(1) I show that you can superpose solutions of different k, discretely or continuously.
(2) I do the Smythian Form analysis in Part C and clarify the m=0 case in Part D.
(3) I show that J_z asym does not go away if you superpose.
(4) Much of this stuff is quoted now in Section 7. 2
(5) I never resolved the paradox regarding superposition mentioned in "low freq discussion".
Reviewed on 10.8.14.

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I thought this would be obvious, but after fiddling a bit, it seems less obvious.

Preliminary Question:

If you consider the single-k solution and the Helm equation and the div E = 0 equation and the two boundary conditions, does this system determine the SCALE of the solutions or not?

Answer: Here is the App D solution where I is replaced by $2\pi\omega\,(a/k)\,N_0$

$$E_z(r,m) = (1/4)\,\eta_m\,2\pi\omega\,(a/k)\,N_0\,R_{dc}\,(a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad (**)$$

$$E_r(r,m) = (j/4)\,\eta_m\,2\pi\omega\,(a/k)\,N_0\,R_{dc}\,(ak)\,g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\,2\pi\omega\,(a/k)\,N_0\,R_{dc}\,(ak)\,h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

We know that $N_0 = q/(2\pi a)$ and $q = CV$ so you can replace $N_0 = [C/2\pi a]\,V$. Then each E_i is proportional to V. Since you can apply any V you want to a TL, you can have any scale you want for the solution set.

Alternative Answer: Here is the App D solution in its usual form,

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

We know that $I = V/Z_0$ where $Z_0 = \sqrt{(R+j\omega L)/(G+j\omega C)}$. Once again, we end up with each E_i being proportional to V, and so yes we have that same scaling idea.

Comparing the two Forms

It must be true that $I = 2\pi\omega\,(a/k)\,N_0 = 2\pi a\,(\omega/k)\,[C/2\pi a]\,V = V/Z_0$. Then it must be true that

$$2\pi a\,(\omega/k)\,[C/2\pi a] = 1/Z_0$$

or

$$(\omega/k)\,[C] = 1/Z_0$$

or

$$k/Z_0 = \omega C$$

Is this true? We can install k and Z_0 to get

$$k/Z_0 = -j\sqrt{(R+j\omega L)(G+j\omega C)} * \sqrt{(G+j\omega C)/(R+j\omega L)} = -j\,(G+j\omega C)$$

But in App D for the CP BC I assume that G = 0, so then

$$k/Z_0 = -j\,(0+j\omega C) = -j^2\omega C = \omega C$$

and so the two forms do in fact compare OK. I show in separate paradox doc that if you allow G > 0, then you have a new $N_0(\text{new}) = N_0(\text{old}) * (\xi_d/\varepsilon_d)$ requiring that $k/Z_0 = \omega C'$ which is also true, just another internal check of the theory that works!

Conclusion: Solution E_i are proportional to V, and you could double V to double all solutions.

Part A: First shot at the superposition of k idea (discrete sum of two solutions)

Here I show that you can superpose two solutions of Helm to get a new Helm solution. This is done in θ space, so there is no m label on the coefficients C(k,ω) and C(k',ω). I then treat the n(θ) boundary condition incorrectly which gives the incorrect requirement that C(k,ω) + C(k',ω) = 1. In m space, the superposition has θ replaced by m, but the coefficients of course do not depend on m since they are the same coefficients used in θ space.

1. Suppose I know that

$(\nabla^2 + \beta^2)\ \mathbf{E}(r,\theta,z,t;\ k) = 0$ single-k solution for k

$(\nabla^2 + \beta^2)\ \mathbf{E}(r,\theta,z,t;\ k') = 0$ single-k solution for k'

Then create this linear combination solution

$\mathbf{E}(r,\theta,z,t)^{(3)} = C(k,\omega)\mathbf{E}(r,\theta,z,t;\ k) + C(k',\omega)\ \mathbf{E}(r,\theta,z,t;\ k')$ // dim(C) = 1 for discrete sum

We then find that

$(\nabla^2 + \beta^2)\ \mathbf{E}(r,\theta,z,t)^{(3)} = C(k)\ (\nabla^2 + \beta^2)\ \mathbf{E}(r,\theta,z,t;\ k) + C(k')\ (\nabla^2 + \beta^2)\ \mathbf{E}(r,\theta,z,t;\ k')$

$= C(k,\omega)\ 0 + C(k',\omega)\ 0 = 0$

Thus, our linear combination is also a solution of the full vector Helmholtz equation. The two simple solutions have these forms,

$\mathbf{E}(r,\theta,z,t;\ k) = e^{j(\omega t - kz)}\ \mathbf{E}(r,\theta;\ k)$

$\mathbf{E}(r,\theta,z,t;\ k') = e^{j(\omega t - k'z)}\ \mathbf{E}(r,\theta;\ k')$

and then

$\mathbf{E}(r,\theta,z,t)^{(3)} = C(k,\omega)\ e^{j(\omega t - kz)}\ \mathbf{E}(r,\theta;\ k) + C(k',\omega)\ e^{j(\omega t - k'z)}\ \mathbf{E}(r,\theta;\ k')$

2. Go to partial waves: I think each of the three functions

$\mathbf{E}(r,\theta,z,t;\ k)$ $\mathbf{E}(r,\theta,z,t;\ k')$ $\mathbf{E}(r,\theta,z,t)^{(3)}$

goes to partial waves in the obvious manner. Thus we end up with

$\mathbf{E}(r,m,z,t)^{(3)} = C(k,\omega)\ \mathbf{E}(r,m,z,t;\ k) + C(k',\omega)\ \mathbf{E}(r,m,z,t;\ k')$.

Notice that the coefficients *do not* depend on m, they just come through from the θ-space forms.

3. What happens to the **boundary conditions**? The new boundary conditions must be in θ space

$E_r(r=a-\varepsilon,\theta)^{(3)} = (j\omega/\sigma)\ n(\theta)$

$E_\theta(r=a,\theta)^{(3)} = 0$

But if we write out the expansions, we end up with

$C(k,\omega)\ E_r(r=a,\theta,z,t;\ k) + C(k',\omega)\ E_r(r=a,\theta,z,t;\ k') = (j\omega/\sigma)\ n(\theta)$

$C(k,\omega)\ E_\theta(r=a,\theta,z,t;\ k) + C(k',\omega)\ E_\theta(r=a,\theta,z,t;\ k') = 0$

The above is wrong because we really need some $n(\theta,z) = n(\theta)\, r(z)$ on the right.
I do this better below. That is, for a single-k you could cancel e^{-jkz} on both dies, but not for mult k.

Note: the fields all depend on ω as well, I just don't display that fact.

We can prove the second one by assuming that the single-k solutions each satisty the BC, but the first BC is unclear! I think n(θ) is determined by the electrostatic problem (especially at very low ω), so you cannot adjust n(θ). We then end up assuming that each of the single-k solutions satisfies the same BC

$$E_r(r=a,\theta,z,t;\, k) = (j\omega/\sigma)\, n(\theta)$$
$$E_r(r=a,\theta,z,t;\, k') = (j\omega/\sigma)\, n(\theta)$$

then we are forced to have

$$E_r(r=a-\varepsilon,\theta)^{(3)} = C(k,\omega)\, E_r(r=a,\theta,z,t;\, k) + C(k',\omega)\, E_r(r=a,\theta,z,t;\, k')$$
$$= C(k,\omega)\, (j\omega/\sigma)\, n(\theta) + C(k',\omega)\, (j\omega/\sigma)\, n(\theta)$$
$$= [\, C(k,\omega) + C(k',\omega)]\, (j\omega/\sigma)\, n(\theta)$$

The only way to rescue things is to require that

$$C(k,\omega) + C(k',\omega) = 1$$ wrong for reason shown in red above

If I insist on this restriction, then $\mathbf{E}(r,m,z,t)^{(3)}$ does indeed satisfy both boundary conditions as well as the vector Helmholtz equation!

I think the derived B fields would be the obvious linear combinations, and the verification of Maxwell's equations would then go through as before.

4. What happens to quantity I when there is a linear combination solution? The simple form $N_0 = (k/2\pi\omega a)$ no longer applies. So rather than updating I right now, let's just not use I in our solution forms for the single-k solutions. That is to say, replace I by $2\pi\omega\, (a/k)\, N_0$ in those solutions, which then read:

$$E_z(r,m) = (1/4)\, \eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$
$$E_r(r,m) = (j/4)\, \eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$
$$E_\theta(r,m) = (1/4)\, \eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (ak)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Part B: Generalization from 2-term superposition to continuous superposition

Here I just state the candidate continuous superposition called $\mathbf{E}^{(3)}$. This is an explicit integral with arbitrary coefficient C(k,ω). As happened in Part A, in going from θ to m space, the C(k,ω) continue to have no m index. I then show that the double expansion $\Sigma_m \int dk$ for $E_z(r,\theta,z)^{(3)}$ has the exact Smythian atomic superposition form you would expect with a certain specific $c_m(k,\omega)$ = [stuff(m)] C(k.ω).

5. **Generalization** of the results: Here we must mimic the above sum of two terms to get:

$$\mathbf{E}(r,\theta,z,t)^{(3)} = \int dk\, C(k,\omega)\, \mathbf{E}(r,\theta,z,t;\, k) = \int dk\, C(k,\omega)\, e^{j(\omega t - kz)}\ \mathbf{E}(r,\theta;\, k)$$

$$\mathbf{E}(r,m,z,t)^{(3)} = \int dk\, C(k,\omega)\, \mathbf{E}(r,m,z,t;\, k) = \int dk\, C(k,\omega)\, e^{j(\omega t - kz)}\ \mathbf{E}(r,m;\, k)$$

$\int dk\, C(k,\omega) = 1$ in order to meet the CP BC. this line wrong, ignore it

Note: Now dim(C) = m, since we require dim[dk C] = 1.

More specifically [note that I is not used] using boxed result (**) above

$$E_z(r,m,z,t)^{(3)} = \int dk\, C(k,\omega) \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] e^{j(\omega t - kz)}\ (1/4)\, \eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\ (a\beta')\, J_m(\beta' r)$$

or

$$E_z(r,m,z)^{(3)} = (1/4)\, N_m\, 2\pi\omega a^2\, R_{dc} \int dk\, C(k,\omega) \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] e^{-jkz}\, (\beta'/k)\, J_m(\beta' r)$$

$$= \int dk\, c_m(k,\omega)\, e^{-jkz}\, J_m(\beta' r)$$

Added: dim [(1/4) N_m $2\pi\omega a^2$ R_{dc}] = Cou/m^2 m^2/sec ohm/m = amp-ohm/m = volt/m OK

and going back to θ space

$$E_z(r,\theta,z)^{(3)} = \Sigma_m \int dk\, c_m(k,\omega)\, \{\, e^{-jkz}\, J_m(\beta' r)\, e^{jm\theta}\,\}$$

Note Added: dim[dk c_m] = (1/m) dim[c_m] = V/m so must have dim[c_m] = V.

which does have my general Smythian structure with m and k dependence of the coefficient. And

$$c_m(k,\omega) = (1/4)\, N_m\, 2\pi\omega a^2\, R_{dc} \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] (\beta'/k)\ ]\ C(k,\omega)$$

$$= (1/2)\, N_m\, (\omega/\sigma) \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] (\beta'/k)\, C(k,\omega)$$

Dimensional analysis: We know that dim(c_m) = V. Look now at the RHS and see if OK or not OK:

dim(RHS) = Cou/m^2 * sec^{-1} ohm-m * m = Cou * sec^{-1} ohm = amp-ohm = volt OK

Part C: Smythian Approach to the Continuous Superposition

As an alternate approach, here I start with the Smythian form with unknown coefficient $d_m(k,\omega)$ and I try to determine that coefficient. I apply the Helm equations to this Smythian form (for $m \neq 0$) and I show that one must have $d_m(k,\omega) = C_{zm} = (1/2j)(\beta'/k)\, K_m$ where K_m has to be determined from the BC's. A new feature is that the CP BC now has this form

$$E_r(r=a-\varepsilon,\theta,z)^{(3)} = (j\omega/\sigma)\, n(\theta,z)$$

where

$$n(\theta,z) = n(\theta)r(z)$$

and r_k is the FT of r(z) (in the single-k world, we could just cancel e^{-jkz} on both sides). Dealing then with the two BC's I find this new value for K_m (which has an extra factor r_k)

$$K_m = 2\,(j\omega/2\sigma)\, N_m\, r_k \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \qquad \text{// } r_k \text{ is the new feature (new factor)}$$

so

$$d_m(k,\omega) = (1/2j)(\beta'/k)\, K_m = (1/2j)(\beta'/k)\, 2\,(j\omega/2\sigma)\, N_m\, r_k \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$= (1/2)(\beta'/k)(\omega/\sigma)\, N_m\, r_k \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

So the superposition coefficient r_k is directly related to the z shape of the surface charge along the line. Setting d_m here to c_m of the previous section, I find that $C(k,\omega) = r_k$ and I show this earlier as well. Thus, we now have an interpretation of the arbitrary coefficient $C(k,\omega)$

5a. Most General Form for Appendix D.

Now suppose *from the very start* that we tried this as a most general form for $E_z(r,\theta,z)^{(3)}$ $d_m(k,\omega)$ and I try to determine that coefficient. [since dim(dk) = m^{-1}, we must have dim[d_m] = volts]

$$E_z(r,\theta,z)^{(3)} = \Sigma_m \int dk\, d_m(k,\omega)\, \{\, e^{-jkz}\, J_m(\beta' r)\, e^{jm\theta}\,\}$$

where $d_m(k,\omega)$ is completely arbitrary. Since $\{\, e^{-jkz}\, J_m(\beta' r) e^{jm\theta}\,\}$ is the proper atomic form for the scalar E_z Helmholtz equation, I would argue that this expression for $E_z(r,m,z)^{(3)}$ is *the most general form possible* for a solution of this particular scalar Helmholtz equation. [agreed]

What is the m-space version of this equation? Can write

$$\Sigma_m \, E_z(r,m,z)^{(3)} \, e^{jm\theta} = E_z(r,\theta,z)^{(3)}$$

from which I would conclude that

$$E_z(r,m,z)^{(3)} = \int dk \, d_m(k,\omega) \, e^{-jkz} \, J_m(\beta' r)$$

If we track this through the Appendix D steps, what happens? I have no "most general form" for the other two components. Here are the four equations

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \partial_z^2 + \beta^2\ ]\, E_z(r,\theta,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_r(r,\theta,z)^{(3)} - (2/r^2)\,\partial_\theta E_\theta(r,\theta,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_\theta(r,\theta,z)^{(3)} + (2/r^2)\,\partial_\theta E_r(r,\theta,z)^{(3)} = 0$$

$$\partial_r\,(r\,E_r(r,\theta,z)^{(3)}) + \partial_\theta\, E_\theta(r,\theta,z)^{(3)} + r\,\partial_z\, E_z(r,\theta,z)^{(3)} = 0$$

I can surely convert all this to m-space to get

$$[\partial_r^2 + (1/r)\,\partial_r - (1/r^2)\,m^2 + \partial_z^2 + \beta^2\ ]\, E_z(r,m,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_r(r,m,z)^{(3)} - (2/r^2)\,jmE_\theta(r,m,z)^{(3)} = 0 \qquad \text{/ 2nd eq}$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_\theta(r,m,z)^{(3)} + (2/r^2)\,jmE_r(r,m,z)^{(3)} = 0$$

$$\partial_r\,(r\,E_r(r,m,z)^{(3)}) + jm\,E_\theta(r,m,z)^{(3)} + r\,\partial_z\,E_z(r,m,z)^{(3)} = 0$$

I would then rewrite the last equation as

$$-jm\,E_\theta(r,m,z)^{(3)} = \partial_r\,(r\,E_r(r,m,z)^{(3)}) + r\,\partial_z\,E_z(r,m,z)^{(3)}$$

and then I could sub this into the 2nd equation to get

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_r(r,m,z)^{(3)} + (2/r^2)\,[\partial_r\,(r\,E_r(r,m,z)^{(3)}) + r\,\partial_z\,E_z(r,m,z)^{(3)}] = 0$$

which can be rewritten,

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2 + (2/r^2)\,[\partial_r\,(r\,..\,)]\, E_r(r,m,z)^{(3)} = -(2/r)\,\partial_z\,E_z(r,m,z)^{(3)}$$

or

$$[r^2\partial_r^2 + r\partial_r - m^2 + r^2\partial_z^2 - 1 + \beta^2 r^2 + (2)\,[\partial_r\,(r\,..\,)]\,]\, E_r(r,m,z)^{(3)} = -2r\,\partial_z\,E_z(r,m,z)^{(3)}$$

or

$$[r^2\partial_r^2 + r\partial_r -(m^2+1) + r^2\partial_z^2 + \beta^2r^2 + (2)(r\partial_r + 1)]\, E_r(r,m,z)^{(3)} = \ldots$$

or

$$[r^2\partial_r^2 + r\partial_r -(m^2+1) + r^2\partial_z^2 + \beta^2r^2 + 2r\partial_r + 2]\, E_r(r,m,z)^{(3)}$$

or

$$[r^2\partial_r^2 + 3r\partial_r -(m^2-1) + r^2\partial_z^2 + \beta^2r^2]\, E_r(r,m,z)^{(3)}$$

or

$$[r^2\partial_r^2 + 3r\partial_r +(1-m^2) + r^2\partial_z^2 + \beta^2r^2]\, E_r(r,m,z)^{(3)} = -2r\,\partial_z\, E_z(r,m,z)^{(3)}$$

I could at this point install the most general form for E_z to get

$$[r^2\partial_r^2 + 3r\partial_r +(1-m^2) + r^2\partial_z^2 + \beta^2r^2]\, E_r(r,m,z)^{(3)}$$

$$= -2r\,\partial_z \int dk\, d_m(k,\omega)\, e^{-jkz}\, J_m(\beta' r)$$

$$= -2r \int dk\, d_m(k,\omega)\, (-jk)\, e^{-jkz}\, J_m(\beta' r)$$

$$= +2j\, r \int dk\, d_m(k,\omega)\, k\, e^{-jkz}\, J_m(\beta' r)$$

Now of course we want to solve this for $E_r(r,m,z)^{(3)}$. It seems that just to have a chance in terms of z, you would have to make this assumption about $E_r(r,m,z)^{(3)}$

$$E_r(r,m,z)^{(3)} = \int dk\, e^{-jkz}\, F_m(r)$$

where we then want to solve for $F_m(r)$. OK, if we put this in, the above equation reads

$$[r^2\partial_r^2 + 3r\partial_r +(1-m^2) + r^2\partial_z^2 + \beta^2r^2] \int dk\, e^{-jkz}\, F_m(r) = +2j\, r \int dk\, d_m(k,\omega)\, k\, e^{-jkz}\, J_m(\beta' r)$$

$$\int dk\, e^{-jkz}\, [r^2\partial_r^2 + 3r\partial_r +(1-m^2) + r^2(-k^2) + \beta^2r^2]\, F_m(r) = \int dk\, e^{-jkz}\; 2j\, r\, d_m(k,\omega)\, k\, J_m(\beta' r)$$

This would then require, using completeness of the e^{-jkz}, that

$$[r^2\partial_r^2 + 3r\partial_r +(1-m^2) + \beta'^2r^2]\, F_m(r) = 2j\, r\, d_m(k,\omega)\, k\, J_m(\beta' r)$$

This is the same as (D.2.7) if I set $d_m(k,\omega) = C_{zm}$. I can then make use of Appendix D to solve this equation for $F_m(r)$ and the answer will be

$$F_m(r) = a_m\, x^{-1}\, J_m(\beta' r) + \frac{K_m}{2}\, J_{m+1}(\beta' r) \qquad \text{// but don't yet know constants } a_m \text{ and } K_m$$

At this point I then know that

$$E_r(r,m,z)^{(3)} = \int dk\, e^{-jkz}\, F_m(r)$$

$$= \int dk\, e^{-jkz} \left[a_m\, x^{-1}\, J_m(\beta' r) + \frac{K_m}{2}\, J_{m+1}(\beta' r)\right]$$

[Note: since dim(dk) = m^{-1}, the a_m and K_m here will differ in dimensions from App D coeffs.]

I think the same thing will happen when I solve for E_θ, so the answer is going to be

First summary of the E field solutions (D.2.21)

$$E_z(r,m,z)^{(3)} = \int dk\, e^{-jkz} \left[-j\,(\beta'/k)\,\frac{K_m}{2}\, J_m(x)\right] \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(r,m,z)^{(3)} = \int dk\, e^{-jkz} \left[a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x)\right] \qquad \beta'^2 = \beta^2 - k^2 \qquad \textit{(D.2.11)}$$

$$jE_\theta(r,m,z)^{(3)} = \int dk\, e^{-jkz} \left[-a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)\right] \qquad \textit{(D.2.15)}$$

where

$$d_m(k,\omega) = C_{zm} = (1/2j)(\beta'/k)\, K_m \quad .$$

Note however that coefficients a_m and K_m are not yet determined because we have not yet considered the boundary conditions. So the above solutions are like (D.2.21) "first summary"

So let's look at the BC's. Things are suddenly very different now because of the z dependence. I guess we have to say,

$E_r(r=a-\varepsilon,\theta,z)^{(3)} = (j\omega/\sigma)\, n(\theta,z)$ // new idea of n(θ,z)

$E_r(r=a-\varepsilon,m,z)^{(3)} = (j\omega/\sigma)\, N_m(z)$ // dim[n(θ,z)] = dim[N_m(z)] = Cou/m^2

Dim checks: dim(n) = Cou/m^2 and dim(ω/σ) = $\sec^{-1}$ m-ohm so
dim [(jω/σ) n(θ,z)] = $\sec^{-1}$ m-ohm * Cou/m^2 = ohm amp /m = volt/m so OK.

This then reads,

$$\int dk\, e^{-jkz} \left[a_m\, x^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a)\right] = (j\omega/\sigma)\, N_m(z)$$

In order to have a chance, you have to be able to write

$$N_m(z) = \int dk\, e^{-jkz}\, N_m(k)$$

and then you find that

$$a_m\, x^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a) = (j\omega/\sigma)\, N_m(k) \quad .$$

The other boundary condition says $jE_\theta(r=a,m,z)^{(3)} = 0$ which then gives

$$-\, a_m\, x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right)\, J_{m+1}(x_a) = 0 \quad .$$

But these last two equations are just (1) and (2) after (D.2.27) where $N_m \rightarrow N_m(k)$. We then get (D.2.28) where N_m has this new meaning, whatever it is. So if we solve the above two equations for the coefficients, we will get (D.2.28) with $N_m \rightarrow N_m(k)$ [which below we see is $N_m\, r_k$]

Pause. How do we relate this all back to the electrostatic n(θ) business? Maybe write

$n(\theta,z) = n(\theta)\, r(z)$ // dim r(z) = 1 Now just go to m-space to get:

$N_m(z) = N_m\, r(z)$ // $\dim[N_m] = \dim[N_m(z)] = Cou/m^2$

and then for any z, we at least know the θ dependence of n(θ,z) is the electrostatic value. I guess this is an ansatz of sorts, but for a given z slice, it has to be true Then

$N_m(z) = N_m\, r(z)$ r(z,ω) to be general // dims OK

Now let $N_m(k)$ be the standard FT of $N_m(z)$,

$N_m(z) = \int dk\, e^{-jkz}\, N_m(k)$ dim $N_m(k)$ = Cou/m k like t

$N_m(k) = (1/2\pi) \int dz\, e^{+jkz}\, N_m(z)$ // consistent with (1.6.8) z like ω

$= (1/2\pi) \int dz\, e^{+jkz}\, N_m\, r(z)$

$= N_m\, [(1/2\pi) \int dz\, e^{+jkz}\, r(z)\,]$

$= N_m\, r_k$ // dim r_k = m . // $Cou/m = Cou/m^2 * m$ OK

Of course I don't know r(z) or r_k . The App D equations apply, but I have to do this:

$$N_m \rightarrow N_m(k) = N_m r_k = N_m/N_0\ N_0\ r_k = \eta_m N_0 r_k$$

$$\eta_m = N_m / N_0 = N_m(k)/ N_0(k) = \text{unchanged}$$

In particular, with this change we get

$$a_m = (j\omega/2\sigma)\ 2m\ N_m \frac{1}{J_{m-1}(x_a)}\ r_k$$

$$\frac{K_m}{2} = (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] r_k$$

This in turn tells us that

$$d_m(k,\omega) = C_{zm} = (1/2j)(\beta'/k)\ K_m$$

$$= (1/2j)(\beta'/k)\ (j\omega/\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] r_k$$

$$= (1/2)(\beta'/k)\ (\omega/\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] r_k \qquad \text{// no factor of a}$$

Bug? The first line, $N_m \rightarrow N_m(k) = N_m r_k$ replaces something that was Cou/m^2 with something that is Cou/m. Is that OK? That will change the dims of a_m and K_m !

OK, then

$$E_z(r,m,z)^{(3)} = \int dk\ e^{-jkz}\ (1/4)\ \eta_m\ 2\pi\omega\ (a/k)\ N_0\ r_k\ R_{dc}\ (a\beta')\ f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,m,z)^{(3)} = \int dk\ e^{-jkz}\ (j/4)\ \eta_m\ 2\pi\omega\ (a/k)\ N_0\ r_k\ R_{dc}\ (ak)\ g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m,z)^{(3)} = \int dk\ e^{-jkz}\ (1/4)\ \eta_m\ 2\pi\omega\ (a/k)\ N_0\ r_k\ R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Bug? How can you take the original single-k E_i expression and multiply by r_k which has dim = m and still end up with dimE_z = volts/m ? The reason is that you have also added dk which has dim = m^{-1}.

But now we are stuck with this new unknown r_k . So I guess just think of it as what I used to call C(k), a generic weighting function (apart from a constant, see below). So we now have

$$E_i(r,m,z)^{(3)} = \int dk\ e^{-jkz}\ C(k,\omega)\ E_i(r,m)$$

and I am then back to my original method (but without the condition $\int$ dk C(k,ω) = 1). Comparing to the above for E_z says

$C(k,\omega) = r_k$ // both have dimensions of meters

Fact: If you assume the most general form for $E_z(r,m,z)^{(3)}$

$$E_z(r,m,z)^{(3)} = \int dk\, d_m(k,\omega)\, e^{-jkz}\, J_m(\beta' r)$$

then in order to solve the Helm equations and the BC's, you end up with just linear combinations of the App D E_i form for all three components

$$E_i(r,m,z)^{(3)} = \int dk\, e^{-jkz}\, r(k,\omega)\, E_i(r,m)$$

where the $E_i(r,m)$ are the exact Appendix D solutions. The CP BC is then

$$E_r(r=a-\varepsilon,m,z)^{(3)} = (j\omega/\sigma)\, N_m(z) = (j\omega/\sigma)\, N_m\, r(z)$$

$$E_r(r=a-\varepsilon,\theta,z)^{(3)} = (j\omega/\sigma)\, n(\theta)\, r(z) \qquad n(\theta,z) = n(\theta)\, r(z)$$

where r(z) is the way charge density varies in z as this disturbance moves on the transmission line.

$$r(z) = \int dk\, e^{-jkz}\, r(k) = \int dk\, e^{-jkz}\, C(k,\omega)$$

As before, we would argue that at large ω we have

$C(k,\omega) \rightarrow \delta(k - k_1)$ k_1 = the usual function

in which case

$$r(z) = e^{-jk1z}$$

and we are back to the regular Appendix D stuff.

Conclusion: I think k superposition of Appendix D is reasonable.

Added question: How is r_k related to C(k,ω) of the previous section?

Answer: In the above I show that

$$d_m(k,\omega) = (1/2j)(\beta'/k)\, K_m^{new} = (1/2j)(\beta'/k)\, 2\, (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] r_k$$

$$= (1/2)(\beta'/k)(\omega/\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] r_k$$

But in the previous section I showed that

$$c_m(k,\omega) = (1/2)\, N_m\, (\omega/\sigma) \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] (\beta'/k)\, C(k,\omega)$$

But we must have $d_m = c_m$ since both expansions are the same, which requires,

$$(1/2)\, N_m\, (\omega/\sigma) \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] (\beta'/k)\, C(k,\omega)$$

$$= (1/2)(\beta'/k)(\omega/\sigma)\, N_m\, r_k \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

or

$$C(k,\omega) = r_k$$

in agreement with earlier. review is OK to here.

Part D: Repeat Part C with focus on the m = 0 partial wave

I was concerned that something might be wrong with my m = 0 App D analysis, so here I do it all as a special case. The result is that the App D m=0 results are correct. So Part C is really valid for all m. I do the single-k case and the continuous superposition as well here.

I just copy paste and edit from section 5a as far as is possible. A few lines down for m = 0 says

$$E_z(r,m,z)^{(3)} = \int dk\, d_m(k,\omega)\, e^{-jkz}\, J_m(\beta' r)$$

so

$$E_z(r,0,z)^{(3)} = \int dk\, d_0(k,\omega)\, e^{-jkz}\, J_0(\beta' r)$$

I then write out the 4 equations in config space, and I then convert to m space to get

$$[\partial_r^2 + (1/r)\,\partial_r - (1/r^2)\, m^2 + \partial_z^2 + \beta^2\ ]\, E_z(r,m,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_r(r,m,z)^{(3)} - (2/r^2)\, jmE_\theta(r,m,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r - (1/r^2)m^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_\theta(r,m,z)^{(3)} + (2/r^2)\, jmE_r(r,m,z)^{(3)} = 0$$

$$\partial_r (r\, E_r(r,m,z)^{(3)}) + jm\, E_\theta(r,m,z)^{(3)} + r\, \partial_z\, E_z(r,m,z)^{(3)} = 0$$

Now set m = 0 everywhere to get

$$[\partial_r^2 + (1/r)\, \partial_r + \partial_z^2 + \beta^2\ ]\, E_z(r,0,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r + \partial_z^2 - (1/r^2) + \beta^2]\, E_r(r,0,z)^{(3)} = 0$$

$$[\partial_r^2 + (1/r)\partial_r + \partial_z^2 - (1/r^2) + \beta^2]\, E_\theta(r,0,z)^{(3)} = 0$$

$$\partial_r (r\, E_r(r,0,z)^{(3)}) + r\, \partial_z\, E_z(r,0,z)^{(3)} = 0$$

Hmm! Very different from $m \neq 0$. The Helm equations are all separated when m = 0. Let's go ahead and use the ansatz of e^{-jkz} to get $\partial_z \rightarrow -jk$, then we have

$$[\partial_r^2 + (1/r)\, \partial_r - k^2 + \beta^2\ ]\, E_z(r,0,z)^{(3)} = 0$$
$$[\partial_r^2 + (1/r)\partial_r - k^2 - (1/r^2) + \beta^2]\, E_r(r,0,z)^{(3)} = 0$$
$$[\partial_r^2 + (1/r)\partial_r - k^2 - (1/r^2) + \beta^2]\, E_\theta(r,0,z)^{(3)} = 0$$
$$\partial_r (r\, E_r(r,0,z)^{(3)}) - r\, jk\, E_z(r,0,z)^{(3)} = 0$$

or

$$[\partial_r^2 + (1/r)\, \partial_r + \beta'^2\ ]\, E_z(r,0,z)^{(3)} = 0$$
$$[\partial_r^2 + (1/r)\partial_r - (1/r^2) + \beta'^2]\, E_r(r,0,z)^{(3)} = 0$$
$$[\partial_r^2 + (1/r)\partial_r - (1/r^2) + \beta'^2]\, E_\theta(r,0,z)^{(3)} = 0$$
$$\partial_r (r\, E_r(r,0,z)^{(3)}) - r\, jk\, E_z(r,0,z)^{(3)} = 0$$

or

$[r^2\partial_r^2 + r\, \partial_r + r^2\beta'^2\ ]\, E_z(r,0) = 0$	solution is $J_0(\beta' r)$
$[r^2\partial_r^2 + r\partial_r - 1 + r^2\beta'^2]\, E_r(r,0) = 0$	solution is $J_1(\beta' r)$
$[r^2\partial_r^2 + r\partial_r - 1 + r^2\beta'^2]\, E_\theta(r,0) = 0$	solution is $J_1(\beta' r)$
$\partial_r (r\, E_r(r,0)) - r\, jk\, E_z(r,0) = 0$	relates solutions

I am back to the single-k world right now, just want to see what happens here. So

$E_z(r,0) = A_z\, J_0(\beta' r)$ // $A_z = A_z(k,\omega)$ I suppose
$E_r(r,0) = A_r\, J_1(\beta' r)$
$E_\theta(r,0) = A_\theta\, J_1(\beta' r)$

Now what happens when you stuff the first two forms into the div E = 0 above?

$$\partial_r (r\, A_r\, J_1(\beta' r)) - r\, jk\, A_z\, J_0(\beta' r) = 0$$

Change to $x = \beta' r$ so $dx = \beta' dr$ and $\partial_x = (1/\beta')\partial_r$ and then $\partial_r = \beta'\, \partial_x$ so

$$\beta'\, \partial_x\, ((x/\beta')\, A_r\, J_1(x)) - (x/\beta')\, jk\, A_z\, J_0(\beta' r) = 0$$

$A_r \partial_x (x J_1(x)) - A_z(jk/\beta') x J_0(x) = 0$

I know from Spiegel 24.21 that $\partial_x (x J_1(x)) = x J_0(x)$ so the above says

$A_r x J_0(x) - A_z(jk/\beta') x J_0(x) = 0$

$[A_r - A_z(jk/\beta')] x J_0(x) = 0$

and so we find that the coefficients must be related by

$A_r = (jk/\beta') A_z$. // this then is from div E = 0

At this point, our solutions take this form

$E_z(r,0) = A_z J_0(\beta' r)$
$E_r(r,0) = (jk/\beta') A_z J_1(\beta' r)$
$E_\theta(r,0) = A_\theta J_1(\beta' r)$

It is then time for the BC's. We write

$E_r(r=a,0) = (j\omega/\sigma) N_0$
$E_\theta(r=a,0) = 0$

The first says

$(jk/\beta') A_z J_1(\beta' a) = (j\omega/\sigma) N_0$

The second says

$A_\theta J_1(\beta' a) = 0 \quad \Rightarrow \quad A_\theta = 0$

Wow, I think this is all new stuff. So we then find that

$(jk/\beta') A_z J_1(\beta' a) = (j\omega/\sigma) N_0$

$(k/\beta') A_z J_1(\beta' a) = (\omega/\sigma) N_0$

$A_z = (\beta'/k) (\omega/\sigma) N_0 / J_1(\beta' a)$

$A_r = (jk/\beta') A_z = (jk/\beta') (\beta'/k) (\omega/\sigma) N_0 / J_1(\beta' a) = j (\omega/\sigma) N_0 / J_1(\beta' a)$

The resulting m = 0 fields are

$E_z(r,0) = A_z J_0(\beta' r) = (\beta'/k)(\omega/\sigma) N_0 J_0(\beta' r)/ J_1(\beta' a)$

$E_r(r,0) = A_r J_1(\beta' r) = j(\omega/\sigma) N_0 J_1(\beta' r)/ J_1(\beta' a)$ (**)

$E_\theta(r,0) = 0$

Is this different from my existing Appendix D for m = 0, or is it the same? Here are my existing App D solutions for general m

$E_z(r,m) = (1/4)\eta_m 2\pi\omega (a/k) N_0 R_{dc} (a\beta') f_m$ $\quad f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$

$E_r(r,m) = (j/4)\eta_m 2\pi\omega (a/k) N_0 R_{dc} (ak) g_m$ $\quad g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

$E_\theta(r,m) = (1/4)\eta_m 2\pi\omega (a/k) N_0 R_{dc} (ak) h_m$ $\quad h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

I know from below (D.11.5) that

$f_0 = 2\frac{J_0(x)}{J_1(x_a)}$ $\quad g_0 = 2\frac{J_1(x)}{J_1(x_a)}$ $\quad h_0 = 0$

Thus, my Appendix D existing solutions are

$E_z(r,0) = (1/4) 2\pi\omega (a/k) N_0 R_{dc} (a\beta') 2\frac{J_0(x)}{J_1(x_a)}$

$E_r(r,0) = (j/4) 2\pi\omega (a/k) N_0 R_{dc} (ak) 2\frac{J_1(x)}{J_1(x_a)}$

$E_\theta(r,0) = 0$

or

$E_z(r,0) = \pi\omega a^2 (\beta'/k) N_0 R_{dc} \frac{J_0(x)}{J_1(x_a)}$

$E_r(r,0) = (j)\pi\omega a^2 N_0 R_{dc} \frac{J_1(x)}{J_1(x_a)}$

$E_\theta(r,0) = 0$

or

$E_z(r,0) = \pi\omega a^2 (\beta'/k) N_0 1/(\sigma\pi a^2) \frac{J_0(x)}{J_1(x_a)}$

$E_r(r,0) = (j)\pi\omega a^2 N_0 1/(\sigma\pi a^2) \frac{J_1(x)}{J_1(x_a)}$

$E_\theta(r,0) = 0$

or

$E_z(r,0) = (\omega/\sigma)(\beta'/k) N_0 \frac{J_0(x)}{J_1(x_a)}$

$$E_r(r,0) = j\,(\omega/\sigma)\; N_0 \frac{J_1(x)}{J_1(x_a)}$$

$$E_\theta(r,0) = 0$$

These agree exactly with the results found more directly in (**) above. So I guess I don't have some major m = 0 malfunction in Appendix D. Then the superposition work of Part C is valid for m = 0.

Part E: The J_z Asymmetry Paradox is still present with a Superposed Solution

6. The J_z Asymmetry Paradox.

First, in the single-k scenario we have

$$\frac{E_z(r,m)}{E_z(r,0)} = \eta_m \frac{f_m(x,x_a)}{f_0(x,x_a)} \qquad \text{where k has some value}$$

As $\omega \rightarrow 0$, we get that $\beta' = jkr$ which we can insert into the ratio above's RHS. Now matter what you assume about k as $\omega \rightarrow 0$, the ratio is non-zero, and that is the "single-k J_z symmetry paradox".

With superposition we have instead from above,

$$E_z(r,m,z)^{(3)} = (1/4)\,\eta_m R_{dc} \int dk\; 2\pi\omega\,(a/k)\,N_0 C(k,\omega)\,(a\beta')\,f_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}$$

$$E_r(r,m,z)^{(3)} = (j/4)\,\eta_m R_{dc} \int dk\; 2\pi\omega\,(a/k)\,N_0 C(k,\omega)\,(ak)\,g_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}$$

$$E_\theta(r,m,z)^{(3)} = (1/4)\,\eta_m R_{dc} \int dk\; 2\pi\omega\,(a/k)\,N_0 C(k,\omega)\,(ak)\,h_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}$$

or

$$E_z(r,m,z)^{(3)} = (1/4)\,N_m R_{dc} 2\pi\omega a \int dk\; (1/k)\,C(k,\omega)\,(a\beta')\,f_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}$$

$$E_r(r,m,z)^{(3)} = (j/4)\,N_m R_{dc} 2\pi\omega a \int dk\; (1/k)\,C(k,\omega)\,(ak)\,g_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}$$

$$E_\theta(r,m,z)^{(3)} = (1/4)\,N_m R_{dc} 2\pi\omega a \int dk\; (1/k)\,C(k,\omega)\,(ak)\,h_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}$$

Now take this ratio and cancel common factors,

$$\frac{E_z(r,m,z)}{E_z(r,0,z)} = \eta_m \frac{\int dk\; C(k,\omega)\,(\beta'/k) f_m(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}}{\int dk\; C(k,\omega)\,(\beta'/k) f_0(\sqrt{\beta^2-k^2}\,r)\,e^{-jkz}} \qquad (**) \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

Comment: Note that $C(k,\omega)$ itself has no m subscript, and cannot have one.

Note: If I assume that $C(k,\omega) = \delta(k-k)$ (so to speak) the above becomes

$$\frac{E_z(r,m,z)}{E_z(r,0,z)} = \eta_m \frac{f_m(\beta' r)}{f_0(\beta' r)} \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

which agrees with my new Chapter 7 start. This thing has the J_z asym problem as described in Chapter 7. But now let's return to the continuous superposition and see if that fixes the problem.
My paradox-removal requirement is that the ratio (**) $\rightarrow 0$ as $\omega \rightarrow 0$. In that limit, $\beta \rightarrow 0$ so our requirement then is this: [$\beta' = \lim \sqrt{\beta^2 - k^2} = \sqrt{-k^2} = jk$, so $\beta'/k = j$

$$\int dk\ C(k,0)\, j\, f_m(jkr)\, e^{-jkz} = 0 \qquad m \neq 0 \qquad F(r,m,z) = 0$$

$$\int dk\ C(k,0)\, j\, f_0(jkr)\, e^{-jkz} \neq 0 \qquad m = 0$$

The first line requires that C(k,0) = 0 as the only way really to get F(r,m,z) to vanish for arbitrary values of r, m and z. But this makes the second line above be not true!

Note: in the superposition scenario, we don't think about what k does as $\omega \rightarrow 0$. Here k is just an integration variable. We do know that $C(k,0) = r_k$ at $\omega = 0$ which is the FT of r(z).

Conclusion: Thus the J_z paradox does NOT go away with superposition of different k solutions. No matter how you superpose, you still have J_z asymmetry.

Reflection Example:

In the reflection scenario you have

$$C(k,\omega) = A(k_1)\delta(k-k) + B(k_1)\,\delta(k+k) \qquad \text{so to speak}$$

This is just a special case of superposition. We would then have to have

$$A(k_1)\, f_m(jk_1 r)\, e^{-jkz} + B(k_1)\, f_m(-jk_1 r)\, e^{+jkz} = 0 \qquad m \neq 0$$

$$A(k_1)/k_1\, f_0(jk_1 r)\, e^{-jkz} + B(k_1)/k_1\ f_0(-jk_1 r)\, e^{+jkz} \neq 0 \qquad m = 0$$

There is no way to make the first line be true unless both $A(k_1) = 0$ and $B(k_1) = 0$. That is why I found long ago that reflection does not "fix" the J_z asym problem.

Eliminate the two terms m and -m possible escape hatch.

7. Let's eliminate the usual m and -m escape hatch. The two terms in the expansion are

$$\text{terms}(m,\theta) = E_z(r,m,z)^{(3)}\, e^{jm\theta} + E_z(r,-m,z)^{(3)}\, e^{-jm\theta} \qquad \text{for some } m > 0$$

$$E_z(r,m,z)^{(3)} = (1/4)\,\eta_m\, I\, R_{dc} \int dk\; C(k,\omega)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz} \qquad \eta_{-m} = \eta_m \qquad f_{-m} = f_m$$

$$\text{terms}(m,\theta) = [\,(1/4)\, N_m\, R_{dc} 2\pi\omega a \int dk/k * \beta' C(k,\omega)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}\,]\, e^{jm\theta}$$
$$+ [\,(1/4)\, N_{-m}\, R_{dc} 2\pi\omega a \int dk/k * \beta' C(k,\omega)\, f_{-m}(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}\,]\, e^{-jm\theta}$$

$$= [\,(1/4)\, N_m\, R_{dc} 2\pi\omega a \int dk/k * \beta' C(k,\omega)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}\,]\, [e^{jm\theta} + e^{-jm\theta}]$$

$$= [\,(1/4)\, N_m\, R_{dc} 2\pi\omega a \int dk/k * \beta' C(k,\omega))\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}\,]\, 2\cos(m\theta)$$

Now consider

$$\text{terms}(m=0,\theta) = E_z(r,0,z)^{(3)}\, e^0 = E_z(r,0,z)^{(3)} \qquad \text{// only one term for m = 0}$$

Then if we include both the m and -m terms *together* we have

$$\frac{E_z(r,\theta,z)^{2_mterms}}{E_z(r,\theta,z)^{m=0_term}} = \eta_m \frac{[\int dk/k\, \sqrt{\beta^2-k^2}\, C(k,\omega)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}\,]\, 2\cos(m\theta)}{\int dk/k\, \sqrt{\beta^2-k^2}\, C(k,\omega)\, f_0(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}}$$

$$= \eta_m\, 2\cos(m\theta) \frac{[\int dk/k\, \sqrt{\beta^2-k^2}\, C(k,\omega)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}\,]}{\int dk/k\, \sqrt{\beta^2-k^2}\, C(k,\omega)\, f_0(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}}$$

As $\omega \to 0$ we set $\sqrt{\beta^2-k^2} = jk$ to get

$$\frac{E_z(r,\theta,z)^{2_mterms}}{E_z(r,\theta,z)^{m=0_term}} = \eta_m\, 2\cos(m\theta) \frac{\int dk\; j\, C(k,0)\, f_m(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}}{\int dk\; j\, C(k,0)\, f_0(\sqrt{\beta^2-k^2}\, r)\, e^{-jkz}}$$

But this is the exact same ratio found above with an extra 2 cos(mθ) on the outside, and we already showed that said ratio is not 0 !

Conclusion: Allowing superposition of k values does NOT resolve the J_z asymmetry paradox!

This enhances my Appendix D mystery! There is no possible superposition of solutions that makes this asymmetry go away!! It has nothing to do with the specific function k(ω) of Appendix Q, for example.

8. **The Infinite B_r Paradox.** Start with

$$\text{curl } \mathbf{E}(r,\theta,z) = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$

and in particular

$$[\text{curl } \mathbf{E}(r,\theta,z)]_r = r^{-1}\partial_\theta E_z - \partial_z E_\theta$$

Now apply this to our superposed solution, so it says

$$[\text{curl } \mathbf{E}(r,\theta,z)^{(3)}]_r = r^{-1}\partial_\theta E_z(r,\theta,z)^{(3)} - \partial_z E_\theta(r,\theta,z)^{(3)}$$

Now install the superposed solutions

$$\mathbf{E}(r,\theta,z,t)^{(3)} = \int dk\, C(k,\omega)\, e^{j(\omega t - kz)}\ \mathbf{E}(r,\theta;k)$$

$$\mathbf{E}(r,\theta,z)^{(3)} = \int dk\, C(k,\omega)\, e^{-jkz}\ \mathbf{E}(r,\theta;k)$$

$$E_i(r,\theta,z)^{(3)} = \int dk\, C(k,\omega)\, e^{-jkz}\ E_i(r,\theta;k)$$

We then get

$$[\text{curl } \mathbf{E}(r,\theta,z)^{(3)}]_r =$$

$$r^{-1}\partial_\theta \int dk\, C(k,\omega)\, e^{-jkz}\ E_z(r,\theta;k) - \partial_z \int dk\, C(k,\omega)\, e^{-jkz}\ E_\theta(r,\theta;k)$$

I think it is OK to go to partial waves and write

$$[\text{curl } \mathbf{E}(r,m,z)^{(3)}]_r =$$

$$r^{-1}jm \int dk\, C(k,\omega)\, e^{-jkz}\ E_z(r,m;k) - \partial_z \int dk\, C(k,\omega)\, e^{-jkz}\ E_\theta(r,m;k)$$

Now install the known single-k solutions,

$$E_z(r,m) = (1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,m) = (j/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

to get

$$[\text{curl } \mathbf{E}(r,m,z)^{(3)}]_r =$$

$$= r^{-1} jm \int dk\, C(k,\omega)\, e^{-jkz} \{(1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (a\sqrt{\beta^2-k^2})\, f_m(\sqrt{\beta^2-k^2}\, r)\}$$

$$- \partial_z \int dk\, C(k,\omega)\, e^{-jkz} \{(1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (ak)\, h_m(\sqrt{\beta^2-k^2}\, r)\}$$

$$= r^{-1} jm\, N_m\, 2\pi\omega\, R_{dc}(1/4)a^2 \int dk\, C(k,\omega)\, e^{-jkz} \{(1/k)\,(\sqrt{\beta^2-k^2})\, f_m(\sqrt{\beta^2-k^2}\, r)\}$$

$$- N_m\, 2\pi\omega\, R_{dc}(1/4)a^2\, \partial_z \int dk\, C(k,\omega)\, e^{-jkz} \{h_m(\sqrt{\beta^2-k^2}\, r)\}$$

Divide both sides by (jω) to get

$$B_r(\omega) = r^{-1} m\, N_m\, 2\pi\, R_{dc}(1/4)a^2 \int dk\, C(k,\omega)\, e^{-jkz} \{(1/k)\,(\sqrt{\beta^2-k^2})\, f_m(\sqrt{\beta^2-k^2}\, r)\}$$

$$+ jN_m\, 2\pi\, R_{dc}(1/4)a^2 \int dk\, C(k,\omega)\, (-jk) e^{-jkz} \{h_m(\sqrt{\beta^2-k^2}\, r)\}$$

Now we want to take $\omega \to 0$. This gives,

$$B_r(0; r,z,m) = r^{-1} m\, N_m\, 2\pi\, R_{dc}(1/4)a^2 \int dk\, C(k,0)\, e^{-jkz} \{(1/k)\,(jk)\, f_m(jkr)\}$$

$$+ jN_m\, 2\pi\, R_{dc}(1/4)a^2 \int dk\, C(k,0)\, (-jk) e^{-jkz} \{h_m(jkr)\}$$

But now there is nothing trying to blow up the right hand side! In this superposition model, there is no connection between k and ω, and that is where the problem arose before. In the large ω limit, we have instead I think that $C(k,\omega) = \delta(k-k_1)$ so to speak where k_1 is our formula. Then

$$B_r(\omega) = r^{-1} m\, N_m\, 2\pi\, R_{dc}(1/4)a^2\, e^{-jkz} \{(1/k)\,(\sqrt{\beta^2-k^2})\, f_m(\sqrt{\beta^2-k^2}\, r)\}$$

$$+ jN_m\, 2\pi\, R_{dc}(1/4)a^2\, (-jk) e^{-jkz} \{h_m(\sqrt{\beta^2-k^2}\, r)\}$$

and NOW you have to worry about $k = k(\omega)$. Rewrite for small ω,

$$B_r(\omega) = r^{-1} m\, N_m\, 2\pi\, R_{dc}(1/4)a^2\, e^{-jkz} \{(1/k)\,(jk)\, f_m(jkr)\}$$

$$+ jN_m\, 2\pi\, R_{dc}(1/4)a^2\, (-jk) e^{-jkz} \{h_m(jkr)\}$$

So OK, where is the problem now??

$$f_m = (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$$
$$g_m = (r/a)^{|m|+1} + (r/a)^{|m|-1} \qquad g_0 = 2\,(r/a)$$
$$h_m = (r/a)^{|m|+1} - (r/a)^{|m|-1} \qquad h_0 = 0 \quad . \qquad \text{(D.11.6)}$$

where now $\beta' = jk$ as $\omega \rightarrow 0$ so we then have

$$B_r(\omega) = r^{-1} m\, N_m\, 2\pi\, R_{dc} (1/4) a^2\, e^{-jkz} \{ (1/k)\,(jk)\,(r/a)^{|m|}\,(|m|+1)\,(2/jka) \} (r/a)^{|m|}\,(|m|+1)$$

$$+ jN_m\, 2\pi\, R_{dc} (1/4) a^2\, (-jk) e^{-jkz} \{ (r/a)^{|m|+1} - (r/a)^{|m|-1} \}$$

The second term has just k and goes then as $\sqrt{\omega}$ and so is not a problem.
The first term now has 1/k and goes as $1/\sqrt{\omega}$ and THAT is the problem, as always.

Conclusion: The superposition solution of k values can make the infinite B_r paradox go away.