The curl E problem PhL 8.18.14

Reviewed on 9.10.14.

In both Appendix D and Plan C Revised (reflection), I compute B fields from Maxwell's equation

$$\mathbf{B} = -(1/j\omega)\ \text{curl}\ \mathbf{E} \qquad // \ \text{curl}\ \mathbf{E} = -\partial_t \mathbf{E}$$

I am then interested in the limit of things as $\omega \to 0$. For any physical situation including at DC, the B fields must be finite. Therefore, the following fact must be true.

Theorem 1

Suppose as $\omega \to 0$ that $[\text{curl E}]_i \to \omega^s [\text{stuff}]_i$ where stuff is non-infinite. Note that

$$B_i = +(j/\omega)[\text{curl E}]_i = j[\text{stuff}]_i\ \omega^{s-1}.$$

In this case one of the following must apply:
(1) if $s > 1$, then $B_i = 0$
(2) if $s = 1$, then $B_i = j[\text{stuff}]_i$ which could be either finite or 0
(3) if $s < 1$, then $B_i = \infty$ and we have a **non-physical situation**.

Now, the general equation for the curl is as follows: (cylindrical coordinates, partial waves, and e^{-jkz} assumption on all field components)

$$\text{curl}\ \mathbf{E}(r,\theta,z) = \hat{\mathbf{r}}\,[\,r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$

$$\text{curl}\ \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\,r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\,[-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,]$$

or

$$[\text{curl}\ \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + jkE_\theta$$

$$[\text{curl}\ \mathbf{E}(r,m,z)]_\theta = -jkE_r - \partial_r E_z = A\omega + B\sqrt{\omega} = \sqrt{\omega}\,[\,B + A\sqrt{\omega}\,] = B\sqrt{\omega}$$

$$[\text{curl}\ \mathbf{E}(r,m,z)]_z = r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r$$

Let's first look at our Appendix D E fields written out for the case of small ω,

$$\begin{aligned} E_z(r,m) &= (1/2)\,\eta_m\, I\, R_{dc}\ (r/a)^{|m|}\ (|m|+1) \\ E_r(r,m) &= (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}] \\ E_\theta(r,m) &= (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}] \end{aligned} \qquad (D.11.7)$$

$$E_z(r,0) = I\, R_{dc}$$

$E_r(r,0) = (j/2)\, I\, R_{dc}\, (ak)\, (r/a)$
$E_\theta(r,0) = 0$ // low ω E fields

where things are expressed in terms of current I. We know that at z = 0 we have

$I = V/Z_0$

where

Fact 8: In the low frequency limit with G = 0, both components diverge as $1/\sqrt{\omega}$: (Q.8)

$\mathrm{Re}(Z_0) \approx (1/\sqrt{2})\sqrt{R/C}\ \ 1/\sqrt{\omega}$
$\mathrm{Im}(Z_0) \approx -(1/\sqrt{2})\sqrt{R/C}\ \ 1/\sqrt{\omega}$

The above has been replaced by the following, which is in fact exactly the same :

Fact 8: The small ω limit for $Z_0(\omega)$, assuming $\omega_d = 0$, is given by (Q.8.6)

$\mathrm{Re}(Z_0) \approx \sqrt{R_{dc}/(2C)}\ \ 1/\sqrt{\omega} + (1/2)\,[L_{dc}/\sqrt{2R_{dc}C}\,]\,\sqrt{\omega} + O(\omega^{3/2})$
$\mathrm{Im}(Z_0) \approx -\sqrt{R_{dc}/(2C)}\ \ 1/\sqrt{\omega} + (1/2)\,[L_{dc}/\sqrt{2R_{dc}C}\,]\,\sqrt{\omega} + O(\omega^{3/2})$

This says that

$Z_0 = (1/\sqrt{2})\sqrt{R/C}\ \ 1/\sqrt{\omega}\,(1-j) = \sqrt{R/C}\ \ 1/\sqrt{\omega}\ \ e^{-j\pi/4}$

$1/Z_0 = \sqrt{C/R}\ \ e^{+j\pi/4}\,\sqrt{\omega}$

and thus,

$I = V/Z_0 = V\ \sqrt{C/R}\ \ e^{+j\pi/4}\,\sqrt{\omega}$

Note Added: In lines doc we have $I = 2\pi\omega\,(a/k)\,N_0$. Using k as given just below we get

$I = 2\pi\omega\,(a/k)\,N_0 = 2\pi\omega a N_0[e^{+j\pi/4}/(\sqrt{RC}\,\sqrt{\omega}\,)] = 2\pi a N_0\, e^{+j\pi/4}\sqrt{\omega}\,/\sqrt{RC}$

Comparing this to the above expression for I gives

$V\ \sqrt{C/R}\ \ e^{+j\pi/4}\,\sqrt{\omega} = 2\pi a N_0\, e^{+j\pi/4}\sqrt{\omega}\,/\sqrt{RC}$

or

$V\ \sqrt{C/R} = 2\pi a N_0\,/\sqrt{RC}$

or

$$V\ \sqrt{C/R}\ \sqrt{RC}\ =\ 2\pi a N_0$$

or

$$V\ C\ =\ 2\pi a N_0$$

or

$$q = 2\pi a N_0$$

and this agrees with (D.1.8), where $N_0 = \langle n(\theta) \rangle$

Moreover, we also have

Fact 4: In the low frequency limit with G = 0 , (Q.4)

$$\mathrm{Re}(k)\ \approx\ \sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2}$$
$$\mathrm{Im}(k)\ \approx\ -\sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2} \qquad \omega << R/L$$

The above has been replaced by the following, which is again exactly the same :

Fact 4: The small ω limit for k(ω), assuming $\omega_d = 0$, is given by (Q.4.9)

$$\mathrm{Re}(k) \approx +\sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (\ 1 -\ \tan_L/2)\ + O(\omega^{3/2})$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (\ 1 + \tan_L/2)\ + O(\omega^{3/2})$$

Thus we can write

$$k = \sqrt{RC/2}\ \ \omega^{1/2}(1\text{-}j) = \sqrt{RC}\ \ e^{-j\pi/4}\sqrt{\omega} \qquad I = V/Z_0 = V\ \sqrt{C/R}\ \ e^{+j\pi/4}\sqrt{\omega}$$

Notice then that

$$Ik = V\ \sqrt{C/R}\ \ e^{+j\pi/4}\sqrt{\omega}\ *\sqrt{RC}\ \ e^{-j\pi/4}\sqrt{\omega}\ \ =\ \omega\ V\ C.$$

We can then rewrite our small ω field expressions as

$$E_z(r,m)\ = (1/4)\ \eta_m\ V\ R_{dc}\ \{2\ \sqrt{C/R}\ \ e^{+j\pi/4}\ \ (r/a)^{|m|}\ (|m|+1)\ \sqrt{\omega}\ \}$$
$$E_r(r,m)\ = (1/4)\ \eta_m\ V\ R_{dc}\ \{(Ca)\ j\ [(r/a)^{|m|+1} +\ (r/a)^{|m|-1}]\ \omega\} \qquad (D.11.7)$$
$$E_\theta(r,m) =\ (1/4)\ \eta_m\ V\ R_{dc}\ \{(Ca)\ \ \ [(r/a)^{|m|+1} -\ \ (r/a)^{|m|-1}]\ \omega\}$$

$$E_z(r,0)\ = V\ R_{dc}\ \ \{\sqrt{C/R}\ \ e^{+j\pi/4}\ \sqrt{\omega}\ \}$$
$$E_r(r,0)\ = V\ R_{dc}\ \ \ \{C\ (j/2)\ r\ \omega\}$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

Notice that all E field components are non-infinite for any $\omega \geq 0$. We can simplify the above by writing things this way

$$E_z(r,m) = A_z(r,m)\sqrt{\omega}$$
$$E_r(r,m) = A_r(r,m)\,\omega \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = A_\theta(r,m)\,\omega$$

$$E_z(r,0) = A_z(r,0)\sqrt{\omega}$$
$$E_r(r,0) = A_r(r,0)\,\omega$$
$$E_\theta(r,0) = A_\theta(r,0)\,\omega \qquad A_\theta(r,0) = 0 \text{ it happens} \qquad \text{// low } \omega \text{ E fields}$$

where all the coefficients are finite (non-zero and non-infinite).

So now we reconsider our three curl E components stated earlier, where we replace $k = \sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}$

$$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + j\sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}\; E_\theta$$

$$[\text{curl } \mathbf{E}(r,m,z)]_\theta = -j\sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}\; E_r - \partial_r E_z$$

$$[\text{curl } \mathbf{E}(r,m,z)]_z = r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r$$

Now let's look at our three curl E components, one at a time. First,

$$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + j\sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}\; E_\theta$$

$$= r^{-1}jm\, A_z(r,m)\sqrt{\omega} + j\sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}\; A_\theta(r,m)\,\omega$$

$$= A\omega^{1/2} + B\omega^{3/2} = \omega^{1/2}[A + B\omega] = A\omega^{1/2}$$

For m = 0 we have instead, since $A_\theta(r,0) = 0$,

$$[\text{curl } \mathbf{E}(r,0,z)]_r = 0 + 0 = 0$$

Thus, according to Theorem 1, this component of curl E gives $B_r = \infty$ for m > 0, unphysical. s = 1/2.

Next, we have

$$[\text{curl } \mathbf{E}(r,m,z)]_\theta = -j\,[\sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}\,]E_r - \partial_r E_z \qquad \text{// matches (D.4.7) for } B_\theta$$

$$= -j\sqrt{RC}\; e^{-j\pi/4}\sqrt{\omega}\; A_r(r,m)\,\omega - \partial_r\, A_z(r,m)\sqrt{\omega}$$

$$= A\,\omega^{3/2} + B\,\omega^{1/2} = B\,\omega^{1/2} = \text{in violation}$$

$$[\text{curl } \mathbf{E}(r,0,z)]_\theta = -j\sqrt{RC}\ e^{-j\pi/4}\sqrt{\omega}\ E_r - \partial_r E_z$$

$$= -j\sqrt{RC}\ e^{-j\pi/4}\sqrt{\omega}\ A_r(r,0)\,\omega - \partial_r A_z(r,0)\sqrt{\omega}$$

But it happens that $A_z(r,0) = V R_{dc} \{\sqrt{C/R}\ e^{+j\pi/4}\}$ = constant, so $\partial_r A_z(r,0) = 0$ and we get

$$= A\,\omega^{3/2} \quad = \text{NOT in violation}$$

Finally, we have

$$[\text{curl } \mathbf{E}(r,m,z)]_z = r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r$$

$$= r^{-1}\partial_r(r\,A_\theta(r,m)\,\omega) - r^{-1}jm\,A_r(r,m)\,\omega$$

$$= \omega\,[r^{-1}\partial_r(r\,A_\theta(r,m)\,) - r^{-1}jm\,A_r(r,m)]$$

and

$$[\text{curl } \mathbf{E}(r,0,z)]_z = r^{-1}\partial_r(rE_\theta) - r^{-1}j0E_r = r^{-1}\partial_r(rE_\theta) = \omega\,[r^{-1}\partial_r(r\,A_\theta(r,0)\,] = \omega\,0$$

So this component is OK for $m \geq 0$.

Conclusion: For Appendix D fields we find that as $\omega \to 0$.

	$m > 0$	$m = 0$
$[\text{curl } \mathbf{E}(r,m,z)]_r$	$B_r \to \infty$	$B_r \to 0$
$[\text{curl } \mathbf{E}(r,m,z)]_\theta$	$B_\theta \to \infty$	$B_\theta \to 0$
$[\text{curl } \mathbf{E}(r,m,z)]_z$	$B_z \to$ finite	$B_z \to 0$

Example:

$$B_\theta(r,m) = +(j/\omega)\,[\text{curl } E]_\theta = (j/\omega)\,B\,\omega^{1/2} = j\,B\,\omega^{-1/2} \to \infty$$

I would say **this is a huge problem** with Appendix D that I failed to notice until recently!! These two fields are infinite in the $\omega \to 0$ limit:

$B_r(r,m)$ and $B_\theta(r,m)$ for $m > 0$

Review of 9/10/14 is fine from start down to here, I agree with everything said so far.

Discussion. Here are the assumptions that go into Appendix D:

1. e^{-jkz} dependence of everything on z (ansatz)

2. the two boundary conditions (ignoring Debye term) [see separate doc re-studying Debye]

3. the expressions for k(ω) and Z_0 as obtained from the TL equations. This is done in the Appendix K network model and in Chapter 4. Note that

network model => TL equations => $Z_0 = \sqrt{\mathbf{z}/\mathbf{y}}$ and $k = -j\sqrt{\mathbf{yz}}$
physics model => TL equations => $Z_0 = \sqrt{\mathbf{z}/\mathbf{y}}$ and $k = -j\sqrt{\mathbf{yz}}$

but in the physics model, we had to do "averaging" and this makes things hazy.

In App D we obtain solutions for the E fields which satisfy the E Helmholtz equations and div E = 0 and the two boundary conditions. When the B fields are obtained, we satisfy ALL four of Maxwell's equations.

So OK, is there some physical reason that $B_\theta(r,m) \rightarrow \infty$ for all $r \le a$ as $\omega \rightarrow 0$ in my infinite transmission line made from two cylinders? You would think just the opposite, that as $I \rightarrow 0$, the B_θ field would surely vanish! Since $J_z \rightarrow 0$ for all r (as $\sqrt{\omega}$) where would B_θ be coming from?? In fact ALL E_i fields are going to 0. All currents are then $J_i = 0$. How can you have $B_\theta = \infty$ if there are no currents to create the B_θ field.

Rephrase this question: What causes $[\text{curl } E]_\theta \rightarrow I$ as $\omega \rightarrow 0$?

$$[\text{curl } \mathbf{E}(r,m,z)]_\theta = -jk\, E_r - \partial_r E_z$$

with

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \sim \omega \qquad \text{(D.11.7)}$$
$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\, (|m|+1) \quad \sim \omega^{1/2}$$

The problem arises from the second term $-\partial_r E_z$ which has $I \sim \sqrt{\omega}$ dependence on ω, and thus we get that $-\partial_r E_z \sim \sqrt{\omega}$ causing $[\text{curl } \mathbf{E}(r,m,z)]_\theta \sim \sqrt{\omega}$ causing $B_\theta \sim \omega^{-1/2}$.

Can I see this problem in the non-small-ω limit expressions? [yes, as shown below]

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

I want to ponder the quantity $-\partial_r E_z$. It already has $\sqrt{\omega}$ from the I, and another $\sqrt{\omega}$ from β', but what happens with $\partial_r f_m(r)$? Should be easy to do:

$$\partial_r f_m = dx/dr\ d/dx\ f(x) = \beta'\ f'_m(x) = \beta' \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] * J'_m(x)$$

$$= (\beta'/2) \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \{ J_{m-1}(x) - J_{m+1}(x) \} \ .$$

Now for small x and x_a the leading term in each bracket is the one selected below,

$$\approx (\beta'/2) \left[\frac{1}{J_{m+1}(x_a)}\right] \{ J_{m-1}(x) \} \sim \beta' (x_a)^{-m-1} x^{m-1} = \beta' (x/x_a)^{m-1} x_a^{-2} \sim \beta' \beta'^{-2}$$

$$\sim 1/\beta' \sim 1/\sqrt{\omega} \ .$$

Then I have shown that

$$-\partial_r E_z \sim \sqrt{\omega}\ \sqrt{\omega}\ \ 1/\sqrt{\omega} \sim \sqrt{\omega}$$

which agrees with the small ω field expression results, and which results in $B_\theta \rightarrow \infty$.

Comment 1. The problem arises because $B_\theta \sim (1/\omega)\ \partial_r E_z$ and $E_z \sim I\ r^m \sim \sqrt{\omega}\ r^m$. At small β, the E Helmholtz equations are $(\nabla^2 + \beta^2)\mathbf{E} = 0$ and in particular $(\nabla^2 + \beta^2)\ E_z = 0$ which is then $\nabla^2\ E_z \approx 0$ which is in fact the 3D scalar Laplace equation. If we assume e^{-jkz} then $\partial_z \rightarrow -jk$ and we then end up with a 2D scalar Laplace equation involving polar variables r and θ. We know from Stakgold that the solutions of such an equation always have the form $r^m\ e^{jm\theta}$, and therefore we *expect* to have $E_z(r,m) \sim r^m$ and that is exactly what we are getting as just shown on the first line of this comment. Our issue then is not with the r^m part, it is with the *coefficient*. We solved the $(\nabla^2 + \beta^2)\mathbf{E} = 0$ Helm equations in App D with the ansatz e^{-jkz} and got the general form

$$E_z(r,m) = -j\ (\beta'/k) \frac{K_m}{2}\ J_m(x) \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(r,m) = a_m\ x^{-1}\ J_m(x) + \frac{K_m}{2}\ J_{m+1}(x) \qquad \beta'^2 = \beta^2 - k^2 \qquad \textit{(D.2.11)}$$

$$jE_\theta(r,m) = -a_m\ x^{-1}\ J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad \textit{(D.2.15)}$$

in terms of two coefficients a_m and K_m. Since $J_m(x) = (x/2)^m/m!$ and $x = \beta' r$, we see at once

$$E_z(r,m) = -j\ (\beta'/k) \frac{K_m}{2}\ (x/2)^m/m! \sim (\beta'/k) \frac{K_m}{2}\ \beta'^{\,m}\ r^m \sim (\beta')^{m+1}\ (1/k) \frac{K_m}{2}\ r^m$$

There is the expected r^m factor just explained. In order to get $B_\theta(r,m)$ finite, coefficient K_m just do exactly the right thing. But from my two boundary conditions I get

$$\frac{K_m}{2} = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \sim \omega \frac{1}{J_{m+1}(x_a)} \sim \omega\, (\beta' a)^{-m-1} \sim \omega\, (\beta')^{-m-1}$$

and then we get

$$E_z(r,m) = (\beta')^{m+1}\, (1/k)\, \frac{K_m}{2}\, r^m = (\beta')^{m+1}\, (1/k)\, \omega\, (\beta')^{-m-1}\, r^m = (1/k)\, \omega\, r^m = (\omega/k) r^m$$

and *once again* you have $(\omega/k) \sim \sqrt{\omega}$ and so $E_z(r,m) \sim \sqrt{\omega}\ r^m$ which causes $B_\theta = \infty$. Therefore, the only way to "fix" this problem is to change K_m to some other form, and that means one must change at least one of the boundary conditions.

Perhaps that answer is that the ansatz e^{-jkz} is not valid for small ω. But then why should we expect it to be valid for larger ω?

In partial waves where $\partial_\theta \rightarrow$ jm and also $\partial_z \rightarrow$ -jk, this equation reads

$$[r^2 \partial_r^2 + r\, \partial_r - m^2 + r^2\, \beta'^2]\, E_z(r,m) = 0 \ . \qquad \text{(D.1.15)}$$

Review of 9/10/14 is fine from start down to here, I agree with everything said so far.

Update 9.10.14.

Appendix Q has been rewritten, but it makes no difference for the curl E problem since the super low ω limits are the same. Of course I didn't really know that till I did it Here is the problem in a nutshell:

I am proposing adding the following verbatim to Section D.11.

Another low-frequency anomaly and its implications.

We consider here a sequence of steps which leads to a contradictory result. The scenario here is a transmission line one of whose conductors is a round wire, but it is helpful just to think of both conductors being round wires as studied in Chapter 6. In this case we know for example that $\eta_m = -\eta_m$ where η_m are the partial wave moments of a round wire surface charge density.

1. For monochrome fields, Maxwell tells us that (in any coordinate system),

$$B_i = +(j/\omega)\, [\text{curl } \mathbf{E}]_i$$

2. The low ω and low k limit of the Appendix D partial wave **E** field is this:

$$\begin{aligned} E_z(r,m) &= (1/2)\, \eta_m\, I\, R_{dc}\, (r/a)^{|m|}\, (|m|+1) \qquad m>0 \\ E_r(r,m) &= (j/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)} \\ E_\theta(r,m) &= (1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}] \end{aligned}$$

These expressions incorporate the ansatz assumption that $\mathbf{E}(r,\theta,z) = \mathbf{E}(r,\theta,0)\, e^{-jkz}$ which is assumed at the very start of Appendix D.

3. In cylindrical coordinates, curl **E** is given by

$$\text{curl } \mathbf{E}(r,\theta,z) = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,] .$$

With the above mentioned ansatz we get $\partial_z \rightarrow -jk$. If at the same time we convert to partial waves where $\partial_\theta \rightarrow jm$, the above curl expression becomes,

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\, r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\,[-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,]$$

or

$$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + jkE_\theta$$
$$[\text{curl } \mathbf{E}(r,m,z)]_\theta = -jkE_r - \partial_r E_z$$
$$[\text{curl } \mathbf{E}(r,m,z)]_z = r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r .$$

For example, inserting the low-ω **E** field expressions from above gives,

$$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + jkE_\theta$$

$$= r^{-1}jm\,[(1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)] + jk\,[(1/4)\,\eta_m\, I\, R_{dc}\, (ak)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,]$$

$$= j\,(1/4)\,\eta_m\, R_{dc}\,\{\, 2\, I\, r^{-1}m\, (r/a)^{|m|}\,(|m|+1) + a\,k^2 I\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\} .$$

4. If we assume that the k appearing above for the internal round wire **E** field is the same as the k which appears in the network model of Appendix **, and in the large ω theory of Chapters 4 and 5, namely,

$$k = -j\sqrt{\mathbf{zy}} = -j\,\sqrt{(R+j\omega L)(G+j\omega C)}\,,$$

then using the parameter model of Appendix Q we know that, for $G_{dc} = C\,\omega_d = 0$,

Fact 4: The small ω limit for k(ω), assuming $\omega_d = 0$, is given by *(Q.4.9)*

$$\text{Re}(k) \approx +\sqrt{R_{dc}C/2}\;\sqrt{\omega}\,(\,1 - \tan_L/2) + O(\omega^{3/2})$$
$$\text{Im}(k) \approx -\sqrt{R_{dc}C/2}\;\sqrt{\omega}\,(\,1 + \tan_L/2) + O(\omega^{3/2})$$

We might as well assume a vacuum dielectric, so than both $G_{dc} = 0$ and $\tan_L = 0$. In this case, we find that. as $\omega \rightarrow 0$,

$$k = \sqrt{R_{dc}C/2}\;\omega^{1/2}(1-j) = \sqrt{R_{dc}C}\;\; e^{-j\pi/4}\sqrt{\omega} \qquad \text{as } \omega \rightarrow 0$$

so

$$k^2 = R_{dc}C\, e^{-j\pi/2}\omega = -jR_{dc}C\omega\,. \qquad \text{// as also in (D.11.2)}$$

Now from (D.2.31) we can determine how I depends on ω for small ω ($N_0 = \langle n(\theta) \rangle$),

$$I = 2\pi\omega\,(a/k)\,N_0 = 2\pi\omega a N_0 [e^{+j\pi/4} / (\sqrt{R_{dc}C}\,\sqrt{\omega}\,)] = 2\pi a N_0\, e^{+j\pi/4}\sqrt{\omega}\,/\sqrt{R_{dc}C}$$

which says $I \sim \sqrt{\omega}$ for small ω. As expected, $I \to 0$ since $Z_0 \to \infty$ as $\omega \to 0$. Then also,

$$k^2 I = -jR_{dc}C\omega * 2\pi a N_0\, e^{+j\pi/4}\sqrt{\omega}\,/\sqrt{R_{dc}C} = \text{constant} * \omega^{3/2}.$$

5. Installing these expressions for k^2 and $k^2 I$ into $[\text{curl } \mathbf{E}(r,m,z)]_r$ gives: (all for m > 0)

$$[\text{curl } \mathbf{E}(r,m,z)]_r = j\,(1/4)\,\eta_m\,R_{dc}\,\{\,2\,I\,r^{-1}\,m\,(r/a)^{|m|}\,(|m|+1) + a\,k^2 I\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\}$$

$$= m\,a(r,|m|)\,\sqrt{\omega} + b(r,|m|)\,\omega^{3/2} \quad . \qquad // \ \eta_m = \eta_{-m} = \eta_{|m|} \text{ for Ch 6 TL}$$

6. From Step 1 we then find that, as $\omega \to 0$,

$$B_r(r,m) = +(j/\omega)\,[\text{curl } E]_r = (j/\omega)\,[m\,a(r,|m|)\sqrt{\omega} + b(r,|m|)\,\omega]$$

$$= j\,m\,a(r,|m|)\,/\sqrt{\omega} + j\,b(r,|m|) \approx j\,m\,a(r,|m|)\,/\sqrt{\omega} \quad \text{as } \omega \to 0$$

$$\to \infty \qquad \text{for } m \neq 0$$

so in the DC limit of the theory, we end up with an infinite radial magnetic field! More generally we find,

	m > 0	m = 0
$[\text{curl } \mathbf{E}(r,m,z)]_r$	$B_r \to \infty$	$B_r \to 0$
$[\text{curl } \mathbf{E}(r,m,z)]_\theta$	$B_\theta \to \infty$	$B_\theta \to 0$
$[\text{curl } \mathbf{E}(r,m,z)]_z$	$B_z \to$ finite	$B_z \to 0$

7. Do the +m and - m terms cancel?

$$B_r(r,\theta) = B_r(r,m)\,e^{jm\theta} + B_r(r,-m)\,e^{-jm\theta} \qquad // \text{ m and -m terms only}$$

$$= j\,a(r,|m|)/\sqrt{\omega} * [\,m\,e^{jm\theta} + (-m)\,e^{-jm\theta}]$$

$$= j\,m\,a(r,|m|)/\sqrt{\omega} * [\,2j\sin(m\theta)] \qquad \to \infty \text{ still}$$

So cancellation does not save the day.

8. Conclusions: In looking for traveling wave solutions of the form $e^{j(\omega t - kz)}$ going down a transmission line, the assumption that $k = -j\sqrt{\mathbf{zy}} = -j\sqrt{(R+j\omega L)(G+j\omega C)}$ for ultra-low ω leads to the contradiction of infinite B fields as $\omega \to 0$ (the DC limit of the theory). The assumption $k = -j\sqrt{\mathbf{zy}}$ follows from the transmission line equations (4.11.14/15) or (K.5/6) and from the requirement that the interior and exterior

wave solutions must match at the round wire surfaces. We conclude that the transmission line equations are not valid at ultra-low frequencies. These equations were derived in Chapters 4 and 5 only for "large" ω, although the network model of Appendix K supports the transmission line equations all the way down to ω → 0. We conclude therefore that the network model is invalid at ultra-low frequencies as a representation of the physics of an actual transmission line.

Redo everything for the G> 0 case. Start with

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\, (|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\, R_{dc}$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\, (ak)\, (r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

And write these as

$$E_z(r,m) = A_m\, I\, (r/a)^{|m|}$$
$$E_r(r,m) = B_m\, I\, k\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = B_m\, I\, k\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = A_0\, I$$
$$E_r(r,0) = B_0\, I\, k$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

Rewrite using $I\, k = 2\pi\omega\, a\, N_0$

$$E_z(r,m) = A_m\, I\, (r/a)^{|m|}$$
$$E_r(r,m) = B_m\, 2\pi\omega\, a\, N_0\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = B_m\, 2\pi\omega\, a\, N_0\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = A_0\, I$$
$$E_r(r,0) = B_0\, 2\pi\omega\, a\, N_0$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

Now use $I = 2\pi(\omega/k)\, a\, N_0$, so once again

$$E_z(r,m) = A_m'\, (\omega/k)\, (r/a)^{|m|}$$
$$E_r(r,m) = B_m\, 2\pi\omega\, a\, N_0\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = B_m\, 2\pi\omega\, a\, N_0\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = A_0'\, (\omega/k)$$

$E_r(r,0) = B_0\ 2\pi\omega\ a\ N_0$

$E_\theta(r,0) = 0$ // low ω E fields

The dependences are the same for m = 0 or m > 0 really. Meanwhile,

$[\text{curl } \mathbf{E}(r,m,z)]_r = r^{-1}jmE_z + jkE_\theta = (\omega/k) + k\omega$

$[\text{curl } \mathbf{E}(r,m,z)]_\theta = -jkE_r - \partial_r E_z = k\omega + (\omega/k)$

$[\text{curl } \mathbf{E}(r,m,z)]_z = r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\ . = \omega + \omega$

So if $k \rightarrow k_0$, then *all* these things are proportional to ω and we do *avoid* all infinite B fields

B_r = constant

B_θ = constant

B_z = constant