Confusion about the meaning of n_c PhL 9.30.14

I think this stuff is all OK, see conclusion in red at the end. Reviewed again 10.8.14, OK.

Recall from Fig 1.7 that we have

$$n_c(\mathbf{x},\omega) = (\xi_1/\varepsilon_1)\, n_s(\mathbf{x},\omega)\ . \qquad (1.5.17)$$

Now, since

$$(\xi_d/\varepsilon_d) = \rightarrow (\varepsilon_d - j\sigma_d/\omega)/\varepsilon_d \rightarrow -j(\sigma_d/\varepsilon_d)\,(1/\omega) = (G_{dc}/C)\,(1/j\omega)$$

we find that n_c becomes infinite as $\omega \rightarrow 0$. What on earth does that mean??

I look back at the conducting capacitor example. At DC, battery feeds constant current, why would n_c go infinite? I am reading through that discussion, and it seems OK to this point:

$$J_{c2n}(\mathbf{x},\omega) = (\xi_1/\varepsilon_1)\,(j\omega)\, n_s(\mathbf{x},\omega)\ . \qquad \xi_1 \equiv \varepsilon_1 + \sigma_1/j\omega = \text{complex dielectric constant} \qquad (1.5.16)$$

because we $\omega \rightarrow 0$, the right side here is finite, as expected. I then make this statement:

"If we observe the conduction current J_{c2n} flowing through a unit-area loop (red in figure), we can write $J_{c2n} = \partial_t n_c$ where n_c is the total amount of conduction charge flowing through that unit-area loop per unit time. "

I think something is wrong here. One could say that dn_c is the amount of conduction charge flowing through the loop in time dt, but I don't think you can claim that n_c is the charge flowing in 1 second. I think n_c is the total integrated charge flowing through the loop from time t = 0 to time t. But maybe I can just define n_c as $J_{c2n} = \partial_t n_c$ and not try to interpret it?

Consider I = dQ/dt . Here dQ flows through a wire in time dt. Q itself is the integrated charge over some time, meaning $Q(t) = \int^t I(t')dt'$.

So maybe a better claim would be

" Quantity n_c is the total amount of charge that flows through the red loop from some initial time up to time t. It's units are charge/area. "

Now go back to $J_{c2n} = \partial_t n_c$ and that is then OK. Then go do $J_{c2n} = j\omega n_c$ and what does that mean?

When I talk about $\partial_t V$ I don't seem to have this confusion, perhaps because V is not an "extensive quantity". I think of something like $V = V_0 e^{j\omega t}$ and then $\partial_t V \rightarrow j\omega V$. At $\omega = 0$, I have $V = V_0$. Maybe I have a problem with Fourier Transforms, ignoring constant things as appear in Laplace Transforms.

I will just go back and review this entire Section with commentary in red:

Consider the situation at a general boundary between dielectric (region 1) and conductor (region 2) where there exists a surface charge density n_s :

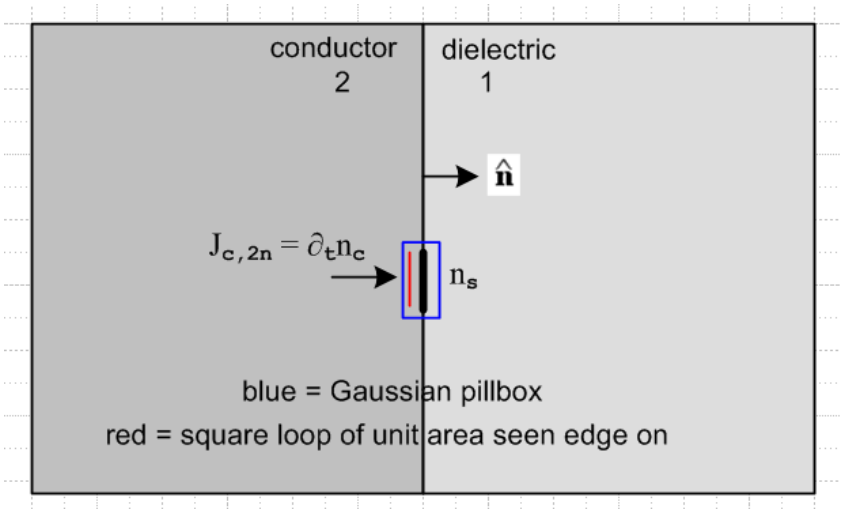


Fig 1.7

In (1.1.18) it was shown that div $[\mathbf{J_d} + \mathbf{J_c}] = 0$ where $\mathbf{J_c} = \sigma\mathbf{E}$ is the conduction current and $\mathbf{J_d}$ the displacement current $\partial_t\mathbf{D} = \varepsilon\partial_t\mathbf{E}$. The divergence theorem (1.1.30) then says

$$0 = \int_V \text{div}\,[\mathbf{J_d} + \mathbf{J_c}]\,dV = \int_S [\mathbf{J_d} + \mathbf{J_c}] \bullet d\mathbf{S} \ .$$

Comment: This "total current" is continuous at a boundary, and I really do show that in 1.1.18. There is no time element here.

Applied to the blue pillbox which straddles the boundary in the figure, we find

$$J_{d1n} + J_{c1n} = J_{d2n} + J_{c2n}$$

where n means normal component. Writing this out,

$$\varepsilon_1\partial_t E_{1n} + \sigma_1 E_{1n} = \varepsilon_2\partial_t E_{2n} + \sigma_2 E_{2n} \approx \sigma_2 E_{2n} = J_{c2n}$$

since σ_2 is huge inside the conductor. Therefore,

$$J_{c2n} = \sigma_1 E_{1n} + \varepsilon_1\partial_t E_{1n} \quad . \qquad (1.5.14)$$

The left equation is exact, but then I do make an approximation, things are in the time domain. In the ω domain I would be saying

$$\varepsilon_1 j\omega E_{1n} + \sigma_1 E_{1n} = \varepsilon_2 j\omega E_{2n} + \sigma_2 E_{2n} \approx \sigma_2 E_{2n} = J_{c2n}$$
$$(\varepsilon_1 j\omega + \sigma_1) E_{1n} = (\varepsilon_2 j\omega + \sigma_2) E_{2n} \approx \sigma_2 E_{2n} = J_{c2n}$$
$$j\omega(\varepsilon_1 + \sigma_1/j\omega) E_{1n} = j\omega(\varepsilon_2 + \sigma_2/j\omega) E_{2n} \approx \sigma_2 E_{2n} = J_{c2n}$$
$$j\omega \xi_1 E_{1n} = j\omega\, \xi_2 E_{2n} \approx \sigma_2 E_{2n} = J_{c2n}$$

I don't think dropping the ε_2 term is a problem here, especially near $\omega = 0$!

The final line just says J_{c2n} has the two contributions shown, and you could write that as

$$J_{c2n} = \sigma_1 E_{1n} + \varepsilon_1 j\omega E_{1n} \quad . \tag{1.5.14}$$

When $\omega \rightarrow 0$, you are left just with the DC leakage current, as you would expect.

Meanwhile, Gauss's Law (1.1.33) states that

$$\text{div}\,(\varepsilon \mathbf{E}) = \rho \quad \Leftrightarrow \quad \int_V \rho\, dV = \int_S \varepsilon \mathbf{E} \bullet d\mathbf{S} \,. \tag{1.1.33}$$

Applied to the same blue pillbox we find

$$n_s = \varepsilon_1 E_{n1} - \varepsilon_2 E_{n2} \approx \varepsilon_1 E_{n1}$$

since $E_{n2} \approx 0$ inside the conductor. Thus,

$$E_{n1} = n_s/\varepsilon_1 \quad \text{and then} \quad J_{cn1} = \sigma_1 E_{n1} = n_s(\sigma_1/\varepsilon_1) \quad . \tag{1.5.15}$$

Notice in the first line that it is div **D** = ρ. On the second line I do another approximation which is that E is very small inside conductor, I think that is OK since I always use it. Dielectric is 1. I end up with a way to relate the dielectric conduction current to the surface charge n_s. Note that (σ_1/ε_1) = G/C from 4.4.10, so the last equation says $J_{cn1} = n_s(G/C)$. I think this is correct and OK.

Then (1.5.14) can be written as

$$J_{c2n} = \sigma_1 E_{1n} + \varepsilon_1 \partial_t E_{1n} = (\sigma_1 + \varepsilon_1 \partial_t) E_{1n} = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 \partial_t) n_s$$

or, writing out the arguments,

$$J_{c2n}(\mathbf{x},t) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 \partial_t) n_s(\mathbf{x},t) \,.$$

This is the *conductor's* current expressed in terms of surface charge n_s and ε_1 and σ_1. All is OK here. The current has two components as expected. The leakage is $(\sigma_1/\varepsilon_1)n_s$ as shown in (1.5.15). The second term is the capacitative current.

In the frequency domain with rules (1.5.2) this becomes

$$J_{c2n}(\mathbf{x},\omega) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 j\omega) n_s(\mathbf{x},\omega)$$

$$= (1/\varepsilon_1)\,(j\omega)(\varepsilon_1 + \sigma_1/j\omega)\, n_s(\mathbf{x},\omega)$$

$$= (\xi_1/\varepsilon_1)\,(j\omega)\, n_s(\mathbf{x},\omega)\ . \qquad \xi_1 \equiv \varepsilon_1 + \sigma_1/j\omega \ = \text{complex dielectric constant} \qquad (1.5.16)$$

Nothing new added here, it is OK.

If we observe the conduction current J_{c2n} flowing through a unit-area loop (red in figure), we can write $J_{c2n} = \partial_t n_c$ where n_c is the total amount of conduction charge flowing through that unit-area loop per unit time. Thus we have

$$\partial_t n_c(\mathbf{x},t) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 \partial_t) n_s(\mathbf{x},t)$$

or

$$j\omega\, n_c(\mathbf{x},\omega) = (\xi_1/\varepsilon_1)\,(j\omega)\, n_s(\mathbf{x},\omega)$$

or

$$n_c(\mathbf{x},\omega) = (\xi_1/\varepsilon_1)\, n_s(\mathbf{x},\omega)\ . \qquad (1.5.17)$$

where is our result claimed at the start that $n_s = (\varepsilon_d/\xi_d)\, n_c$. Note that:

OK, here is where I may have a problem of sorts. I think I am defining $n_c(t)$ in this manner really,

$$n_c(\mathbf{x},t) \equiv \int_0^t J_{c2n}(\mathbf{x},t')\, dt'$$

where here I arbitrarily integrate from some time t = 0. There is nothing wrong with such a definition, so it is certainly then OK to say $J_{c2n}(\mathbf{x},t) = \partial_t n_c(\mathbf{x},t)$. We then end up with (1.5.17) which is the key result of this section, and all is well. The only confusion is in the *interpretation* of n_c .

In the time domain, n_c is the total charge passing through the unit red rectangle from time t = 0 to time t = t. There is no confusion about what that means. The confusion is how you translate that concept into the frequency domain.

I now end this section of commentary and return to the main doc.

At this point I wrote some notes called " Those constant terms in Laplace Transforms.doc". I consider Fourier Transforms as well and I show that:

Fact: Let F(x) have a Fourier transform f(ω).
Let $G(x) = \partial_x F(x)$.
Then: $g(\omega) = j\omega\, f(\omega)$ *provided that* $F(\pm\infty) = 0$.

So there is a *condition* on the little $\partial_t \rightarrow j\omega$ rule! I don't think I ever mention this in lines doc.

Suppose we have

$$n_c(t) = \int_0^t dt' J_{c2n}(t') \qquad \Leftrightarrow \qquad J_{c2n}(t) = \partial_t n_c(t)$$

Let's now assume that

$$J_{c2n}(t) = J_{c2n}(t=0)\, e^{j\omega_1 t} = J_{c2n0}\, e^{j\omega_1 t}$$

as a monochrome probe (I guess it runs all the time). Then we have

$$J_{c2n}(\omega) = J_{c2n0}\, 2\pi\delta(\omega-\omega_1)$$

for the official Fourier Transform. The spectrum is *all* at the one frequency ω_1, and $\omega_1 = 0$ is as good as any other value of ω_1. Now consider

$$n_c(t) = \int_0^t dt' J_{c2n}(t') = \int_0^t dt' J_{c2n0}\, e^{j\omega_1 t'} = J_{c2n0}\, [e^{j\omega_1 t} - 1]/ (j\omega_1)$$

$$n_c(t=\varepsilon) = J_{c2n0}\, [1 + j\omega_1\varepsilon - 1]/ (j\omega_1) = J_{c2n0}\, \varepsilon$$

$$n_c(t=0) = 0$$

This all seems reasonable since integration of n_c starts at t = 0. The second term "-1" is in there exactly to cause the integrated charge to be 0 at t = 0, as shown.

We shall now compute $n_c(\omega)$ as a standard Fourier Transform, using $n_c(t)$ shown above,

$$n_c(\omega) = \int_{-\infty}^{\infty} dt\, n_c(t)\, e^{-j\omega t} = \int_{-\infty}^{\infty} dt\, \{J_{c2n0}\, [e^{j\omega_1 t} - 1]/ (j\omega_1)\} e^{-j\omega t}$$

$$= J_{c2n0} / (j\omega_1) * \int_{-\infty}^{\infty} dt\, [e^{j\omega_1 t} - 1]\, e^{-j\omega t}$$

$$= J_{c2n0} / (j\omega_1) * [\ 2\pi\delta(\omega-\omega_1) - 2\pi\, \delta(\omega)\ ]$$

Notice that the "second term" has created a second term here as well which is the $\delta(\omega)$ term.

Comment: The second term "- 1" is really illegal for a FT for functions, but is legal for distribution theory. It creates $\delta(\omega)$. A spectrum $\delta(\omega)$ always corresponds to a constant in the t domain.

Let's go the other direction to see what this dual term spectrum means:

$$n_c(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, e^{+j\omega t}\, n_c(\omega)$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\omega\, e^{+j\omega t}\{ J_{c2n0} / (j\omega_1) * [\ 2\pi\delta(\omega-\omega_1) - 2\pi\,\delta(\omega)\ ]\}$$

$$= J_{c2n0} / (j\omega_1) * \int_{-\infty}^{\infty} d\omega\, e^{+j\omega t}\{\ [\ \delta(\omega-\omega_1) - \delta(\omega)\ ]\}$$

$$= J_{c2n0} / (j\omega_1) * [\ e^{+j\omega 1t} - 1]$$

and as expect, the second term in ω space creates that second term in t-space.

We can also compute (we really already did this above)

$$J_{c2n}(\omega) = \int_{-\infty}^{\infty} dt\, J_{c2n}(t)\, e^{-j\omega t} = \int_{-\infty}^{\infty} dt\ \{ J_{c2n0}\, e^{j\omega 1t} \} e^{-j\omega t} = J_{c2n0}\, 2\pi\delta(\omega-\omega_1)$$

So we now have this interesting pair of equations

$$n_c(\omega) = J_{c2n0} / (j\omega_1) * [\ 2\pi\delta(\omega-\omega_1) - 2\pi\,\delta(\omega)\ ]$$

$$J_{c2n}(\omega) = J_{c2n0}\, 2\pi\delta(\omega-\omega_1)$$

As long as we avoid ω = 0, we can say that

$$J_{c2n}(\omega) = j\omega\, n_c(\omega)$$

where we cancel out the two delta functions. I think this is pretty much OK.

Lets back up and replace the t = 0 endpoint with t = -∞ as a start time.

We then get these equations,

$$n_c(t) = \int_{-\infty}^{t} dt'\, J_{c2n}(t') \qquad \Leftrightarrow \qquad J_{c2n}(t) = \partial_t n_c(t)$$

$J_{c2n}(t) = J_{c2n0}\, e^{j\omega 1t}$ going on forever

$J_{c2n}(\omega) = J_{c2n0}\, 2\pi\delta(\omega-\omega_1)$

$$n_c(t) = \int_{-\infty}^{t} dt'\, J_{c2n}(t') = \int_{-\infty}^{t} dt'\, J_{c2n0}\, e^{j\omega 1t'} = J_{c2n0}\,[e^{j\omega 1t} - 0]/\,(j\omega_1) = J_{c2n0}\,[e^{j\omega 1t}]/\,(j\omega_1)$$

In this last line, we do our usual trick to say that $e^{j\omega 1(-\infty)} = 0$ by adding a tiny imag part to ω_1.

$$n_c(\omega) = \int_{-\infty}^{\infty} dt\, n_c(t)\, e^{-j\omega t} = \int_{-\infty}^{\infty} dt\, \{J_{c2n0}\, [e^{j\omega 1 t}]/(j\omega_1)\} e^{-j\omega t}$$

$$= J_{c2n0} / (j\omega_1) * \int_{-\infty}^{\infty} dt\, [e^{j\omega 1 t}]\, e^{-j\omega t}$$

$$= J_{c2n0} / (j\omega_1) * [\ 2\pi\delta(\omega-\omega_1)\ ]$$

and now that extra $\delta(\omega)$ term is gone. Let's go the other direction to see what this dual term spectrum means:

$$n_c(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, e^{+j\omega t}\, n_c(\omega)$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\omega\, e^{+j\omega t}\{\ J_{c2n0} / (j\omega_1) * [\ 2\pi\delta(\omega-\omega_1)\ ]\}$$

$$= J_{c2n0} / (j\omega_1) * \int_{-\infty}^{\infty} d\omega\, e^{+j\omega t}\{\ [\ \delta(\omega-\omega_1)]\}$$

$$= J_{c2n0} / (j\omega_1) * [\ e^{+j\omega 1 t}]$$

and we recover where we started. Finally

$$n_c(\omega) = J_{c2n0} / (j\omega_1) * [\ 2\pi\delta(\omega-\omega_1\ ]$$

$$J_{c2n}(\omega) = J_{c2n0}\, 2\pi\delta(\omega-\omega_1)$$

$$\Rightarrow \quad J_{c2n}(\omega) = j\omega\, n_c(\omega)$$

So either way, I think I am "comfortable" with this last equation.

Interpret in terms of ω? Well, suppose we define script variables as unscripted over $2\pi\delta$. Then scripted are the "line strengths" (but only one line, so line strength):

$$\mathcal{J}_{c2n}(\omega_1) = j\omega_1\, \boldsymbol{n}_c(\omega_1) = j\omega_1 * J_{c2n0} / (j\omega_1) = J_{c2n0}$$

Or writing ω as the name of the single line (as I always do in lines doc)

$$\mathcal{J}_{c2n}(\omega) = j\omega\, \boldsymbol{n}_c(\omega) = j\omega * J_{c2n0} / (j\omega) = J_{c2n0}$$

So, as $\omega \rightarrow 0$, $\mathcal{J}_{c2n}(\omega) = J_{c2n0}$.

Comment and Question: Consider our starting point in the time domain

$J_{c2n}(t) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1\partial_t)\, n_s(t)$.

In the ω domain this reads

$J_{c2n}(\omega) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 j\omega)\, n_s(\omega) = (\xi_1/\varepsilon_1)\,(j\omega)\, n_s(\omega) \equiv j\omega\, n_c(\omega)$

where $n_c(\omega)$ is defined as shown. This is then precisely our equation above that $J_{c2n}(\omega) = j\omega\, n_c(\omega)$. The comment is this: if you just look at $J_{c2n}(\omega) = j\omega\, n_c(\omega)$, you might be tempted to say that in the time domain this reads $J_{c2n}(t) = \partial_t n_c(t)$. But the real equation is $J_{c2n}(t) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1\partial_t)\, n_s(t)$ as shown above. The question then is this: what is going on here? Are these consistent?

$\partial_t n_c(t) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1\partial_t)\, n_s(t)$

$\partial_t[n_c(t) - n_s(t)] = (\sigma_1/\varepsilon_1)$.

$n_c(t) - n_s(t) = (\sigma_1/\varepsilon_1)t + K$

$n_c(t) = n_s(t) + (\sigma_1/\varepsilon_1)t + K$

So n_c has the capacitative $n_s(t)$ component, and then it has a linear-in-time component due to the ongoing constant leakage. It we were to magically "start the leakage" at time t = 0, then we would have

$n_c(t) = n_s(t) + (\sigma_1/\varepsilon_1)\, t$

I think this is in fact OK, and therefore these two equations *are* consistent with each other,

$J_{c2n}(t) = \partial_t n_c(t)$ $\qquad$ $J_{c2n}(\omega) = j\omega\, n_c(\omega)$

$J_{c2n}(t) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1\partial_t)\, n_s(t)$ $\qquad$ $J_{c2n}(\omega) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 j\omega)\, n_s(\omega) = j\omega(\xi_1/\varepsilon_1)\, n_s(\omega)$

$n_c(\omega) = (\xi_1/\varepsilon_1)\, n_s(\omega)$ $\qquad$ $\xi_1 = \varepsilon_1 + \sigma_1/j\omega$

Now what does all this say as $\omega \to 0$ in the ω domain? Look at each of the three equations, starting with the middle one:

$J_{c2n}(\omega) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 j\omega)\, n_s(\omega) \quad \to \quad (\sigma_1/\varepsilon_1)\, n_s(\omega) \quad$ as $\omega \to 0$

This says in the time domain that $J_{c2n}(t) = (\sigma_1/\varepsilon_1)\, n_s(t) = (\sigma_1/\varepsilon_1)\, n_s$ = constant leakage. Next,

$n_c(\omega) = (\xi_1/\varepsilon_1)\, n_s(\omega) = [\, 1 + (\sigma_1/\varepsilon_1)/j\omega\,]\, n_s(\omega)$

Now I think of $n_s(t) = n_{s0}\, e^{j\omega t}$ with n_{s0} being some finite quantity (capacitor problem). This means that the line strength $n_s(\omega)$ is also some finite number, in fact just this n_{s0}. So $n_c(\omega) \to \infty$ as $\omega \to 0$. How should I interpret that fact? We go back to

$$J_{c2n}(\omega) = j\omega\, n_c(\omega) \qquad \to \qquad (\sigma_1/\varepsilon_1)\, n_s(\omega) = (\sigma_1/\varepsilon_1) n_{s0}$$

As $\omega \to 0$, the only way for $J_{c2n}(\omega) \to$ finite is if $n_c(\omega) \to \infty$, otherwise we could get $J_{c2n}(\omega) \to 0$. So this says that it is a *requirement* that $n_c(\omega) \to \infty$ as $\omega \to 0$. You can see what the plot looks like:

$$n_c(\omega) = [\, 1 + (\sigma_1/\varepsilon_1)/j\omega \,]\, n_{s0}$$

I just altered lines doc to add the notion of a line strength. Then consider

$$n_s(t) = n_{s0}\, e^{j\omega t}$$

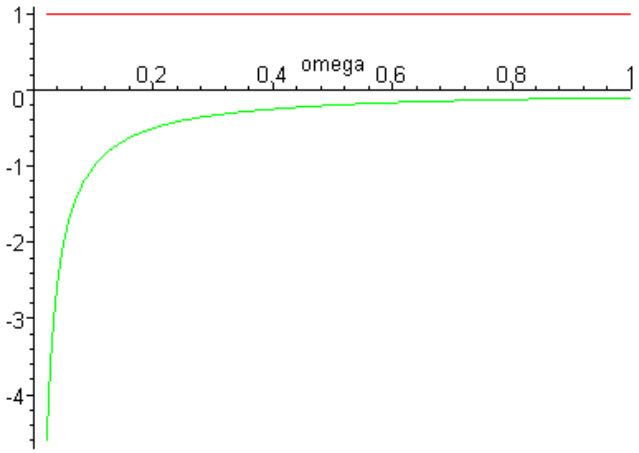
Then the Fourier transform will be

$$n_s(\omega') = n_{s0}\, 2\pi\delta(\omega'-\omega)$$

and then the "line strength" is just n_{s0} . Then (*) has line strength $n_c(\omega)$ on the left.

$$n_c(\omega) = [\, 1 + (\sigma_1/\varepsilon_1)/j\omega \,]\, n_{s0} = [\, 1 - j\,(\sigma_1/\varepsilon_1)/\omega \,]\, n_{s0} = [\, 1 - jA/\omega \,]\, B$$

```
restart; alias(I=I,j=sqrt(-1)):
nc := (1 -j*A/omega)*B:
A := .0001:    B := 1:
plot([Re(nc),1000*Im(nc)],omega = 0..1);
```



Notice that $\mathrm{Im}(n_c(\omega))$ only blows up if $\sigma_1 \neq 0$. I guess I no longer have a problem with n_c blowing up in this way. As $\omega \to 0$, we know (line strengths)

$$J_{c2n}(\omega) = (1/\varepsilon_1)(\sigma_1 + \varepsilon_1 j\omega)\, n_{s0} \qquad \to \qquad (\sigma_1/\varepsilon_1)\, n_{s0} \qquad \text{as } \omega \to 0$$

approaches its leakage value which is a small constant, as $\omega \rightarrow 0$. Since $J_{c2n}(\omega) = j\omega\, n_c(\omega)$, we must have $n_c(\omega)$ blow up as $1/\omega$. It's "line strength" gets stronger and stronger as $\omega \rightarrow 0$ in order to maintain $J_{c2n}(\omega)$.

Conclusion of this doc: Everything about n_c in lines doc is OK except the paragraph about interpretation which I will right now fix.

OLD
If we observe the conduction current J_{c2n} flowing through a unit-area loop (red in figure), we can write $J_{c2n} = \partial_t n_c$ where n_c is the total amount of conduction charge flowing through that unit-area loop per unit time. Thus we have

NEW
If we observe the conduction current J_{c2n} in the conductor just below the surface and flowing through a unit-area loop (red in Fig 1.7), we can write $J_{c2n} = \partial_t n_c$ where $n_c(t)$ is the total amount of conduction charge flowing through that unit-area loop from some initial time to the current time t:

$$J_{c2n}(\mathbf{x},t) = \partial_t n_c(\mathbf{x},t) \qquad\qquad n_c(\mathbf{x},t) = \int_{t_0}^{t} dt'\, J_{c2n}(\mathbf{x},t')$$

Thus we have from the first line of (1.5.16),