Transmission Line at very low Frequency PhL 8.16.14

Reviewing this doc on 9.10.14.

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1. Basic facts for traveling wave with complex k.

Imagine driving an infinite transmission line, two large solid copper cylinders that go off the infinity. We drive at very low ω. Let's assume for the moment that the network model applies in this regime, so that the traditional formula $k \equiv -j\sqrt{(R+j\omega L)(G+j\omega C)}$ applies. Assume also that G = 0. Then we know that
[this is the same limit as found in the updated App Q]

Fact 4: In the low frequency limit with G = 0 , (Q.4)

$$\text{Re}(k) \approx \sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2}$$
$$\text{Im}(k) \approx -\sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2} \qquad \omega << R/L$$

Now put that above aside for a moment, and consider a wave going down the line with complex k. We then assume our usual form $e^{j(\omega t - kz)}$ and we find that

$$\exp[\, j(\omega t - \{k_r + jk_i\}z] = \exp[+k_i z]\exp[\, j(\omega t - k_r z)]$$

Therefore, in talking about a phase front, only k_r plays a role, as I have long suspected. The phase velocity is determined in this manner:

$$\omega t - k_r z = \text{constant}$$

Apply dt to get

$$\omega = k_r(dz/dt) \qquad \Rightarrow \qquad dz/dt = \omega/k_r \equiv v_\varphi = \text{velocity of phase fronts}$$

Meanwhile, the wavelength of a wave is determined from

$$\lambda = 2\pi/k_r$$

Here k_r is the number of radians of wave phase per meter [since phase = $\omega t - k_r z$ = constant], so that $1/k_r$ is the meters per radian of phase. The number of meters per full wave is then 2π times this latter.

Finally, the decay distance [that is, the 1/e dropoff distance] is $\Delta z = 1/(-k_i)$.

We now summarize these facts:

Traveling Wave with Complex k

$v_\varphi = \omega/k_r$ = velocity of phase front going down the line

$\lambda = 2\pi/k_r$ = distance between fronts of the same phase = wavelength

$\Delta z = 1/(-k_i)$ = decay distance for 1/e decay of the wave

[I think everything in the box is correct]

2. Apply these basic facts to a TL operating at low ω

Now having determined the above for general complex k, we apply it to our transmission line k where

$k_r = \sqrt{RC/2}\ \omega^{1/2}$
$k_i = -\sqrt{RC/2}\ \omega^{1/2}$

We then find that

$v_\varphi = \omega/k_r = \omega / [\sqrt{RC/2}\ \omega^{1/2}] = \sqrt{\frac{2}{RC}}\ \omega^{1/2}$ phase velocity

$\lambda = 2\pi/k_r = 2\pi / [\sqrt{RC/2}\ \omega^{1/2}] = 2\pi \sqrt{\frac{2}{RC}}\ \omega^{-1/2}$ wavelength

$\Delta z = 1/(-k_i) = 1/k_r = \sqrt{\frac{2}{RC}}\ \omega^{-1/2}$ decay distance

As $\omega \rightarrow 0$, we find that

$v_\varphi \rightarrow 0$ as $\sqrt{\omega}$ phase velocity goes to zero, an unexpected result

$\lambda \rightarrow \infty$ as $1/\sqrt{\omega}$ wavelength goes to ∞, an expected result

$\Delta z \rightarrow \infty$ decay distance goes to ∞, expected result since know $I \rightarrow 0$

The feature unexpected by me is that the phase velocity slows down to 0 ! This should have some implications below.

Now I claim that

$$V(z) = V(0)\, e^{j(\omega t - kz)}$$

$$q(z) = q(0)\, e^{j(\omega t - kz)}$$

$$q(z) = C\, V(z)$$

So the driving voltage source V(0) determines q(0), and so we know q(z) going down the line. This in turn determines n(θ,z) going down the line, from the usual 2D electrostatic solution.

[everything in Section 2 above seems correct]

3. Where does n(θ) come from?

Now we come to a big question: In this low frequency regime, "where does n(θ) come from? "

I argued in Section D.9 (c), and I quote,

"The first hypothesis might be that the individual electrons which make up n(θ) simply travel at v_d in the z direction down the conductor surface, and n(θ) is not fed by any radial currents inside the conductor. In this case one would have $K_z^D(\theta) = v_d\, n(\theta)$. But we know this is not what happens. Apart from the massive energy required to achieve relativistic electron velocities, we know from Appendix N.1 that the electrons in the Debye layer in fact drift along at something like ~ 1 mm/sec, just as do the regular conduction electrons in the conductor bulk. "

Now in our TL very low ω limit, recall that $v_\varphi \rightarrow 0$ and so q(z) moves very slowly in the z direction. in this case, both arguments made above are invalid. Suppose v_φ = .01 mm/sec. Then the Debye layer drift could easily supply that flow. Also, no relativistic issues. And of course the bulk can also carry charges at this slow rate.

[the linear charge density q(z) *appears* to move at phase velocity v_φ just as do all other transmission line quantities. This does not mean any actual physical charged particles are moving at v_φ .]

Comment: This may be the Breakthrough I have been long looking for!![hah!] In the $\omega \rightarrow 0$ limit, if the surface charge is supplied by the Debye or bulk layer, then you have to include this in the boundary condition, and I do NOT include this because I argued that all charge came from charge pumping by J_r. So maybe my BC is only valid when v_φ is very large.

[current flow in the bulk only gets to the surface via J_r, and that is why J_r appears in n(θ). But current flowing along the surface is already there, at the surface, so this would be a J_z Debye current]

I guess I will now go off and rethink my section on Debye surface currents! [see separate doc]
[I have already reviewed that separate doc, it was not enlightening]

4. Rewrite Section 6.5 (d) using the correct k instead of β_{d0} :

[I am going to read the existing lines doc Section 6.5 right now, 9.10.14:
(a) seems OK
(b) I correctly claim β = β' at large ω ("low loss"). This section seems all OK
(c) OK, but I am always talking large ω where k = β_{d0}.
(d) I still assume "skin effect regime" so large ω so k = β_{d0}. But perhaps the point made in the following rewrite is that I *could* make this section apply for general k and not just for β_{d0}. I start with (6.5.14) which is presumably valid for all ω , and same for (6.5.15). But THEN I go to skin effect limit on next few equations. OK , I just now edited the lines doc version so it looks more like the stuff right below, but in the end we really have k = β_{d0} so it does not matter much.

(d) The relationship between $J_z(a,\theta)$ and n(θ) obtained from div E = 0

[see notes just above]

Here, assuming the skin effect regime and making a few assumptions, we obtain (6.5.13) directly from the div **E** = 0 equation and the charge pumping boundary condition, just to provide some intuition about the linkage between J_z and n(θ).

The charge pumping boundary condition of Appendix D says (r = a means r = a-ε)

$$E_r(r=a,\theta) = (j\omega/\sigma)\, n(\theta) \qquad \textit{(D.2.24)} \qquad (6.5.14)$$

so the pattern of n(θ) is directly mapped to $E_r(r=a,\theta)$ at the surface. But we are interested in E_z since our current density of interest is $J_z = \sigma E_z$. The condition div **E** = 0 in cylindrical coordinates reads,

$$\partial_r (r\, E_r(r,\theta,z)) + \partial_\theta E_\theta(r,\theta,z) + r\, \partial_z E_z(r,\theta,z) = 0 \ .$$

For r = a we know from (3.7.0) that $E_\theta(a,\theta) = 0$ and $\partial_\theta E_\theta(r,\theta) = 0$, so for r just below the surface we expect

$$\partial_r (r\, E_r(r,\theta,z)) + r\, \partial_z E_z(r,\theta,z) \approx 0 \qquad \text{// near r = a}$$

$$\partial_r (r\, E_r(r,\theta)) - jk\, r\, E_z(r,\theta) \approx 0 \qquad \text{// using } \partial_z \rightarrow -jk \text{, see (D.1.16), then cancel } e^{jkz} \text{ factors}$$

$$E_z(r,\theta) \approx (1/jk)\,(1/r)\,\partial_r (r\, E_r(r,\theta)) \ . \qquad \text{// near r = a} \qquad (6.5.15)$$

For the symmetric-environment round wire, we know from (2.2.29) that

$$E_z(r) = E_z(a)\,\frac{J_0(\beta r)}{J_0(\beta a)}\ . \qquad (2.2.29)$$

Taking the large argument limits of the two Bessel functions using (2.3.3) and (2.3.6), we find that in the skin effect regime,

$$E_z(r,\theta) \approx E_z(a,\theta)\sqrt{\frac{a}{r}}\ e^{(r-a)/\delta}\ e^{j(r-a)/\delta} \qquad (6.5.16)$$

which we note has the same general form as the simple result (2.1.8) with x = a-r which is $e^{-x/\delta}\ e^{-jx/\delta}$ and is also consistent with (2.3.7) for magnitude. If we blindly assume this same equation applies to $E_r(r,\theta)$ and $E_r(a,\theta)$, then

$$E_r(r,\theta) \approx E_r(a,\theta)\sqrt{\frac{a}{r}}\ e^{(r-a)/\delta}\ e^{j(r-a)/\delta}\ . \qquad (6.5.17)$$

Then (6.5.15) says,

$$E_z(a,\theta) = (1/jk)\,E_r(a,\theta)\left[\,(1/r)\,\partial_r\left(r\sqrt{\frac{a}{r}}\ e^{(r-a)/\delta}\,e^{j(r-a)/\delta}\right)\right]\Big|^{r=a}$$

$$= (1/jak)\,E_r(a,\theta)\,[1/2 + (1+j)(a/\delta)\,] \qquad (6.5.18)$$

where the derivative is done by Maple,

```
f := r*sqrt(a/r)*exp((r-a)/delta)*exp(j*(r-a)/delta);
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$$f := r\sqrt{\frac{a}{r}}\,e^{\left(\frac{r-a}{\delta}\right)}\,e^{\left(\frac{j\,(r-a)}{\delta}\right)}$$

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g := (1/r)*diff(f,r):
r := a;
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$$r := a$$

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g;
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$$\frac{\frac{1}{2}+\frac{a}{\delta}+\frac{a\,j}{\delta}}{a}$$

If in (6.5.18) we use the boundary condition (6.5.14) that $E_r(a,\theta) = (j\omega/\sigma)\,n(\theta)$, the result is

$$E_z(a,\theta) = (1/jak)\,\{(j\omega/\sigma)\,n(\theta)\}\,[1/2 + (1+j)a/\delta\,]$$

$$= (1/jak)\,\{(j\omega/\sigma)\,n(\theta)\}\,(1+j)a/\delta\,] \qquad \text{// ignore 1/2 relative to } a/\delta$$

$$= (1/jk)\,\{(\omega/\sigma)\,n(\theta)\}\,(j-1)/\,\delta\,]$$

$$= (1/jk)\ \{(\omega/\sigma)\ n(\theta)\}\ \beta\ ] \qquad // (6.5.8)$$

$$= (-j\omega/\sigma)\ (\beta/k)\ n(\theta)\ . \qquad (6.5.19)$$

Putting this into (6.5.16) then gives

$$E_z(r,\theta) \approx (-j\omega/\sigma)\ (\beta/k)\ n(\theta) \sqrt{\frac{a}{r}}\ e^{(r-a)/\delta}\ e^{j(r-a)/\delta} \qquad (6.5.20)$$

which agrees with our earlier result (6.5.13) quoted from Appendix D. Although we just guessed at the form (6.5.17), that form is verified in box (D.10.13). The bottom line here is that E_z (and thus J_z) "tracks" $n(\theta)$ for its θ dependence.

[review OK to here]

5. So how does this discussion go for low ω instead of high skin depth ω?

We certainly get to this early step, which comes just from div **E** = 0 inside the conductor,

$$E_z(r,\theta)\ \approx\ (1/jak)\ \partial_r\ (r\ E_r(r,\theta))\ . \qquad // \text{ near } r = a \qquad (6.5.15)$$

This is my little "linkage" between E_r and E_z. This is only valid below the Debye layer, by the way, since otherwise there is some free charge to worry about and then div **E** = ρ/ε_0. Another concern inside the Debye layer is that Ohm's Law is replaced by **J** = σ**E** - D grad ρ where again ρ is that free charge, and this would imply that $J_r \neq \sigma E_r$ in the Debye layer.

So let's assume we are close to r = a, but not so close that we are in the Debye layer. Then the above linkage ought to be valid, and is also valid for current densities,

$$J_z(r,\theta)\ \approx\ (1/jak)\ \partial_r\ (r\ J_r(r,\theta))$$

The CPBC says (here for fun I include the Debye term)

$$j\omega\ n(\theta)\ =\ J_r(a-\lambda_D,\theta)\ \ + jk\ \lambda_d\ J_z(a,\theta)$$

At low ω, *not only* is λ_d tiny, being 10^{-10} m, but k is *also* small, since

$$\text{Re}(k)\ \approx\ \sqrt{RC/2}\ \omega^{1/2}$$
$$\text{Im}(k)\ \approx\ -\sqrt{RC/2}\ \omega^{1/2} \qquad [\text{ correct for G = 0 and new App Q or old App Q }]$$

[So I have a solid arm-waving argument (λ_d and k *both* tiny as $\omega \rightarrow 0$) why Debye does not matter]

It is hard for me to imagine that the Debye term shown above is significant, but somehow it might be. So let's ignore the Debye term for now and write

$$j\omega\, n(\theta) = J_r(a,\theta)$$

where r = a is the same location appearing in the linkage equation above. So we now have:

$$J_z(r,\theta) \approx (1/jak)\, \partial_r(r\, J_r(r,\theta)) \qquad \text{div } E = 0$$

$$J_r(a,\theta) = j\omega\, n(\theta) \qquad \text{CPBC}$$

Now assume that n(θ) has a monstrous huge peak at θ = π because our two cylinders are almost touching each other! Recall n(θ) plots from bipolar doc showing such peaks. The CPBC certainly says that, for any ω > 0,

Fact 1: $J_r(a,\theta)$ is a very strong function of θ going around the cylinder perimeter.

Now since the world is a smooth place, we can presume I think that

Fact 2: $J_r(r,\theta)$ is a very strong function of θ going around the cylinder perimeter for $r \leq a$ but close to a.

By strong function in the above two Facts, I mean that the *scale* of the function depends strongly on n(θ) so that the function $J_r(r,\theta)$ has a strong peak at θ = π. Where n(θ) is very large, we need lots of radial pumping to provide the surface charge density!

Now we take this strong function of θ and we have

$$\partial_r\, (r\, J_r(r,\theta)) = J_r(a,\theta) + a\, [\partial_r J_r(r,\theta)\,]|^{r=a}$$

This is the sum of two terms. The first term has a very strong peak at θ = π. We are not sure what the second term is doing. It just seems exceedingly likely that the second term also has a strong peak at θ = π. Since we must have $J_r(0,\theta) = 0$ on the center line, it seems likely that J_r increases monotonically to its value on the surface and thus $\partial_r J_r$ will be positive, so we add to the first term. This is a generic argument. We can of course peek at the solution of Appendix D which is this, for small ω,

$$J_r(r,\theta) = (j/2)\, I\, \sigma\, R_{dc}\, (ak)\, \{(r/a) + \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} + (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\, \} \qquad \text{(D.11.8)}$$

$$J_r(a,\theta) = (j/2)\, I\, \sigma\, R_{dc}\, (ak)\, \{1 + 2\, \Sigma_{m=1}^{\infty}\, \eta_m\, \cos(m\theta)\, \}$$

$$\partial_r\, J_r(r,\theta) = (j/2)\, I\, \sigma\, R_{dc}\, (ak)\, \{(1/a) + (1/a)\Sigma_{m=1}^{\infty}\, [(m+1)(r/a)^m + (m-1)(r/a)^{m-2}]\, \eta_m\, \cos(m\theta)\, \}$$

$$a\, [\partial_r J_r(r,\theta)\,]|^{r=a} = (j/2)\, I\, \sigma\, R_{dc}\, k\, \{1 + \Sigma_{m=1}^{\infty}\, [(m+1) + (m-1)]\, \eta_m\, \cos(m\theta)\, \}$$

$$= (j/2)\, I\, \sigma\, R_{dc}\, (ak)\, \{1 + 2\, \Sigma_{m=1}^{\infty}\, m\, \eta_m\, \cos(m\theta)\, \} \qquad \eta_m = (-1)^m\, e^{-m|\xi|}$$

Both functions $J_r(a,\theta)$ and $a\, [\partial_r J_r(r,\theta)\,]|^{r=a}$ have strong positive peaks at θ = π since there

$$J_r(a,\pi) = (j/2)\, I\, \sigma\, R_{dc}\ (ak)\ \{1 + 2\, \Sigma_{m=1}^{\infty}\, e^{-m|\xi|}\}$$

$$a\, [\partial_r J_r(r,\pi)\,]|^{r=a} = (j/2)\, I\, \sigma\, R_{dc}\ (ak)\ \{1 + 2\, \Sigma_{m=1}^{\infty}\, m\, e^{-m|\xi|}\}$$

and if anything the second peak is stronger.

My conclusion is this:

Fact 3: $\partial_r(r\, J_r(r,\theta))$ is strongly peaked at $\theta = \pi$.

Therefore, I conclude that, from my linkage condition near the surface,

Fact 4: $J_z(r,\theta)$ is strongly peaked at $\theta = \pi$, since our linkage above says,

$$J_z(a,\theta) \approx (1/jak)\, \partial_r\, (r\, J_r(r,\theta))$$

This then says that $J_z(a,\theta)$ is highly asymmetric around the perimeter of the cylinder. From that, I would conclude that $J_z(r,\theta)$ is highly asymmetric all across the cross section. This leads to

This conclusion applies for our infinite transmission line and shows that J_z stays asymmetric right to the limit $\omega \rightarrow 0$. The conclusion does not directly apply to shorted cylinders since $\partial_z \rightarrow jk$ does not apply in that case, so one has then to implement the "reflection scenario" in order to get a conclusion for that situation. This process of course injects new uncertainty.

[I agree with all these conclusions -- my theory without Debye in BC says J_z is asymmetric at low ω, and I give a good argument above for why Debye should be ignored especially as $\omega \rightarrow 0$.]