**Another Relaxed Pondering of Appendix D $J_z$ Asymmetry** PhL 8.12.14

Reviewing this doc on 9.10.14


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## 1. The High ω explanation of why $J_z$ tracks n(θ)

My most basic explanation linking $J_z$ to n(θ) appears in Section 6.5 (d).

Question: why do I replace $\partial_z$ by $\beta_{d0}$ here? As shown in (1.5.1b), this is just $\beta_d$ in the special case that G = 0 which means $\sigma_d = 0$. Fine. And

$$\beta_{d0}^2 = \omega^2 \mu_d \varepsilon_d$$

But I think really I should have $\partial_z \rightarrow$ k, not $\rightarrow \beta_{d0}$ since the ansatz z dependence is k.

$$jk = a + jb = \sqrt{zy} = \sqrt{(R+j\omega L)(G+j\omega C)}$$

so then

$$k = -j\sqrt{(R+j\omega L)(j\omega C)}$$ if G = 0, this is NOT the same as $\beta_{d0} = \omega/v_d$.

I am making no assumptions about ω here, so I guess I better fix things up:

For r = a we know from (3.7.0) that $E_\theta(a,\theta) = 0$ and $\partial_\theta E_\theta(r,\theta) = 0$, so for r just below the surface we expect

$$\partial_r (r\, E_r(r,\theta,z)) + r\, \partial_z E_z(r,\theta,z) \approx 0 \qquad \text{// near r = a}$$

$$\partial_r (r\, E_r(r,\theta)) - jk\, r\, E_z(r,\theta) \approx 0 \qquad \text{// using } \partial_z \rightarrow -jk, \text{ see (D.1.16), then cancel } e^{jkz} \text{ factors}$$

$$E_z(r,\theta) \approx (1/jk)\,(1/r)\,\partial_r (r\, E_r(r,\theta)) \; . \qquad \text{// near r = a} \qquad (6.5.15)$$

BUT, I have at the start specified the "skin effect regime", so in that case yes k = $\beta_{d0}$ as shown in (Q.2). So that rescues things at least for large ω.

OK, I read the section and I agree that it shows that $E_z(r,\theta)$ "tracks" n(θ) in the high ω limit.

[ I think the above section is OK ]

**2. So OK, what happens for small ω to this same argument?**

I think the argument begins the same way and we get as shown above

$$E_z(r,\theta) \approx (1/jk)\,(1/r)\,\partial_r\,(r\,E_r(r,\theta))$$

and then

$$E_z(r,m) \approx (1/jk)\,(1/r)\,\partial_r\,(r\,E_r(r,m))$$

$$= (1/jk)\,(1/r)\,\partial_r\,(r\,[(j/4)\,\eta_m\,I\,R_{dc}\,(ak)\,g_{m]})$$

$$= (1/jk)\,(j/4)\,\eta_m\,I\,R_{dc}\,(ak)\,(1/r)\,\partial_r\,(r\,g_m(r))$$

$$= (1/4)\,\eta_m\,I\,R_{dc}\,(a)\,(1/r)\,\partial_r\,(r\,g_m(r))$$

Now comes the asymmetry ratio

$$E_z(r,m)/\,E_z(r,0) = \eta_m\,\partial_r\,(r\,g_m(r))\,/\,\partial_r\,(r\,g_0(r))$$

I could compute this stuff for general ω, but instead lets go to small ω where

$$g_m = (r/a)^{m+1} + (r/a)^{m-1} \qquad g_0 = 2\,(r/a)$$

Let x = r/a for the moment. Then

$$\partial_r = \partial/\partial r = \partial/\partial x^*\,\partial x/\partial r = \partial_x * (1/a) = (1/a)\partial_x$$

$$\partial_r(r\,F(x)) = F(x) + r\,\partial_r\,F(x) = F(x) + r\,(1/a)\partial_x\,F(x) = F(x) + x\,\partial_x F(x)$$

Then we have

$$\partial_r(r\,g_m(x)) = g_m(x) + x\,\partial_x\,g_m(x)$$

$$= [x^{m+1} + x^{m-1}] + x\partial_x[x^{m+1} + x^{m-1}]$$

$$= [x^{m+1} + x^{m-1}] + x[\,(m+1)x^m + (m-1)x^{m-2}]$$

$= [x^{m+1} + x^{m-1}] + [(m+1)x^{m+1} + (m-1)x^{m-1}]$

$= [(m+2)x^{m+1} + (m)x^{m-1}]$

Next,

$\partial_r(r\, g_0(x)) = g_0(x) + x\, \partial_x g_0(x)$

$= (2x) + x\, \partial_x(2x) = 2x + 2x*1 = 4x$

The asym ratio is then

$E_z(r,m)/ E_z(r,0) = \eta_m \partial_r (r\, g_m(r)) / \partial_r (r\, g_0(r))$

$= [(m+2)x^{m+1} + (m)x^{m-1}] / 4x$

$= (1/4)[(m+2)x^m + (m)x^{m-2}]$

$= (1/4)[(m+2)(r/a)^m + (m)(r/a)^{m-2}]$

For small ω, then, this ratio does not vanish and thus we have asym $J_z$. For r ≈ a which is where the path above is justified, we then get

$= (1/4)[(m+2) + (m)] = (1/4)(2m+2)$

so the argument again leads to $J_z$ asym.

[ I agree, my theory does lead to $J_z$ being asym at low ω ]

**3. Do it again keeping $E_\theta$ stuff.**

Start with

$\partial_r (r\, E_r(r,\theta,z)) + \partial_\theta E_\theta(r,\theta,z) + r\, \partial_z E_z(r,\theta,z) = 0$

Go to partial waves and use the $\partial_z$ form as above,

$\partial_r (r\, E_r(r,m,z)) + jmE_\theta(r,\theta,z) - rjk\, E_z(r,m,z) = 0$

Cancel the $e^{-jkz}$ factors to get

$\partial_r (r\, E_r(r,m)) + jmE_\theta(r,m) - rjk\, E_z(r,m) = 0$

Solve for $E_z$

$$rjk\, E_z(r,m) = \partial_r (r\, E_r(r,m)) + jmE_\theta(r,m)$$

Install the general solutions to get

$$rjk\, E_z(r,m) = \partial_r (r\, (j/4)\, \eta_m\, I\, R_{dc}\, (ak)\, g_m) + jm(1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, h_m$$

$$= \eta_m\, I\, R_{dc}\, (ak)\, (j/4)\, \partial_r (r\, g_m) + jm(1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, h_m$$

$$= (j/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [\, \partial_r (r\, g_m) + m\, h_m\, ]$$

Thus [ here I divide both sides by k and my limit of interest is $k \rightarrow 0$ ]

$$r\, E_z(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\, (a)\, [\, \partial_r (r\, g_m) + m\, h_m\, ]$$

The ratio now is a little more complicated than before

$$E_z(r,m)/\, E_z(r,0) = \eta_m\, [\, \partial_r (r\, g_m) + m\, h_m\, ] / [\, \partial_r (r\, g_0) + m\, h_0\, ]$$

Above I already computed these two items

$$\partial_r(r\, g_m(x)) = [\, (m+2)\, x^{m+1} + (m)\, x^{m-1}]$$

$$\partial_r(r\, g_0(x)) = 4x$$

Meanwhile, we know that

$$h_m = (r/a)^{m+1} - (r/a)^{m-1} \qquad h_0 = 0$$

So our ratio is then

$$E_z(r,m)/\, E_z(r,0) = \eta_m\, [\, \partial_r (r\, g_m) + m\, h_m\, ] / [\, \partial_r (r\, g_0)\, ]$$

$$= \eta_m\, [\, (m+2)\, x^{m+1} + (m)\, x^{m-1} + m\, ((r/a)^{m+1} - (r/a)^{m-1})] / [4x\, ]$$

$$= \eta_m\, [\, (m+2)\, x^{m+1} + (m)\, x^{m-1} + m\, (x)^{m+1} - mx^{m-1})] / [4x\, ]$$

$$= \eta_m\, [\, (2m+2)\, x^{m+1} + 0] / [4x\, ]$$

$$= \eta_m\, [\, (2m+2)\, x^m\, ]$$

Thus, we <u>still have the asym keeping both terms</u>. And for $x \approx 1$, this replicates the ratio above.

Remember that I want this ratio to $\rightarrow 0$ as $\omega \rightarrow 0$.

[ Above I show that keeping $E_\theta$ in the "divE = 0" argument does not fix $J_z$ asym at $\omega \rightarrow 0$. ]

**4. A Little Review of Appendix D's Verification of things. [ 8/13/14]**

I start off in Appendix D by solving a set of four equations: three Helmholtz for $E_i$ and div E = 0. The solutions for $E_i$ are stated in (D.2.21) in terms of TBD $K_m$ and $a_m$ constants. Are these the most general possible solutions, or have I omitted possible solutions?

I start with the homogeneous $E_z$ equation and reject the $Y_m$ solution since require finite at r = 0 in the wire, so (D.2.4) has only $E_z(r,m) = C_{zm} J_m(\beta' r)$. Then for $E_r$ I get a particular solution and to that I add a homo solution again only with J and not Y. Thus we have (D.2.15). The $E_\theta$ solution first comes from solving div E = 0 and that solution is (D.2.19) and there is no freedom in this solution, no Y that gets thrown out. I then summarize the three solutions in (D.2.21).

At this point, I use Maple to verify that the (D.2.21) solutions do in fact solve the four starting equations. I just updated this code replacing $\beta_d$ by k yesterday. The verification shows that I have at least found "a" solution set for the problem. I later comment on the $E_\theta$ Helmholtz equation to say that it may not be independent of the other 3 equations. I then just note that Maple showed that my obtained solution for $E_\theta$ did cause all four equations to be solved. Probably I could have started off with the θ equation in place of the r equation and obtained the same solution, always rejecting any Y type solution. Note that even $Y_0$ blows up at r = 0.

So at this point I have (D.2.21) and have shown that they solve the 4 equations for any choice of the constants $a_m$ and $K_m$. The next topic is the two boundary conditions, and I reviewed the CPBC yesterday and found it to be good for G = 0. One would think the $E_\theta = 0$ condition would be very good for ω = 0! I then do some very messy algebra to get expressions for $a_m$ and $K_m$ as in (D.2.28). These coefficient solutions involve moments $N_m$ since $N_m$ appears in the CPBC (D.2.25). I then assume ohm's law and integrate $J_z$ to get I, and this results in $N_0 = (k/2\pi\omega a) I$ as in (D.2.31). After more messy algebra, I end up with the "famous" solutions shown in (D.2.33) which involve the $f_m$ type functions. Comparing to (D.2.21) you see that the $a_m\ x^{-1}\ J_m(x) + \frac{K_m}{2}\ J_{m+1}(x)$ combination for $E_r$ now appears as a linear combination of $J_{m+1}(x)$ and $J_{m-1}(x)$. My next task is to verify that this new solution set solves the four equations and I do that in line. While there, I also verify that the two BC's are met. This code was updated a little bit today.

So at this point I know the following: (D.2.33) for $E_i$ solve the three Helmholtz equations, the div E = 0 equation, and satisfy the two boundary conditions. I think that this is the most general solution that works. Could I add constants to these solutions? That would be OK for div E = 0, but would not be OK for the three Helm equations! Have I omitted some other solutions? I don't think so.

My next big step is to compute the B fields using the curl E equation. I am changing the code here and this will require repagination of App D after (D.5.4), a price I can pay. Then staying in the same Maple program, I really do compute the "other three" Maxwell equations and show they are all satisfied by the solution set (D.4.9). I directly verify each of these three Maxwell equations, no question about it!!

At this point, I now know that my solution set for the E and B fields has this property:

They satisfy: div**E** = div**B** = 0, curl **E** = -jω**B** and curl **B** = j ($\beta^2/\omega$) **E**

($\nabla^2+\beta^2$) **E** = 0, ($\nabla^2+\beta^2$) **B** = 0 , + the two boundary conditions

I did not verify that ($\nabla^2+\beta^2$) **B** = 0 but I can prove that right here

curl **B** = j ($\beta^2/\omega$) **E**

curl curl **B** = j ($\beta^2/\omega$) curl **E** = j ($\beta^2/\omega$) [-jω**B**] = $\beta^2$[**B**]

$\nabla(\nabla\bullet B) - \nabla^2\mathbf{B} = \beta^2[\mathbf{B}]$

$-\nabla^2\mathbf{B} = \beta^2[\mathbf{B}]$ => $(\nabla^2+\beta^2)\,\mathbf{B} = 0$

I might at this statement into App D at some point: *******

" The solution set (D.4.9) satisfies the 4 Maxwell equations, the 6 Helmholtz equations, and the 2 boundary conditions. "

This concludes my little review of Appendix D's computation of the E and B fields, and while doing it I updated the code with $\beta_d$ replaced by k.

[ the above discussion seems correct on 9.10.14 ]

**4. What happens to $a_m$ and $K_m$ as ω→ 0 ?**

I have never really studied what the B fields do. In (D.4.9) there are many pieces to worry about. I will do it all right here. First, the coefficients:

$$a_m = (j/4)\,(ak)\,\eta_m\, I\, R_{dc} \;*\; 2m\,\frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = (j/4)\,(ak)\,\eta_m\, I\, R_{dc} \;*\; \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = (j/4)\,(ak)\,\eta_m\, I\, R_{dc} \;*\; \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] \quad .$$

For ω→0 I expect small $x_a$ and then from Section D.11:

First $\frac{K_m}{2}$ for m > 0, and then for m = 0: [ omitting lead factors for the moment ]

$$\frac{K_m}{2} = \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] = \left[\frac{1}{(x_a/2)^{m+1}/(m+1)!} - \frac{1}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= (m+1)!\;(x_a/2)^{-m-1} - (m-1)!\;(x_a/2)^{-m+1}$$

$= (m-1)!\,(x_a/2)^{-m-1}\ [\,(m+1)\,m\ -\ (x_a/2)^2\ ]$

$\approx (m-1)!\,(x_a/2)^{-m-1}\ (m+1)\,m\ = (m+1)!\,(x_a/2)^{-m-1}\ =$ blows up! (2nd term nada)

$\frac{K_0}{2} = 2/J_1(x_a) = 2\,/\,[\,x_a/2] = 4/x_a =$ blows up

First $a_m$ for m > 0, and then for m = 0: [ omitting lead factors for the moment ]

$a_m = 2m\,/\,J_{m-1}(x_a) = 2m\,/\,[(x_a/2)^{m-1}\,/\,(m-1)!\,] = 2m\,(x_a/2)^{-m+1}\,(m-1)!$

$a_0 = 2m\,/\,[-x_a/2] = -2m\,(x_a/2)^{-1}$

This wants to blow up, but m = 0 seems top stop it and we get $a_0 = 0$ as elsewhere.

First $(\frac{K_m}{2} + \frac{a_m}{m})$ for m > 0, and then for m = 0: [ omitting lead factors for the moment ]

This is the same as for $\frac{K_m}{2}$ except there is a minus sign between first and second terms, at least for m>0. Thus

$(\frac{K_m}{2} + \frac{a_m}{m}) = \frac{K_m}{2} = (m+1)!\,(x_a/2)^{-m-1}\ =$ blows up!

What about for m = 0 ? I claim above (D.2.28) that

$(\frac{K_0}{2} + \frac{a_0}{0}) = 0$

**Summary:**

$a_m = (j/4)\,(ak)\,\eta_m\,I\,R_{dc}\ 2m\,(x_a/2)^{-m+1}\,(m-1)!$ m > 0

$a_0 = 0$ m = 0

$\frac{K_m}{2} = (j/4)\,(ak)\,\eta_m\,I\,R_{dc}\ (m+1)!\,(x_a/2)^{-m-1}$ m > 0

$\frac{K_0}{2} = (j/4)\,(ak)\,\eta_m\,I\,R_{dc}\ 4/x_a$ m = 0

$(\frac{K_m}{2} + \frac{a_m}{m}) = (j/4)\,(ak)\,\eta_m\,I\,R_{dc}\ (m+1)!\,(x_a/2)^{-m-1}$ m > 0

$(\frac{K_0}{2} + \frac{a_0}{0}) = 0$ m = 0

So these things are generally blowing up in our ω = 0 limit!

[ Yes, things blow up as ω→0. This was the first sign of this problem, more on it below. ]

**5. What happens to the $B_i$ as ω→ 0 with G = 0 ?**

What then about the B fields?? I have generally neglected these, but I wonder if something bad happens?

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\{ + \left( m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{k}{\beta'}\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \}$$

$$B_\theta(r,m) = (\beta'/\omega)\{ - \left( m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{K_m}{2}\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) J_{m+1}(x) \} \quad , \qquad (D.4.8)$$

First off, I assume G = 0 and from Section D.11 I have

$\beta^2 = -j\omega\mu\sigma$ *(1.5.1d)* for good conductor

$k \approx \sqrt{RC/2}\ \omega^{1/2}(1-j) = \sqrt{RC}\ \omega^{1/2} e^{-j\pi/4}$ *(Q.4)* (highly damped)

$k^2 = RC\omega(-j) = -j\omega RC$

$$\beta'^2 = \beta^2 - k^2 \approx -j\omega\mu\sigma + j\omega RC = -j\omega(\mu\sigma - RC) \qquad (D.11.2)$$

This means that

$(\beta'/\omega) = \sqrt{-j\omega(\mu\sigma - RC)}\ /\ \omega = \sqrt{-j(\mu\sigma - RC)}\ (1/\sqrt{\omega}\ )$ = blows up

$(\beta'/k) = \sqrt{-j\omega(\mu\sigma - RC)}\ /\ [\sqrt{RC}\ \omega^{1/2} e^{-j\pi/4}] = \sqrt{-j(\mu\sigma - RC)}\ /\ [\sqrt{RC}\ e^{-j\pi/4}]$ = constant

$(k/\beta')$ = inverse of the above constant

Meanwhile, again for small ω,

$J_m(x) = (x/2)^m / m!$ $\quad m > 0$

$J_0(x) = 1$ $\quad m = 0$

$J_{m+1}(x) = (x/2)^{m+1} / (m+1)!$

$J_{-1}(x) = -(x/2)$

Now assemble the pieces:

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_m(x)$$

$$= \sqrt{-j(\mu\sigma - RC)}\ (1/\sqrt{\omega}\ )\ [(j/4)\ (ak)\ \eta_m\ I\ R_{dc}\ (m+1)!\ (x_a/2)^{-m-1}]\ (x/2)^m / m!$$

$$= \sqrt{-j(\mu\sigma - RC)}\ (1/\sqrt{\omega}\ )\ [(j/4)\ (a\sqrt{RC}\ \omega^{1/2}\ e^{-j\pi/4})\ \eta_m\ I\ R_{dc}\ (m+1)\ (x_a/2)^{-m-1}]\ (x/2)^m$$

$$= \sqrt{-j(\mu\sigma - RC)}\ \ [(j/4)\ (a\sqrt{RC}\ \ e^{-j\pi/4})\ \eta_m\ I\ R_{dc}\ (m+1)\ (x_a/2)^{-m-1}]\ (x/2)^m$$

$$= \sqrt{-j(\mu\sigma - RC)}\ \ [(j/4)\ (a\sqrt{RC}\ \ e^{-j\pi/4})\ \eta_m\ I\ R_{dc}\ (m+1)\ (x_a/2)^{-1}]\ (x/x_a)^m$$

$$= \sqrt{-j(\mu\sigma - RC)}\ \ [(j/4)\ (a\sqrt{RC}\ \ e^{-j\pi/4})\ \eta_m\ I\ R_{dc}\ (m+1)\ (x_a/2)^{-1}]\ (r/a)^m$$

Bang! We have a problem Houston! Due to the $x_a^{-1}$ factor, this field is blowing up at $\omega = 0$!
On the other hand, we know that $I = 2\pi\omega\ (a/k)\ N_0$ and we can then consider

$$I/x_a = I/(\beta'a) = 2\pi\omega a\ N_0 / (\beta'k)$$

$$= 2\pi\omega a\ N_0 / [\ (\sqrt{-j\omega(\mu\sigma - RC)}\ \sqrt{RC}\ \omega^{1/2}\ e^{-j\pi/4})]$$

$$= 2\pi a\ N_0 / [(\sqrt{-j\ (\mu\sigma - RC)}\ \sqrt{RC}\ e^{-j\pi/4})] \quad = \text{constant}$$

Then we end up with $B_z(r,m)$ = a constant in this limit. The constant is this

$$\sqrt{-j(\mu\sigma - RC)}\ \ [(j/4)\ (a\sqrt{RC}\ \ e^{-j\pi/4})\ \eta_m\ \ R_{dc}\ \ (m+1)\ 2]\ (r/a)^m\ \ 2\pi a\ N_0 / [(\sqrt{-j\ (\mu\sigma - RC)}\ \sqrt{RC}\ e^{-j\pi/4})]$$

$$= \ [(j/4)\ (a\ \eta_m\ \ (1/\sigma\pi a^2)\ (m+1)\ 2]\ (r/a)^m\ \ 2\pi a\ N_0$$

$$= \ [(j)\ (\ \eta_m\ \ (1/\sigma)\ (m+1)\ ]\ (r/a)^m\ \ \ N_0\ = j\ N_m\ \ (1/\sigma)\ (m+1)\ ]\ (r/a)^m$$

Since $\sigma$ is very large, this $B_z$ field is very small, but it approaches this small constant value!

$$B_z(r,0) = (\beta'/\omega)\left(\frac{K_0}{2} + \frac{a_0}{0}\right) J_m(x) = 0$$

so the $B_z$ field exists only in the moments for $m > 0$.

So far, $B_z$ did not blow up! What about the other components? It is a real mess, maybe I need to do this all in Maple. I want to convince myself that B does not blow up.

Another approach would be to start with the small $\omega$ fields for $E_i$

$$E_z(r,m) = (1/2)\ \eta_m\ I\ R_{dc}\ (r/a)^{|m|}\ (|m|+1)$$
$$E_r(r,m) = (j/4)\ \eta_m\ I\ R_{dc}\ (ak)\ [(r/a)^{|m|+1} + \ (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\, R_{dc}$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\ (ak)\ (r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

and compute from these B fields

$$B_r(r,m) = (j/\omega)\,[\text{curl } E]_r = (j/\omega)\,[r^{-1} jm E_z + jk E_\theta]$$

$$B_\theta(r,m) = (j/\omega)\,[\text{curl } E]_\theta = (j/\omega)[-jk E_r - \partial_r E_z]$$

$$B_z(r,m) = (j/\omega)\,[\text{curl } E]_z = (j/\omega)\,[r^{-1}\partial_r(rE_\theta) - r^{-1} jm E_r]\ . \qquad \text{(D.4.7)}$$

Recall $I = 2\pi\omega\,(a/k)\,N_0$ so we can look at the singular nature of each B field. Here for $m > 0$

$$B_r(r,m) \sim (1/\omega)\,[\,I + I k^2] = (1/\omega)\, I\,(A + Bk^2) = (1/\omega)\, 2\pi\omega\,(a/k)\,(A + Bk^2)$$

$$= 2\pi\,(a/k)\,(A + Bk^2) = 2\pi aA/k + 2\pi aBk = 2\pi aA/k = \text{blows up!}$$

Ouch! Even using the I protection, $B_r$ is blowing up at $\omega = 0$ !! Non-physical, something is wrong./

**Conclusion**: I need to do more study of the B fields in Appendix D and compute their behavior in various limits as well! It seems that they blow up as $\omega \to 0$ and that would seem to be a problem with the entire Appendix if true!

[ Above is where I first realized that $B_r$ blows up as $\omega \to 0$. Later this was called the curl E problem and I wrote a separate doc about it. ]

**6. Try to find better expressions for the $B_i$ fields.**

[ The work of this section below is too messy to be useful, and besides, in the curlE doc I find a very simple and direct way to show that $B_r$ and $B_\theta$ blow up at DC, so details below are not needed. ]

I would like to get something like (D.2.33) for the B fields. Here is where I left things off

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\left\{ + \left( m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'} a_m\right) x^{-1} J_m(x) + \frac{k}{\beta'}\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \right\}$$

$$B_\theta(r,m) = (\beta'/\omega)\left\{ - \left( m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'} a_m\right) x^{-1} J_m(x) + \frac{K_m}{2}\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) J_{m+1}(x) \right\} \quad , \qquad \textit{(D.4.8)}$$

$$a_m = (j/4)(ak)\,\eta_m\, I\, R_{dc} \;*\; 2m\,\frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = (j/4)(ak)\,\eta_m\, I\, R_{dc} \;*\; \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = (j/4)(ak)\,\eta_m\, I\, R_{dc} \;*\; \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] \quad .$$

Let's define the following symbol

$$Q = (j/4)(ak)\,\eta_m\, I\, R_{dc}$$

Then we can write

$$a_m = Q\; 2m\,\frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = Q\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = Q\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] \quad .$$

and then to recap:

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\left\{ + \left( m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'} a_m\right) x^{-1} J_m(x) + \frac{k}{\beta'}\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \right\}$$

$$B_\theta(r,m) = (\beta'/\omega)\left\{ - \left( m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'} a_m\right) x^{-1} J_m(x) + \frac{K_m}{2}\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) J_{m+1}(x) \right\} \quad , \qquad \textit{(D.4.8)}$$

Now install coefficients to get

$$B_z(r,m) = (\beta'/\omega)\, Q\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\left\{ + \left( m\, Q\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]\frac{\beta'}{k} - \frac{k}{\beta'} Q\; 2m\,\frac{1}{J_{m-1}(x_a)}\right) x^{-1} J_m(x) \right.$$

$$\left. + \frac{k}{\beta'} Q\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x) \right\}$$

$$B_\theta(r,m) = (\beta'/\omega)\left\{ - \left( m\, Q\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]\frac{\beta'}{k} - \frac{k}{\beta'} Q\; 2m\,\frac{1}{J_{m-1}(x_a)}\right) x^{-1} J_m(x) \right.$$

$$\left. + Q\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) J_{m+1}(x) \right\} \quad , \qquad \textit{(D.4.8)}$$

Now define $S = (\beta'/\omega)\, Q$ so we then have

$$B_z(r,m) = S\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$B_r(r,m) = jS\{ + ( m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{k} - \frac{k}{\beta'} 2m \frac{1}{J_{m-1}(x_a)}) x^{-1} J_m(x)$$

$$+ \frac{k}{\beta'} \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x) \}$$ ok

$$B_\theta(r,m) = S\{ - ( m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{k} - \frac{k}{\beta'} 2m \frac{1}{J_{m-1}(x_a)}) x^{-1} J_m(x)$$

$$+ \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] ( \frac{k}{\beta'} + \frac{\beta'}{k} ) J_{m+1}(x) \}$$ , *(D.4.8)* ok

Pick one and play:

$$B_r(r,m)/jS =$$

$$( m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{k} - \frac{k}{\beta'} 2m \frac{1}{J_{m-1}(x_a)}) x^{-1} J_m(x) + \frac{k}{\beta'} \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x)$$ ok

$$= ( \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{k} - 2 \frac{k}{\beta'} \frac{1}{J_{m-1}(x_a)}) m x^{-1} J_m(x) + \frac{k}{\beta'} \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x)$$ ok

$$= \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{k} m x^{-1} J_m(x)$$ ok 1

$$- \frac{k}{\beta'} 2 \frac{1}{J_{m-1}(x_a)} m x^{-1} J_m(x)$$ ok 2

$$+ \frac{k}{\beta'} \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x)$$ ok 3

$$= \frac{1}{J_{m+1}(x_a)} \{ \frac{\beta'}{k} m x^{-1} J_m(x) + \frac{k}{\beta'} J_{m+1}(x) \}$$ ok

3 1 2

$$+ \frac{1}{J_{m-1}(x_a)} \{ \frac{k}{\beta'} J_{m+1}(x) - \frac{\beta'}{k} m x^{-1} J_m(x) - \frac{k}{\beta'} 2 m x^{-1} J_m(x) \}$$

$$= \frac{1}{J_{m+1}(x_a)} \{ \frac{\beta'}{k} m x^{-1} J_m(x) + \frac{k}{\beta'} J_{m+1}(x) \}$$ ok

$$+ \frac{1}{J_{m-1}(x_a)} \{ \frac{k}{\beta'} J_{m+1}(x) - m x^{-1} J_m(x) ( \frac{\beta'}{k} - 2 \frac{k}{\beta'} ) \}$$

$$= \frac{1}{J_{m+1}(x_a)} \{ \quad J_{m+1}(x) \frac{k}{\beta'} \quad + \quad m\, x^{-1} J_m(x) \frac{\beta'}{k} \} \qquad \text{ok}$$

$$+ \frac{1}{J_{m-1}(x_a)} \{ \; J_{m+1}(x) \frac{k}{\beta'} \; - \; m\, x^{-1} J_m(x) \left(\frac{\beta'}{k} - \frac{2k}{\beta'}\right) \; \}$$

I think the above is as simple as I can make things. I could rearrange in this way

$$= J_{m+1}(x) \; \left[\frac{k}{\beta' J_{m+1}(x_a)} + \frac{k}{\beta' J_{m-1}(x_a)}\right]$$

$$+ m\, x^{-1} J_m(x) \left[ \frac{\beta'}{k J_{m+1}(x_a)} - \left(\frac{\beta'}{k} - \frac{2k}{\beta'}\right) \frac{1}{J_{m-1}(x_a)} \right]$$

$$= J_{m+1}(x) \frac{k}{\beta'} \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right]$$

$$+ m\, x^{-1} J_m(x) \frac{\beta'}{k} \left[ \frac{1}{J_{m+1}(x_a)} - \left(1 - \frac{2k^2}{\beta'^2}\right) \frac{1}{J_{m-1}(x_a)} \right]$$

Going another route we get,

$$= \frac{1}{J_{m+1}(x_a)} \{ \; \frac{k}{\beta'} J_{m+1}(x) \quad + \quad \frac{\beta'}{k} m\, x^{-1} J_m(x) \; \} \qquad \text{ok}$$

$$+ \frac{1}{J_{m-1}(x_a)} \; \{ \frac{k}{\beta'} J_{m+1}(x) \; - 2m\, x^{-1} J_m(x) \left(\frac{\beta'}{2k} - \frac{k}{\beta'}\right) \; \}$$

Is there any way to simplify this result? I do know that

$$(2m/x) J_m(x) = J_{m-1}(x) + J_{m+1}(x) \qquad \text{NIST (10.6.1)}$$

so the above can be rewritten

$$= \frac{1}{J_{m+1}(x_a)} \{ \; \frac{k}{\beta'} J_{m+1}(x) \quad + \quad \frac{\beta'}{2k} 2\, m\, x^{-1} J_m(x) + \; \} \qquad \text{ok}$$

$$+ \frac{1}{J_{m-1}(x_a)} \; \{ \frac{k}{\beta'} J_{m+1}(x) \; - [\, J_{m-1}(x) + J_{m+1}(x)] \left(\frac{\beta'}{2k} - \frac{k}{\beta'}\right) \; \}$$

$$= \frac{1}{J_{m+1}(x_a)} \{ \frac{k}{\beta'} J_{m+1}(x) + \frac{\beta'}{2k} [ J_{m-1}(x) + J_{m+1}(x)] \} \quad \text{ok}$$

$$+ \frac{1}{J_{m-1}(x_a)} \{ \frac{k}{\beta'} J_{m+1}(x) - J_{m-1}(x) (\frac{\beta'}{2k} - \frac{k}{\beta'}) - J_{m+1}(x) (\frac{\beta'}{2k} - \frac{k}{\beta'}) \}$$

$$= \frac{1}{J_{m+1}(x_a)} \{ J_{m+1}(x) (\frac{k}{\beta'} + \frac{\beta'}{2k}) + J_{m-1}(x) \frac{\beta'}{2k} \} \quad \text{ok}$$

$$+ \frac{1}{J_{m-1}(x_a)} \{ J_{m+1}(x) (\frac{2k}{\beta'} - \frac{\beta'}{2k}) - J_{m-1}(x) (\frac{\beta'}{2k} - \frac{k}{\beta'}) \}$$

which does not seem very simple.

Next, try

$$B_\theta(r,m)/S = - ( m [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}] \frac{\beta'}{k} - \frac{k}{\beta'} 2m \frac{1}{J_{m-1}(x_a)}) x^{-1} J_m(x)$$
$$+ [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}] (\frac{k}{\beta'} + \frac{\beta'}{k}) J_{m+1}(x) \qquad \text{// copy from above} \quad \text{ok}$$

$$= ( - m [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}] \frac{\beta'}{k} + \frac{k}{\beta'} 2m \frac{1}{J_{m-1}(x_a)}) x^{-1} J_m(x)$$
$$+ [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}] (\frac{k}{\beta'} + \frac{\beta'}{k}) J_{m+1}(x) \qquad \text{// sign on first line} \quad \text{ok}$$

$$= - m [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}] \frac{\beta'}{k} x^{-1} J_m(x) + \frac{k}{\beta'} 2m \frac{1}{J_{m-1}(x_a)} x^{-1} J_m(x)$$
$$+ [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}] (\frac{k}{\beta'} + \frac{\beta'}{k}) J_{m+1}(x) \qquad \text{// remove () first line} \quad \text{ok}$$

$$= \frac{1}{J_{m+1}(x_a)} \{ -m \frac{\beta'}{k} x^{-1} J_m(x) + (\frac{k}{\beta'} + \frac{\beta'}{k}) J_{m+1}(x) \}$$

$$+ \frac{1}{J_{m-1}(x_a)} \{ +m \frac{\beta'}{k} x^{-1} J_m(x) + 2m \frac{k}{\beta'} x^{-1} J_m(x) - (\frac{k}{\beta'} + \frac{\beta'}{k}) J_{m+1}(x) \} \quad \text{ok}$$

$$= \frac{1}{J_{m+1}(x_a)} \{ (\frac{k}{\beta'} + \frac{\beta'}{k}) J_{m+1}(x) - m \frac{\beta'}{k} x^{-1} J_m(x) \}$$

$$+\frac{1}{J_{m-1}(x_a)}\left\{ -\left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)J_{m+1}(x)+m\frac{\beta'}{k}x^{-1}J_m(x)+2m\frac{k}{\beta'}x^{-1}J_m(x)\right\}$$

$$=\frac{1}{J_{m+1}(x_a)}\left\{ \left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)J_{m+1}(x)-m\frac{\beta'}{k}x^{-1}J_m(x)\right\}$$

$$+\frac{1}{J_{m-1}(x_a)}\left\{ -\left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)J_{m+1}(x)+m\,x^{-1}J_m(x)\left(\frac{\beta'}{k}+\frac{2k}{\beta'}\right)\right\}$$

$$=\frac{1}{J_{m+1}(x_a)}\left\{ J_{m+1}(x)\left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)-m\,x^{-1}J_m(x)\frac{\beta'}{k}\right\}$$

$$+\frac{1}{J_{m-1}(x_a)}\left\{ -J_{m+1}(x)\left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)+m\,x^{-1}J_m(x)\left(\frac{\beta'}{k}+\frac{2k}{\beta'}\right)\right\}$$

Here are my results (need Maple checking for sure! )

$$B_z(r,m)/S = \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$B_r(r,m)/jS = \frac{1}{J_{m+1}(x_a)}\left\{ J_{m+1}(x)\frac{k}{\beta'} + m\,x^{-1}J_m(x)\frac{\beta'}{k}\right\} \quad \text{ok}$$

$$+\frac{1}{J_{m-1}(x_a)}\left\{ J_{m+1}(x)\frac{k}{\beta'} - m\,x^{-1}J_m(x)\left(\frac{\beta'}{k} - \frac{2k}{\beta'}\right)\right\}$$

$$B_\theta(r,m)/S = \frac{1}{J_{m+1}(x_a)}\left\{ J_{m+1}(x)\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) - m\,x^{-1}J_m(x)\frac{\beta'}{k}\right\}$$

$$+\frac{1}{J_{m-1}(x_a)}\left\{ -J_{m+1}(x)\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) + m\,x^{-1}J_m(x)\left(\frac{\beta'}{k} + \frac{2k}{\beta'}\right)\right\}$$

$$S = (\beta'/\omega)\,Q = (\beta'/\omega)(j/4)(ak)\,\eta_m\, I\, R_{dc} = (j/4)\,a\,\eta_m\, I\, R_{dc}\,(k\beta')\,(1/\omega)$$

Now notice that

$$(k\beta')\frac{k}{\beta'} = k^2 \qquad (k\beta')\frac{\beta'}{k} = \beta'^2 \qquad (k\beta')\left(\frac{\beta'}{k} - \frac{2k}{\beta'}\right) = \beta'^2 - 2k^2 \qquad (k\beta')\left(\frac{\beta'}{k} + \frac{2k}{\beta'}\right) = \beta'^2 + 2k^2$$

Now off to the side, let's take a sneak peak at the low-ω limit of things for G = 0.

**7. Show again that $B_z$ = constant**

For low ω and G= 0, we use instead these expressions

$\beta^2 = -j\omega\mu\sigma$ *(1.5.1d)* for good conductor

$k \approx \sqrt{RC/2}\;\omega^{1/2}(1-j) = \sqrt{RC}\;\omega^{1/2}\,e^{-j\pi/4}$ *(Q.4)* (highly damped)

$k^2 = RC\omega(-j) = -j\omega RC$

$\beta'^2 = \beta^2 - k^2 \approx -j\omega\mu\sigma + j\omega RC = -j\omega(\mu\sigma - RC)$ (D.11.2)

Note that

$$\beta'/k = \sqrt{-j\omega(\mu\sigma - RC)} \,/\, [\sqrt{RC}\;\omega^{1/2}\,e^{-j\pi/4}] = \sqrt{-j\,(\mu\sigma - RC)} \,/\, [\sqrt{RC}\;e^{-j\pi/4}]$$

= a constant at low ω where both R and C are constants.

Start with

$$B_z(r,m) = S\ [\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}]\ J_m(x) \qquad J_n(x) = (x/2)^n / n! \quad n \geq 0$$

Keeping only most divergent term we get $S = (j/4)\, a\, \eta_m\, I\, R_{dc}\, (k\beta')\, (1/\omega)$

$$B_z(r,m) = S\ [\frac{1}{J_{m+1}(x_a)}]\ J_m(x) = S\,(x_a/2)^{-m-1}\,(m+1)!\ \ *\,(x/2)^m / m!$$

$$= S\,(x_a/2)^{-m}\,(x_a/2)^{-1}\ \ (x/2)^m\ \ (m+1) = S\,(x/x_a)^m\,(m+1)\,(2/x_a)$$

$$= S\,(2/a)\,(x/x_a)^m\,(m+1)\,(1/\beta')$$

$$= (j/4)\, a\, \eta_m\, I\, R_{dc}\, (k\beta')\, (1/\omega) * (2/a)\,(x/x_a)^m\,(m+1)\,(1/\beta')$$

$$= (j/2)\ \ \eta_m\, I\, R_{dc}\, (k)\, (1/\omega)\, (r/a)^m\,(m+1)$$

$$= (j/2)\ \ \eta_m[\, 2\pi\omega\, (a/k)\, N_0]\, R_{dc}\, (k)\, (1/\omega)\, (r/a)^m\,(m+1)$$

$$= (j/2)\ \ \eta_m[\, 2\pi\, (a)\, N_0]\, R_{dc}\ \ (r/a)^m\,(m+1)$$

$$= (j/2)\ \ \eta_m[\, 2\pi\, (a)\, N_0]\, (1/\sigma\pi a^2)\ \ (r/a)^m\,(m+1)$$

$$= j\ \ \eta_m\ \ N_0\, (1/\sigma a)\ \ (r/a)^m\,(m+1)$$

$$= j\, N_m\, (1/\sigma a)\ \ (r/a)^m\,(m+1)$$

The main point: this $B_z$ field is FINITE as it well should be, and it is a constant value once we have taken the $\omega \to 0$ limit. This was something I wanted to get, is $B_z$ finite.

**8. Show again that $B_r$ and $B_\theta$ diverge at $\omega = 0$**

How about the other components?

$$B_r(r,m)/jS = \frac{1}{J_{m+1}(x_a)}\ \{\ \underset{1}{J_{m+1}(x)\,\frac{k}{\beta'}}\ +\ \underset{2}{m\,x^{-1}J_m(x)\,\frac{\beta'}{k}}\ \}\qquad \text{ok}$$
$$+\ \frac{1}{J_{m-1}(x_a)}\ \{\ \underset{3}{J_{m+1}(x)\,\frac{k}{\beta'}}\ -\ \underset{4}{m\,x^{-1}J_m(x)\,(\frac{\beta'}{k} - \frac{2k}{\beta'})}\ \}$$

Now all the k and β' ratios are constant. The most divergent term has largest index if in bottom and smallest index if on top. The first line should then dominate as $\omega \to 0$ so write

$$B_r(r,m)/jS = \frac{1}{J_{m+1}(x_a)} \{ J_{m+1}(x) \frac{k}{\beta'} + m\, x^{-1} J_m(x) \frac{\beta'}{k} \} \qquad J_n(x) = (x/2)^n / n! \quad n \geq 0$$

The first term has ratio $(x/x_a)^{m+1} = (r/a)^{m+1}$. We then have

$$B_r(r,m)/jS = \frac{k}{\beta'} (r/a)^{m+1} + m \frac{\beta'}{k} x^{-1} J_m(x)/ J_{m+1}(x_a)$$

Looking above, the last ratio is

$$J_m(x)/ J_{m+1}(x_a) = (2/a) (r/a)^m (m+1) (1/\beta')$$

so we then have

$$B_r(r,m)/jS = \frac{k}{\beta'} (r/a)^{m+1} + m \frac{\beta'}{k} x^{-1} (2/a) (r/a)^m (m+1) (1/\beta')$$

$$= \frac{k}{\beta'} (r/a)^{m+1} + m \frac{\beta'}{k} (1/r\beta') (2/a) (r/a)^m (m+1) (1/\beta')$$

$$= \frac{k}{\beta'} (r/a)^{m+1} + m \frac{1}{k\beta'} (1/r) (2/a) (r/a)^m (m+1)$$

$$= m \frac{1}{k\beta'} (1/r) (2/a) (r/a)^m (m+1)$$

where I keep only the divergent second term. Then we get

$$B_r(r,m) = jS\, m \frac{1}{k\beta'} (1/r) (2/a) (r/a)^m (m+1)$$

$$= j[(j/4) a \eta_m I R_{dc} (k\beta') (1/\omega)] m \frac{1}{k\beta'} (1/r) (2/a) (r/a)^m (m+1)$$

$$= -(1/2) \eta_m I R_{dc} (1/\omega) m (1/r) (r/a)^m (m+1)$$

$$= -(1/2) \eta_m [2\pi\omega (a/k) N_0] R_{dc} (1/\omega) m (1/r) (r/a)^m (m+1)$$

$$= - (1/2) \eta_m [2\pi (a/k) N_0] R_{dc} m (1/r) (r/a)^m (m+1)$$

and now we have a problem! The factor 1/k causes $B_r(r,m) \rightarrow \infty$ as $\omega \rightarrow 0$.

These conclusions about $B_z$ and $B_r$ agree with those found earlier by much simpler means, which I summarize right here:

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$B_r(r,m) = (j/\omega)\,[\text{curl } E]_r = (j/\omega)\,[r^{-1}jmE_z + jkE_\theta]$$

$$B_r(r,m) \sim (1/\omega)\,[\,I + I\,k^2] = (1/\omega)\,I\,(A + Bk^2) = (1/\omega)\,2\pi\omega\,(a/k)\,(A + Bk^2)$$

$$= 2\pi\,(a/k)\,(A + Bk^2) = 2\pi aA/k + 2\pi aBk = 2\pi aA/k = \text{blows up!}$$

Similarly,

$$B_\theta(r,m) = (j/\omega)\,[\text{curl } E]_\theta = (j/\omega)[-jkE_r - \partial_r E_z]$$

$$B_r(r,m) \sim (1/\omega)\,[\,k\,kI - I\,] = (1/\omega)\,I\,[\,k^2 - 1\,] = (1/\omega)\,I\,(A + Bk^2) = \text{blows up as above}$$

**Question**: For the infinite transmission line, as $\omega \to 0$, how do we interpret the fact that both magnetic field components $B_r$ and $B_\theta$ become infinite? I saw this problem arise also in my reflection scenario. This would seem to cast a lot of doubt on the theory since if flunks this $\omega \to 0$ reasonableness test.

This problem first appears in (D.4.7) of Appendix D where we compute the B fields. Here is a very *concise statement of the problem*:

$$B_r(r,m) = (j/\omega)\,[\text{curl } E]_r = (j/\omega)\,[r^{-1}jmE_z + jkE_\theta] \approx (j/\omega)\,[r^{-1}jmE_z]$$

$$\approx (j/\omega)\,[r^{-1}jm(1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)] \sim (1/\omega)\,I \sim 1/k \to \infty$$

So now I have *two* problems with Appendix D.

(1) As $\omega \to 0$, $J_z$ remains asymmetric

(2) As $\omega \to 0$, both $B_r$ and $B_\theta$ diverge.

To fix this second problem, we really need to have

$$E_z \sim \omega$$

but we seem to have instead

$$E_z \sim I \sim (\omega/k) \sim \sqrt{\omega}$$

One repair would somehow be to have $k \to$ constant. This is the case when $G \neq 0$.

**9. Just compute curl E from (D.2.33) fields.**

$$[\text{curl } E]_r = [r^{-1} jmE_z + jkE_\theta]$$

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\, (|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}] \qquad \text{assumes } \beta' \to 0$$

$$[\text{curl } E]_r = [r^{-1} jmE_z + jkE_\theta]$$

$$= (1/4)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|} 2\,(|m|+1) + jk\,(1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$= (1/4)\,\eta_m\, I\, R_{dc}\, \{\, 2\,(|m|+1)\,(r/a)^{|m|} + j\,a\,k^2\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\}$$

Even if k → constant, since I → 0 we do get $[\text{curl } E]_r = 0$.

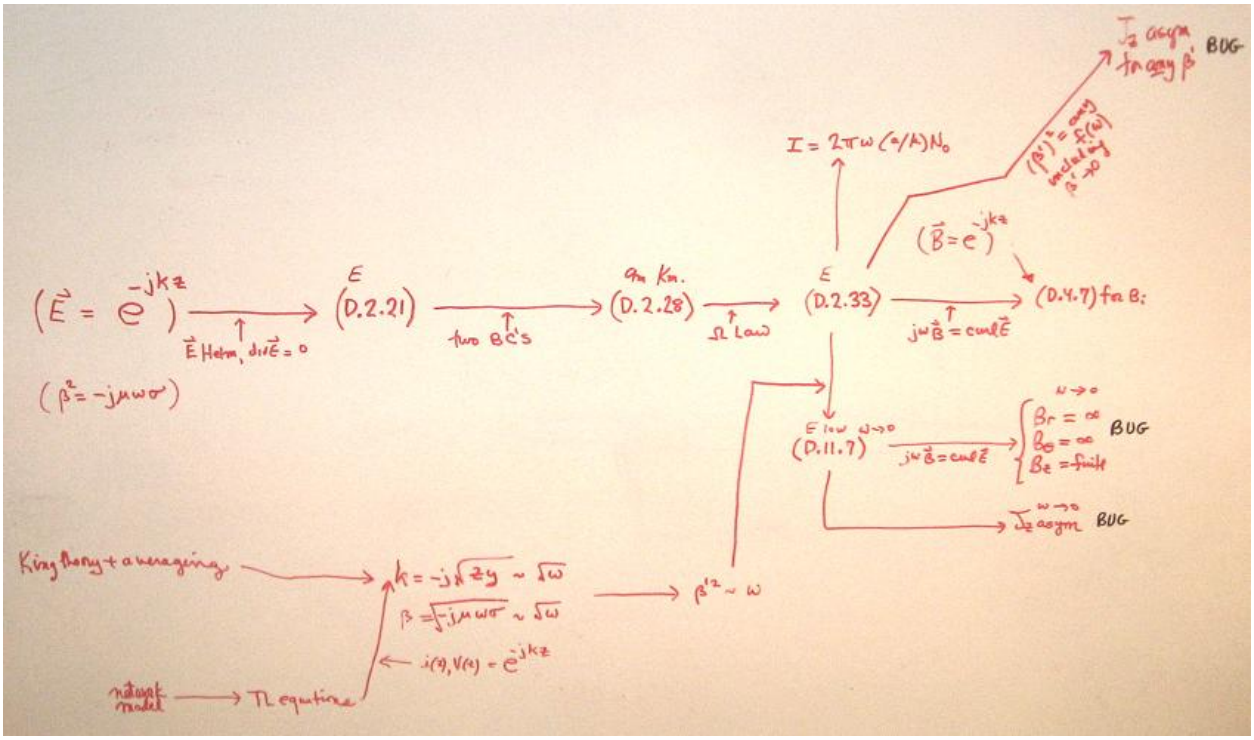
Here are all three curl components

$$[\text{curl } E]_r = [r^{-1} jmE_z + jkE_\theta] \qquad = I + k^2 I$$
$$[\text{curl } E]_\theta = [-jkE_r - \partial_r E_z] \qquad = k^2 I + I$$
$$[\text{curl } E]_z = [r^{-1}\partial_r(rE_\theta) - r^{-1} jmE_r]\,. \qquad = k\,I + kI$$

and they all go to zero as ω→0 assuming this means β' → 0.

Therefore, my solutions (D.2.3) satisfy both div E = 0 and curl E = 0 at DC. The problem is that we really are getting for the r and θ curl components that curl E ~ $\sqrt{\omega}$ and then (1/jω) curl E blows up.
[ agreed, it is a problem ]

**10. Whiteboard drawing**




This shows various paths to bugs.

**11. Problem with k → 0 ?**

In Appendix Q we find that

**Fact 4:** In the low frequency limit with G = 0 , (Q.4)

$$\mathrm{Re}(k) \approx \sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2}$$
$$\mathrm{Im}(k) \approx -\sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2} \qquad \omega << R/L$$

This implies for a $10^8$ meter long transmission line at as you go to DC, there will be no loss! After all, you are assuming that things go as $e^{-jkz}$ and Im(k) is the attenuation constant.

The fact that the above limit vanishes as $\sqrt{\omega}$ is quite obvious from the general form of k :

$$k \equiv -j\sqrt{(R+j\omega L)(G+j\omega C)} \rightarrow -j\sqrt{Rj\omega C}$$

But I always explain this by saying that $I \sim \sqrt{\omega}$ in this limit so there is no loss at ω = 0 since there is no current to make a loss. For very small ω there IS a loss which should be dV/dz = -IR. But I cannot make this simple DC limit work. For example,

$$-jkV(0)e^{-jkz} = -I\,R\,e^{-jkz}$$

$-j\, k\, V(0) = -I\, R$

$-j\, V(0) / I = -R / k = -j\, Z_0$

This last equation is

$R/k = j\, Z_0$

$R = j\, Z_0 k = j\,(1/\sqrt{2})\sqrt{R/C}\;\; 1/\sqrt{\omega}\;\; (1-j) * \sqrt{RC/2}\;\omega^{1/2}\;\; (1-j)$

$= j\,(1/\sqrt{2})^2\, R\; 1/\sqrt{\omega}\;\; (1-j) * \omega^{1/2}\;\; (1-j)$

$= j\,(1/\sqrt{2})^2\, R\;\; (1-j) * (1-j)$

$= j\, R\, e^{-j\pi/2} = j\,(-j)\, R$

$= R$

so OK.

[ So I have discovered the curl E problem in addition to the $J_z$ asym problem. The question remains as to where I should put the blame for these anomalies! ]