Where does the sym J_z solution lie? PhL 9.17.14

Here I try to understand why I don't get the right DC solution from Appendix D. Much of the stuff below ends up in Chapter 7. I run into the infinite B problem perhaps for the first time (perhaps not). See red comments in each section. Reviewed 10.8.14.

1. Smythian form for 2 round wires with I > 0 ? **1**

2. How does the DC solution slip away? **3**

3. Superposition and J_z Asym. **7**

4. Digression on Atoms in Spherical Coords **8**

5. Try starting in App D with ω = 0 ..... **8**

6. Precursor of Section 7.3 on getting the DC solution. **9**

7. On the infinite B problem **14**

1. Smythian form for 2 round wires with I > 0 ?

Here I consider the Smythian superposition solution of Chapter 7.2 and conclude that you need a δ(k) coefficient and then you have J_z asym, and the two parallel wires carrying I > 0 just becomes the same old App D TL problem for single k. This is the superposition paradox I have never resolved. Of course those two wires must be infinitely long, then it really is an infinite TL etc etc.

Thinking here of the two parallel round wires carrying large I and -I. In one of these wires, the E fields it would seem are determined by Appendix D even though this is not a transmission line. That is to say, the solution to the E Helm equation and the two BC's must have this form for E_z : [atomic forms argument]

$$E_z(r,m,z)^{(3)} = \int dk\, e^{-jkz}\, (1/4)\, \eta_m\, 2\pi\omega\, (a/k)\, N_0\, r_k\, R_{dc}\, (a\beta')\, f_m(\beta' r)$$

$$= \eta_m\, 2\pi\omega\, a^2\, N_0\, R_{dc} \int dk\, e^{-jkz}\, (\omega/k)\, r_k\, \beta'\, f_m(\beta' r)$$

where r_k is the "arbitrary weighting function". This result is based on a "most general possible form for E_z" argument [see doc "show that the linear combination..."] along with the two BC's. If you wanted a situation where n(θ,z) = n(θ)r(z) is constant in z, then you need r(z) = constant, which means $r_k = \delta(k)$. If k is then very small, and we are also interested in ω being small, then $\beta' = \sqrt{\beta^2 - k^2}$ will be small in magnitude, and we should be able to use

$$f_m = (r/a)^{|m|}\, (|m|+1)\, (2/\beta' a) \qquad f_0 = 4/(a\beta')$$

This says then that

$$E_z(r,m,z)^{(3)} = \eta_m\, 2\pi\, a\, N_0\, R_{dc} \int dk\, e^{-jkz}\, (\omega/k)\, r_k\, (r/a)^{|m|}\, (|m|+1)\, (2) \qquad m \neq 0$$

$$E_z(r,0,z)^{(3)} = 2\pi\, a\, N_0\, R_{dc} \int dk\, e^{-jkz}\, (\omega/k)\, r_k \quad (4) \qquad m = 0$$

Now somehow we are interested in a double limit where $k \to 0$ to give uniform r(z), and $\omega \to 0$ which we think is associated with this same uniform r(z), that is, a DC situation where nothing moves. So what are we going to do with the ratio (ω/k)? If we were to make our network model for the two wires as in Fig K.5, which looks just at a differential piece dz of our two conductor system. we obtain the TL equations with $k^2 = -\,\mathbf{zy}$. But this then is our usual $k(\omega)$ and we know from App Q that for small ω and G = 0,

$$\mathrm{Re}(k) \approx +\sqrt{R_{dc}C/2}\ \sqrt{\omega}$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C/2}\ \sqrt{\omega}$$

$$k = \sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (1-j) = \sqrt{R_{dc}C}\ \sqrt{\omega}\ e^{-j\pi/2}$$

$$(\omega/k) = e^{+j\pi/2}\, (1/\sqrt{R_{dc}C}\,)\, \sqrt{\omega}\,.$$

Using k = k(ω) based on the network model, we then at least come up with (ω/k) not blowing up. If we now take our limit $r_k = \delta(k)$ we find

$$E_z(r,m,z)^{(3)} = \eta_m\, 2\pi\, a\, N_0\, R_{dc}\, e^{+j\pi/2}\, (1/\sqrt{R_{dc}C}\,)\, \sqrt{\omega}\ (r/a)^{|m|}\, (|m|+1) \quad (2) \qquad m \neq 0$$
$$E_z(r,0,z)^{(3)} = 2\pi\, a\, N_0\, R_{dc}\, e^{+j\pi/2}\, (1/\sqrt{R_{dc}C}\,)\, \sqrt{\omega}\ \ (4) \qquad m = 0$$

But now as $\omega \to 0$, both of these vanish and we get $E_z \equiv 0$. The DC solution we are hoping to find has somehow slipped out of our hands! Somehow it got excluded. But we know that our elusive DC solution does satisfy the E Helm equation, div E = 0, and both BC's! So where did that solution go??? We thought we had a "most general solution" in terms of the harmonics of the scalar E_z Helm equation.

Let's go back at the harmonics we use to construct $E_r(r,\theta,z)$:

$$e^{jm\theta}\, e^{-jkz}\, J_m(\sqrt{\beta^2-k^2}\, r)$$

If you were looking for a constant solution, you would take m = 0 and k = 0 so there is then no variation in either θ or z. The harmonic then becomes

$$e^{jm\theta}\, e^{-jkz}\, J_m(\sqrt{\beta^2-k^2}\, r) \ \to\ J_0(\beta r)$$

But alas, this is NOT a constant solution for E_z. However, if we now take $\omega \to 0$. we get

$$e^{jm\theta}\, e^{-jkz}\, J_m(\sqrt{\beta^2-k^2}\, r) \ \to\ J_0(0) \ = 1$$

and there is your hoped-for constant solution. Note that for $\beta^2 \neq 0$, a constant is NOT a solution of the Helm equation! Only for the Laplace when $\beta^2 = 0$.

2. How does the DC solution slip away?

Here I try to figure out how the constant DC solution "slips away". This led me later to write 7.3.

So why does this constant solution "get lost" in Appendix D? For $\beta = 0$ the solution E_z = constant does satisfy the Helm equation (really Laplace if $\beta^2 = 0$) , and it also satisfies both BC's since $\omega = 0$ goes with $E_r = 0$ and $E_\theta \equiv 0$ makes the other BC just fine. It also satisfies divE = 0.

This missing solution has to be in the m = 0 sector of App D, since we expect no θ asym. That sector has this solution set:

$$E_z(r,0) = (\omega/\sigma)\,(\beta'/k)\,N_0\,\frac{J_0(x)}{J_1(x_a)}$$

$$E_r(r,0) = j\,(\omega/\sigma)\,N_0\frac{J_1(x)}{J_1(x_a)}$$

$$E_\theta(r,0) = 0$$

or

$$E_z(r,0) = (1/\sigma)\,\sqrt{\beta^2-k^2}\,(\omega/k)\,N_0\,\frac{J_0(\sqrt{\beta^2-k^2}\,r)}{J_1(\sqrt{\beta^2-k^2}\,a)}$$

$$E_r(r,0) = j\,(1/\sigma)\,\omega\,N_0\frac{J_1(\sqrt{\beta^2-k^2}\,r)}{J_1(\sqrt{\beta^2-k^2}\,a)}$$

$$E_\theta(r,0) = 0$$

As $\omega \to 0$ we know that $\sqrt{\beta^2-k^2} \to jk$ so we then have

$$E_z(r,0) = (1/\sigma)\,jk\,(\omega/k)\,N_0\,\frac{J_0(jkr)}{J_1(jka)} = (j/\sigma)\,\omega\,N_0\,\frac{J_0(jkr)}{J_1(jka)}$$

$$E_r(r,0) = j\,(1/\sigma)\,\omega\,N_0\frac{J_1(jkr)}{J_1(jka)}$$

$$E_\theta(r,0) = 0$$

Now without doing detail, let's assume that $k \to 0$ in some manner as $\omega \to 0$. Then we can use small args in the Bessel functions and say

$$\frac{J_0(jkr)}{J_1(jka)} = \frac{1}{jka/2}$$

$$\frac{J_1(jkr)}{J_1(jka)} = \frac{jkr/2}{jka/2} = (r/a)$$

We then have

$$E_z(r,0) = (j/\sigma)\,\omega\,N_0\,\frac{J_0(jkr)}{J_1(jka)} = (j/\sigma)\,\omega\,N_0 * \frac{1}{jka/2} = 2\,(1/a\sigma)\,N_0\,(\omega/k)$$

$$E_r(r,0) = (j/\sigma)\,\omega\,N_0\frac{J_1(jkr)}{J_1(jka)} = (j/\sigma)\,\omega\,N_0\,(r/a)$$

$$E_\theta(r,0) = 0$$

Taking the limit now we get

$$E_z(r,0) = 2\,(1/a\sigma)\,N_0\,(\omega/k)$$
$$E_r(r,0) = 0$$
$$E_\theta(r,0) = 0$$

and now we are very close to our hoped-for DC solution, but we have to ponder the ratio (ω/k). We have already assumed that as ω→0 we get k→ 0, but have not specified details of how this happens. \

Question: How do I WANT E_z to come out?

Answer: $E_z = IR_{dc}$

This would require that

$$IR_{dc} = (2/a\sigma)\,N_0\,(\omega/k) = (2/a\sigma)\,q/(2\pi a)\,(\omega/k) = (1/\sigma)\,q/(\pi a^2)\,(\omega/k) = q\,(\omega/k)\,R_{dc}$$

which in turn would require that

$$I = q\,(\omega/k)$$

But why would the total conductor current I be proportional to surface charge q?

Status: I am searching for the missing DC solution. I look in the m = 0 sector, and I am pretty close with the above field set. but I don't know what to say about (ω/k) .

Scenario.

Imagine two parallel conductors which have $R_{dc} = 0$. In this case, V = 0 can drive any current I that you want. You only need a current source. In this case, assuming also that $G_{dc} = 0$, the usual k(ω) formula says that $k = -j\sqrt{(0 + j\omega L)(0 + j\omega C)} = \omega\sqrt{LC} \approx \omega/v_d$. Maybe this is a viable model for the two parallel cylinders. In this case we have exactly $(\omega/k) = v_d$ and then

$$E_z(r,0) = 2\,(1/a\sigma)\,N_0 v_d = (2/a\sigma) * q/(2\pi a) * v_d = q * 1/(\sigma\pi a^2)\,v_d = q\,v_d\,R_{dc}$$
$$E_r(r,0) = 0$$

$E_\theta(r,0) = 0$

In this case we have $J_z(r,0) = q * 1/(\pi a^2)\, v_d$ and then $I = q\, v_d$ just as I found above. But if V = 0, then there would be no surface charge q. Well. perhaps we just assume that $R_{dc} \approx 0$ relative to ωL, though this gets harder as $\omega \to 0$. Could write q = CV and then have

$$E_z(r,0) = C\, v_d\, R_{dc}\, V \qquad I = CV\, v_d. \qquad V = I\,[\,1/(Cv_d)]$$

This last makes it appear that the conductors have $R = 1/(Cv_d)$. The Z_0 formula however gives

$$Z_0 = \sqrt{\omega L/\omega C} = \sqrt{L/C} = \sqrt{LC/C^2} = (1/C)\sqrt{LC} = (1/C)\,(1/v_d) = 1/(Cv_d)$$

in agreement with R just computed. So I seem to have concocted an infinite transmission line which has now a finite current I if you apply a source voltage V and you do this at DC, namely, $I = (Cv_d)\, V$. Of course I have so far only looked at the m = 0 sector.

For $m \neq 0$ we know that

$$E_z(r,m) = (1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

So as done above, let's write $\beta' = \sqrt{\beta^2 - k^2} \to jk$ in the limit $\omega \to 0$ and so again we use the small arg limit of f_m which is $f_m = (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a)$. We then have

$$E_z(r,m) = (1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (a\beta')\, f_m$$

$$= (1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (a\beta')\, (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a)$$

$$= (1/4)\,\eta_m\, 2\pi\omega\, (a/k)\, N_0\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)\,(2)$$

$$= (1/4)\,\eta_m\, 2\pi a\, (\omega/k)\, N_0\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)\,(2)$$

$$= \eta_m\, \pi a\, (\omega/k)\, N_0\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$

But we then have the same (ω/k) factor which we have to set to v_d and then

$$E_z(r,m) = \eta_m\, \pi a\, v_d\, N_0\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$

$$= \eta_m\, \pi a\, v_d * q/(2\pi a) * R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$

$$= (1/2)\,\eta_m\, v_d * q * R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$

$$= (1/2)\,\eta_m\, v_d\, CV\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$

$$= \{(1/2)\,\eta_m\ v_d\,C\,R_{dc}\ (r/a)^{|m|}\ (|m|+1)\}\ V$$

As usual, we end up with J_z asymmetry. Recall from above

$$E_z(r,0) = C\,v_d\,R_{dc}\,V$$

And then

$$E_z(r,m)/\,E_z(r,0) = (1/2)\,\eta_m\ (r/a)^{|m|}\ (|m|+1)$$

and this is the usual result, just appearing now in our special scenario. [so I got nowhere]

Below is another little effort that goes nowhere.

If I assume the usual k(ω) with G > 0, then

$$\mathrm{Re}(k) \approx (\omega/2)\,(R_{dc} + \omega_d L_{dc})\,\sqrt{C/(\omega_d R_{dc})} \ + O(\omega^2) \qquad \omega < \omega_d = (\sigma_d/\varepsilon_d)$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc} C \omega_d}\ [\ 1 + (\tan_I/2)\,(\omega/\omega_d)]\ + O(\omega^2) \qquad G_{dc} = C\omega_d$$

or

$$\mathrm{Re}(k) \approx (\omega/2)\,(R_{dc} + \omega_d L_{dc})\,\sqrt{C/(\omega_d R_{dc})}$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc} C \omega_d}$$

$$k = (\omega/2)\,(R_{dc} + \omega_d L_{dc})\,\sqrt{C/(G_{dc})} \ - j\,\sqrt{R_{dc} G_{dc}}$$

Then

$$(\omega/k) = \omega\ /\ [(\omega/2)\,(R_{dc} + \omega_d L_{dc})\,\sqrt{C/(G_{dc})} \ - j\,\sqrt{R_{dc} G_{dc}}\ ]$$

$$\rightarrow \omega/[\ -j\,\sqrt{R_{dc} G_{dc}}\ ] \ \rightarrow 0$$

and I then get $E_z = 0$. If I assume G = 0, I get instead

$$k = \sqrt{R_{dc} C/2}\ \sqrt{\omega}\ (1-j)$$

and then

$$(\omega/k) = \omega\ /[\sqrt{R_{dc} C/2}\ \sqrt{\omega}\ )(1-j)] \ = \sqrt{\omega}\ /\ [\ \sqrt{R_{dc} C/2}\ )(1-j)] \ \rightarrow 0$$

so in either case I get $E_z = 0$ and the looked-for solution is lost.

Note 1: Something seems wrong with the G > 0 case. There should be some current I for small z, and then it decays away due to the leakage across the dielectric. So I expect to see $E_z(r,0,z)$ be some finite value at z = 0 and then decay away over distance related to $1/\sqrt{R_{dc}C\omega_d}$. This is all at ω = 0. But instead I am seeing $E_z \equiv 0$ at all z due to the (ω/k) factor. [but in 7.2 3 I show $I = V/Z_0 = V\sqrt{G/R}$ which is a fine finite current.]

[so I got nowhere again]

---

3. Superposition and J_z Asym.

So somehow the above expression must have in it somewhere the solution of the two wires with its symmetric J_z. But regardless of the choice of r_k, we can see that

$$\frac{E_z(r,m,z)^{(3)}}{E_z(r,0,z)^{(3)}} = \frac{\int dk\, e^{-jkz}\, (1/4)\, \eta_m\, 2\pi\omega\, (a/k)\, N_0\, r_k\, R_{dc}\ (a\beta')\, f_m(\beta' r)}{\int dk\, e^{-jkz}\, (1/4)\, \eta_0\, 2\pi\omega\, (a/k)\, N_0\, r_k\, R_{dc}\ (a\beta')\, f_0(\beta' r)}$$

$$= \eta_m \frac{\int dk\, e^{-jkz}\, (1/k)\, r_k\, (\beta')\, f_m(\beta' r)}{\int dk\, e^{-jkz}\ \ (1/k)\, r_k\, (\beta')\, f_0(\beta' r)}$$

In order to achieve a uniform r(z) [which we expect for the two long parallel wires near DC] , we know that we must approach $r_k = \delta(k)$ since that turns off the z exponential and since r(z) and r_k are Fourier transforms. If k is small, then $\beta' = \sqrt{\beta^2 - k^2}$ is uncertain, because we are also interested in $\omega \to 0$ which causes $\beta \to 0$. In any event, it does seem that $|\beta'| \to 0$ in this combination of limits, and then we should be justified in using the small arg limits of the f functions, which from D.11 are

$$f_m = (r/a)^{|m|}\, (|m|+1)\, (2/\beta' a) \qquad f_0 = 4/(a\beta')$$

and then the above ratio becomes

$$\frac{E_z(r,m,z)^{(3)}}{E_z(r,0,z)^{(3)}} = \eta_m \frac{\int dk\, e^{-jkz}\, (1/k)\, r_k\, (\beta')\, (r/a)^{|m|}\, (|m|+1)\, (2/\beta' a)}{\int dk\, e^{-jkz}\ \ (1/k)\, r_k\, (\beta')\, 4/(a\beta')}$$

$$= \eta_m \frac{\int dk\, e^{-jkz}\, (1/k)\, r_k\ (r/a)^{|m|}\, (|m|+1)\, (2)}{\int dk\, e^{-jkz}\ \ (1/k)\, r_k\ 4}$$

where we are thinking low ω and low-skewed r_k. Interestingly perhaps, if we set $r_k = \delta(k)$

[nothing new here!]

---

4. Digression on Atoms in Spherical Coords

: Look at the spherical harmonics instead. For the Laplace equation, the harmonics are these (MSp26)

$[\, r^{\ell}, r^{-\ell-1}], P_{\ell}^{m}(\cos\theta), e^{jm\varphi}$ // ignore $r^{-\ell-1}$ if origin is in region of interest

If we want to find a constant in these atoms, we set m = 0 and $\ell = 0$ and we get

$r^{\ell}\ P_{\ell}^{m}(\cos\theta), e^{jm\varphi} \rightarrow 1$

For the scalar Helm equation, however, we get instead,

$[\, j_{\ell}(kr)]\ P_{\ell}^{m}(\cos\theta)\ e^{jm\varphi}$

So in spherical coordinates, in looking for a constant solution, we would have m = 0 and $\ell = 0$ and then we get

$j_{\ell}(kr)P_{\ell}^{m}(\cos\theta)\ e^{jm\varphi} \rightarrow j_0(kr)$

and again, this is NOT a constant solution. But you don't *expect* $(\nabla^2 + k^2)u = 0$ to have a constant solution! That is only for Laplace.

There is something general Sturm-Liouville going on here. Recall the you have two oscillatory and one expo atom. For the spherical coordinates case, it seems that $j_{\ell}(kr)$ is the expo factor! But the plots of these functions look oscillatory. The large r behavior is this,

$$j_n(z) = z^{-1}\sin(z - \tfrac{1}{2}n\pi) + e^{|\Im z|}\,O(z^{-2}),$$

and if k is complex, as in our example above, we have

$j_{\ell}(kr) \sim\ e^{|Im(k)|r}/(kr)^2$ = blows up exponentially, just as r^{ℓ} blows up.

---

5. Try starting in App D with ω = 0.

Plan A. Go through the development of Appendix D at ω = 0 from the start. This means β = 0, and that means that you have a vector Laplace equation instead of a vector Helmholtz equation. Go ahead and work with some particular value of k. The E_z equation will have a solution I guess with partial wave amplitude E_z = constant * r^m . Then as before, use divE = 0 and do all the motions. The goal here is to

see if you can come up with J_z asymmetry directly at $\omega = 0$ without having to take a limit $\omega \to 0$. In the end you can take $k \to 0$ I guess. I have done all this with $\omega \neq 0$ and get $J_m(\beta' r)$ functions, I am just saying "let's try this at $\omega = 0$ from the start". I am of course looking for some mechanism that creates the asym J_z. Now with $R_{dc} = 0$ and $G_{dc} = 0$ I have a scenario that has finite current, and I will get J_z asym right in that scenario! Do this tomorrow. The details of all these appear in "show that linear....". Maybe I will find out that at DC things really are asym and this is a feature of the vector Laplace equation.

9/19/14. I will now carry out Plan A, but I doubt it will reveal anything. If I set $\omega = 0$ at the start, that just means that $\beta' = \sqrt{\beta^2 - k^2} = \sqrt{-k^2} = jk$ (assuming that sign). Everything in App D then goes through exactly the same and we end up with

First summary of the E field solutions (D.2.21)

$$E_z(r,m) = -j(\beta'/k)\frac{K_m}{2} J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x) \qquad \beta'^2 = jk \qquad (D.2.11)$$

$$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad (D.2.15)$$

We then come to the boundary conditions which are now these:

$E_r(r=a,m) = 0$ // since $\omega = 0$
$E_\theta(r=a,m) = 0$

We can use equations (3) and (4) below (D.2.27) which now say

$$\frac{K_m}{2} J_{m+1}(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a) = 0 \qquad (3)$$

$$2 a_m x_a^{-1} J_m(x_a) - \frac{a_m}{m} J_{m+1}(x_a) = 0. \qquad (4)$$

The second equation says $a_m = 0$ for some general value of $x_a = jka$ and general m. Then the first equation says that $K_m = 0$ as well. Very interesting. The conclusion is then

$E_z(r,m) = 0$
$E_r(r,m) = 0$
$E_\theta(r,m) = 0$

so we get no solution at all by assuming $\omega = 0$ and $k \neq 0$. Great. [same old problem that coeffs vanish]

6. Precursor of Section 7.3 on getting the DC solution.

The following is what let to my "proof" of the true DC solution in 7.3.

Suppose we assume both ω and k are 0 at the start, which might be more accurate. In this case, β' = 0 and our equation set is this:

$[\nabla^2\mathbf{E}]_z + 0\ E_z = 0$:

$[r^2\partial_r^2 + r\,\partial_r - m^2\ ]\ E_z(r,m) = 0$ *(D.1.15)*

$[\nabla^2\mathbf{E}]_r + 0\ E_r = 0$:

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\ E_r(r,m) - 2jm\ E_\theta(r,m) = 0$ *(D.1.17)*

$[\nabla^2\mathbf{E}]_\theta + 0\ E_\theta = 0$:

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\ E_\theta(r,m) + 2jmE_r(r,m) = 0$ *(D.1.18)*

div $\mathbf{E} = 0$:

$\partial_r\ [r\ E_r(r,m)] + jmE_\theta(r,m) = 0$ *(D.1.19)*

The acceptable solution to the first equation is this, where for the moment I assume $m \geq 0$:

$E_z(r,m) = A_{zm}\ r^m$ $\qquad r^2m(m-1)r^{m-2} + r\ mr^{m-1} - m^2r^m = r^m[\ m^2 - m + m - m^2] = 0$

As usual, we solve the last for jmE_θ and insert that into the second equation to get

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\ E_r(r,m) + 2\partial_r\ [r\ E_r(r,m)] = 0$

or

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\ E_r + 2\ \{\ r\ \partial_r E_r + E_r\} = 0$

$[r^2\partial_r^2 + 3r\partial_r - (m^2-1)]\ E_r = 0$

Maple shows that the solutions here are r^{m-1} and r^{-m-1} :

```
eq := r^2*diff(f(r),r,r) + 3*r*diff(f(r),r) - (m^2-1)*f(r) = 0;
```

$$eq := r^2\left(\frac{\partial^2}{\partial r^2}f(r)\right) + 3r\left(\frac{\partial}{\partial r}f(r)\right) - (m^2-1)\,f(r) = 0$$

```
dsolve(eq);
```

$$f(r) = _C1\ r^{(m-1)} + _C2\ r^{(-m-1)}$$

This seems then to say that, for $m \geq 1$

$E_r(r,m) = A_r\ r^{m-1}$

For $m = 0$ our second equation is really

$[r^2\partial_r^2 + r\partial_r - 1]\ E_r(r,0) = 0$

But the two solutions here according to Maple are (r+1/r) and (r-1/r),

```
eq := r^2*diff(f(r),r,r) + r*diff(f(r),r) - f(r) = 0;
```

$$eq := r^2\left(\frac{\partial^2}{\partial r^2} f(r)\right) + r\left(\frac{\partial}{\partial r} f(r)\right) - f(r) = 0$$

```
dsolve(eq);
```

$$f(r) = \frac{_C1\,(r^2 - 1)}{r} + \frac{_C2\,(r^2 + 1)}{r}$$

but both these diverge at r = 0. This would seem to imply that $E_r(r,0) = 0$! What did I get in App D?

$$E_r(r,m) = a_m\ x^{-1}\ J_m(x) + \frac{K_m}{2}\ J_{m+1}(x)$$

$$E_r(r,0) = a_0\ x^{-1}\ J_0(x) + \frac{K_0}{2}\ J_1(x) = \frac{K_0}{2}\ J_1(x) \approx \frac{K_0}{2}\ J_1(0) = 0$$

so at least things are consistent. So far then I have

$$E_z(r,m) = A_{zm}\ r^m \qquad m \geq 0$$

$$E_r(r,m) = A_{rm}\ r^{m-1} \qquad m \geq 1$$
$$E_r(r,0) = 0 \qquad m = 0$$

For $m \geq 1$ the div E says, [no z term since k = 0]

$$jmE_\theta(r,m) = -\partial_r\,[r\,E_r(r,m)]$$

$$= -\partial_r\,[r\,A_{rm}\,r^{m-1}] = -A_{rm}\,\partial_r\,[r^m] = -m\,A_{rm}\,r^{m-1}$$

which tells us that

$$jE_\theta(r,m) = -A_{rm}\,r^{m-1}\ .$$

For m = 0 the third equation for E_θ is the same as the second equation for E_r when m = 0, but for this latter we had $E_r = 0$. So here then is our summary

$$E_z(r,m) = A_{zm}\ r^m \qquad m \geq 0$$

$$E_r(r,m) = A_{rm}\ r^{m-1} \qquad m \geq 1$$
$$E_r(r,0) = 0 \qquad m = 0$$

$$jE_\theta(r,m) = -A_{rm}\ r^{m-1} \qquad m \geq 1$$
$$E_\theta(r,0) = 0 \qquad m = 0$$

Now we come to the BC's. They are

$E_r(r=a,m) = 0$ // since $\omega = 0$
$E_\theta(r=a,m) = 0$

For m = 0, since E_r and E_θ are already 0, these BC's are met.

For $m \geq 1$ we get

$A_{rm}\, a^{m-1} = 0 \quad \Rightarrow \quad A_{rm} = 0$
$A_{rm}\, a^{m-1} = 0 \quad \Rightarrow \quad A_{rm} = 0$

Our solution then seems to be

$E_z(r,m) = A_{zm}\, r^m \quad m \geq 0$
$E_r(r,m) = 0 \quad m \geq 0$
$E_\theta(r,m) = 0 \quad m \geq 0$

I think this is a completely new result!. I guess we could integrate to get for m = 0 we get

$$\int dA\, J_z(r,0) = \sigma A_{z0} \int dA = \sigma A_{z0}\, \pi a^2$$

So we have

$$I = \sigma A_{z0}\, \pi a^2 \quad \Rightarrow \quad A_{z0} = I/(\sigma \pi a^2) = I R_{dc}$$

The solution is then

$E_z(r,m) = A_{zm}\, r^m \quad m > 0$
$E_z(r,0) = I R_{dc} \quad m = 0$

$E_r(r,m) = 0 \quad m \geq 0$
$E_\theta(r,m) = 0 \quad m \geq 0$

There is nothing to determine the coefficients A_{zm} for $m \geq 1$!!!

Let's just make sure the 3rd Helm equation is OK.

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\, E_\theta(r,m) + 2jmE_r(r,m) = 0$$

For m > 0 we get

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\, 0 + 2jm\, 0 = 0$ OK

For m = 0 we get

$$[r^2\partial_r^2 + r\partial_r - (0^2+1)]\ 0\ + 2j0\ 0 = 0 \qquad \text{OK}$$

Somehow, it seems that when you solve the problem directly at DC, the problem is underspecified because nothing determines the A_{zm} for m > 0. They could be any numbers you want, and your solution still solves the 3 Helm equations, the div E = 0 equation, and the two BC's. So this equation set seems to be incomplete!

So let's go back to the Maxwell equations. We need this to be respected:

$$\text{curl } \mathbf{E} = -\partial_t \mathbf{B} = 0$$

Start with (taken from "the curl E" doc,

$$\text{curl } \mathbf{E}(r,\theta,z) = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\, r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\,[-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,]$$

Now set k = 0 to get

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\, r^{-1}jmE_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,]$$

But only the E_z field is non-zero so we get

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\, r^{-1}jmE_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r E_z]$$

So we have two equations of interest here

$$r^{-1}jmE_z = 0$$
$$-\partial_r E_z = 0$$

Inserting our E_z we get

$$r^{-1}jm\,A_{zm}\,r^m = 0$$
$$-\partial_r\,A_{zm}\,r^m = 0$$

or

$$m\,A_{zm}r^m = 0$$
$$-mA_{zm}r^{m-1} = 0$$

These both say the same thing

$$m\,A_{zm} = 0$$

and THIS then sets all the other coefficients to 0 !!! Then the problem is determined!

Here I discover that you have to then use curlE = 0 to nail down your DC solution, and this is duly carried out now in Section 7.3.

Fact: The set of equations Helm + div + BC's is incomplete at ω = 0 ! You need to add curl E = 0 to that set. When you do this, you obtain the classical result that J_z = constant on the cross section, so there is in fact no J_z asymmetry!

For ω > 0, the curl E equation is used to find B !

Hypothesis: For ω > 0, the condition corresponding to curl E = 0 for ω = 0 is that B be finite!

7. On the infinite B problem

So now I am off on the issue of understanding how I should be getting curl E = 0 as ω→ 0. I know about the little "theorem", but I did not know much about the B fields at this time.

Now, for ω > 0 and k > 0 I suspect the problem is *over*-specified! Ignoring the curl E equation, we obtain these fields if we assume that both ω and k are very small at the same time,

$$\begin{aligned} E_z(r,m) &= (1/2)\,\eta_m\, I\, R_{dc}\ (r/a)^{|m|}\ (|m|+1) \qquad m>0 \\ E_r(r,m) &= (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)} \\ E_\theta(r,m) &= (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}] \end{aligned}$$

We then find (see curl E doc) that

$$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1} jm E_z + jk E_\theta$$

$$= j\,(1/4)\,\eta_m\, R_{dc}\ \{\ 2\, I\, r^{-1} m\ (r/a)^{|m|}\ (|m|+1) + a\, k^2 I\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\ \}$$

where $I = 2\pi\omega\,(a/k)\,N_0 = 2\pi a\,(\omega/k)\,N_0$ and $N_0 = q/(2\pi a) = CV/(2\pi a)$. The second term is never a problem, we can just set it to zero based on k→0, or more carefully for $G = 0$ $k^2 \sim \omega$ and $I \sim \sqrt{\omega}$. It is the first term that is the problem because

$$B_r(r,m) = +(j/\omega)\,[\text{curl } E]_r = (j/\omega)\, j\,(1/4)\,\eta_m\, R_{dc}\ \{\ 2\, I\, r^{-1} m\ (r/a)^{|m|}\ (|m|+1)$$

Since $I = 2\pi\omega\,(a/k)\,N_0 = 2\pi a N_0\, e^{+j\pi/4}\sqrt{\omega}\,/\sqrt{R_{dc}C}$ we then get $B_r \to \infty$.

Escape Hatch: The notion that $k \sim \sqrt{\omega}$ must be wrong!

My Theorem 1 says that if $[\text{curl } E]_i \rightarrow \omega^s [\text{stuff}]_i$, then we must have $s \geq 1$ to keep B finite. My problem above is that I have $s = 1/2$. The only way out is if this is true:

$$(\omega/k) \rightarrow \omega^s \text{ with } s \geq 1$$

Now suppose $k \rightarrow \omega^r$. Then we have $s = 1\text{-}r$ so we need $1\text{-}r \geq 1$ which means $-r \geq 0$ or $r \leq 0$. But cannot have $r < 0$ since then k blows up which seems completely wrong since we expect no z dependence. Thus we must have $r = 0$.

Fact: if $k \rightarrow$ small constant as $\omega \rightarrow 0$, then B_r will remain finite. [but see Ch 7]

I think that the TL equations are *wrong* as $\omega \rightarrow 0$, but I don't know how to exactly correct them. But whatever that correction is, it must result in $k \rightarrow$ small constant as $\omega \rightarrow 0$. But this in turn says there is some residual z dependence, and that seems wrong!

Fact 4: The small ω limit for $k(\omega)$, assuming $\omega_d = 0$, is given by (Q.4.9)

$$\text{Re}(k) \approx +\sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (1 - \tan_L/2) + O(\omega^{3/2})$$
$$\text{Im}(k) \approx -\sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (1 + \tan_L/2) + O(\omega^{3/2})$$

Fact: If you say $k \rightarrow 0$, that in itself is a contradiction because it implies no decay since $\text{Im}(k) = 0$. But that can only happen in a completely lossless line. So yes, there *is* z dependence at $\omega = 0$ for a real TL. Perhaps the existing imaginary k causes some small real k phase action.

How then can the above limit be right? If $G = 0$, then $I = 0$ and there is no loss, correct.

The $G > 0$ model seems to work better in this sense. We have

Fact 3: The small ω limit for $k(\omega)$, assuming $\omega_d > 0$, is given by (Q.4.6)

$$\text{Re}(k) \approx (\omega/2)\ (R_{dc} + \omega_d L_{dc})\ \sqrt{C/(\omega_d R_{dc})} + O(\omega^2) \qquad \omega < \omega_d = (\sigma_d/\varepsilon_d)$$
$$\text{Im}(k) \approx -\sqrt{R_{dc}C\omega_d}\ [\ 1 + (\tan_L/2)\ (\omega/\omega_d)] + O(\omega^2) \qquad G_{dc} = C\omega_d$$

Then as $\omega \rightarrow 0$ this says

$$\text{Re}(k) = 0 \qquad \Rightarrow \text{ no wave action}$$
$$\text{Im}(k) = -\sqrt{R_{dc}C\omega_d} \qquad \Rightarrow \text{ there is loss going down the line}$$

Also, we end up with B_r = finite! Let's write

$$k = -j\sqrt{R_{dc}G_d} \qquad \text{in the limit } \omega \rightarrow 0. \qquad k^2 = -R_{dc}G_{dc}$$

Then the above says

$I = 2\pi\omega a N_0 / [-j\sqrt{R_{dc}G_d}\,]$

Then go back to

$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + jkE_\theta$

$= j\,(1/4)\,\eta_m\,R_{dc}\,\{\,2\,I\,r^{-1}m\,(r/a)^{|m|}\,(|m|+1) + a\,k^2 I\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\}$

$= j\,(1/4)\,\eta_m\,R_{dc}\,2\pi\omega a N_0 / [-j\sqrt{R_{dc}G_d}\,]\{\,2\,r^{-1}m\,(r/a)^{|m|}\,(|m|+1) - a\,R_{dc}G_{dc}\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\}$

Then divide by jω to get B_r

$B_r = (1/4)\,\eta_m\,R_{dc}\,2\pi a N_0 / [-j\sqrt{R_{dc}G_d}\,]\{\,2\,r^{-1}m\,(r/a)^{|m|}\,(|m|+1) - a\,R_{dc}G_{dc}\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\}$

which is valid only for $m \neq 0$. If $m = 0$ we have instead

$E_z(r,0) = I\,R_{dc}$
$E_r(r,0) = (j/2)\,I\,R_{dc}\,(ak)\,(r/a)$
$E_\theta(r,0) = 0$ // low ω E fields

We then find (see curl E doc) that

$[\text{curl } \mathbf{E}(r,0,z)]_r = r^{-1}j0E_z + jkE_\theta = 0 + 0 = 0$ => $B_r(m=0) = 0$.

Other components?

$[\text{curl } \mathbf{E}(r,0,z)]_\theta = -jkE_r - \partial_r E_z$

$= -jk\,(j/2)\,I\,R_{dc}\,(ak)\,(r/a) - \partial_r\,I\,R_{dc}$

$= -jk_1^2\,(j/2)\,I\,R_{dc}\,r$

$= -jk_1^2\,(j/2)\,2\pi a\,(\omega/k)\,N_0\,R_{dc}\,r$

$= -j\omega\,k_1\,(j/2)\,2\pi a\,N_0\,R_{dc}\,r$

Then get

$B_\theta = -\,k_1\,(j/2)\,2\pi a\,N_0\,R_{dc}\,r = -\,[-j\sqrt{R_{dc}G_d}\,]\,(j/2)\,2\pi a\,N_0\,R_{dc}\,r$

$= [j\sqrt{R_{dc}G_d}\,]\,(j/2)\,2\pi a\,N_0\,R_{dc}\,r$

$= - [\sqrt{R_{dc}G_d}\,]\pi a\, N_0\, R_{dc}\ r$

$= - [\sqrt{R_{dc}G_d}\,]\pi a\, q/2\pi a\, {*}R_{dc}\ r$

$= - [\sqrt{R_{dc}G_d}\,](q/2){*}R_{dc}\ r$

$= - [\sqrt{G_{dc}/R_{dc}}\,](q/2)\ r$

But $Z_0 = \sqrt{R_{dc}/G_{dc}}$ at $\omega = 0$, so the above says

$B_\theta = - [1/Z_0\,]\,(1/2)\,CV\,r$

$B_\theta = - I_{dc}(1/2)\,C\,r$

Needs lost of work!

Well, I guess the time has come to do all the B field limits and check them, probably with Maple. I think this last one above is OK and what you expect inside a wire, perhaps get Z_0 and $q = CV$ etc.

At this time, I had not really computed the B field components in useful detail, but all that later got added to lines doc, and we now how the infinite B section in Chapter 4.