Appendix D for a Parallel Plate Transmission line PhL 3.18.14

Review 5.15.14. Basically I replicate pretty much all of Appendix D replacing the round wire as part of a twin-lead transmission line with fat rectangular conductors with a gap between them, the "parallel plate transmission line". My motivation is to see what happens in terms of J_z asymmetry in this case for $\omega \to 0$. Although I don't say so, I assume that the two plates are infinitely fat, and that the gap is small and the plates are also "high" relative to the gap size (I do say this).

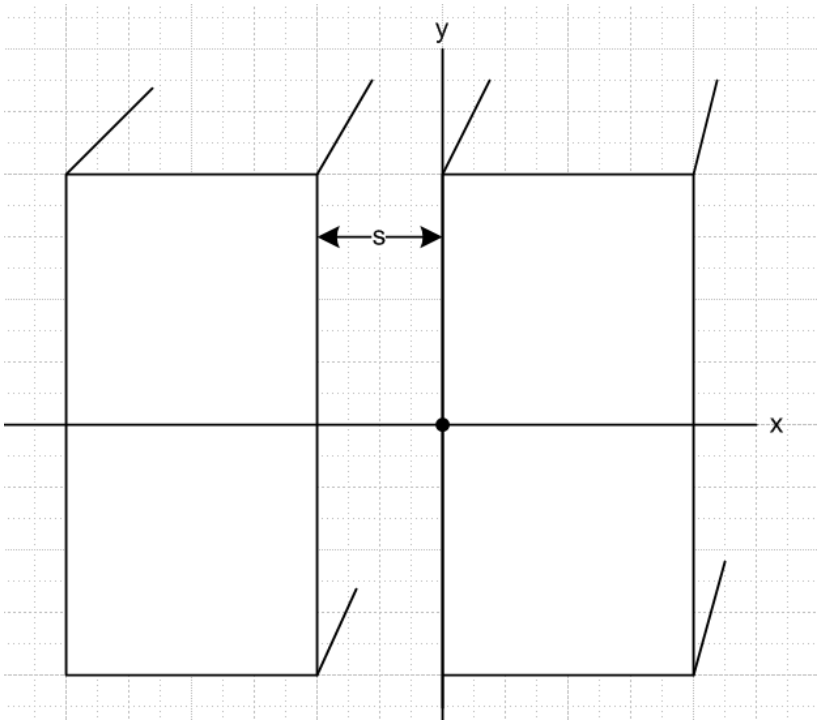
In this geometry, J_z asymmetry means that J_z is larger in both conductors near the gap (I am only studying one conductor I guess). I find that

$$J_z(x) = (1+j)\,(v_d/\delta)\,n \exp(-x/\delta)\ \exp(-j\,x/\delta)$$

which of course is just a usual skin effect situation. You see the current "asymmetry" $\exp(-x/\delta)$. I cheer below and say that as $\omega \to 0$, we have $\delta \to \infty$ and then "the asymmetry goes away at $\omega = 0$". But one could also say that for any $\omega > 0$, no matter how small, there is asymmetry. But it is different from the twin lead case where it seems that that asymmetry pattern is constant as $\omega \to 0$ since it tracks $n(\theta)$. This geometry doesn't really have an $n(\theta)$ because the fat conductors have only a surface facing the gap where I presume we find that $n(\theta)$ = constant. So although this was a valiant effort, I am not sure if the conclusion is very helpful.

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Motivation: I cannot obtain constant J_z at $\omega = 0$ in Appendix D. Maybe PP is a simpler geometry which will shed light on this ongoing problem. Here is the picture, and we shall be interested in the fields inside the right conductor, so we put its left edge at x = 0



D.1 Partial Wave Expansion

(a) The General Method

$$(\nabla^2 - \mu\varepsilon\,\partial_t^2 - \mu\sigma\partial_t)\mathbf{E}(x,y,z,t) = 0\ . \qquad \textit{(1.3.36)}$$

$$\mathbf{E}(x,y,z,t) = e^{j(\omega t - \beta_d z)}\,\mathbf{E}(x,y)\ . \qquad \text{(D.1.1)}$$

$$(\nabla^2 + \beta^2)\,\mathbf{E}(x,y,z,t) = 0 \qquad \text{(D.1.2a)}$$

or

$$[\nabla_{2D}^2 + (\beta^2 - \beta_d^2)]\,\mathbf{E}(x,y = 0 \qquad \nabla^2 = \nabla_{2D}^2 + \partial_z^2 \qquad \text{(D.1.2b)}$$

(b) Partial Wave Expansions

Skip the partial wave expansion!

(c) The Vector Laplacian

The vector Laplacian is this

$$(\nabla^2\mathbf{E})_x = \nabla^2 E_x$$
$$(\nabla^2\mathbf{E})_y = \nabla^2 E_y$$
$$(\nabla^2\mathbf{E})_z = \nabla^2 E_z \qquad \text{(D.1.12)}$$

(d) The three Helmholtz equations and div E = 0

1. The z equation.

$$[\partial_x^2 + \partial_y^2 + \partial_z^2 + \beta^2]\, e^{j(\omega t - \beta_d z)}\, E_z(x,y) \qquad (1)$$

Now assume that the parallel plate capacitor is very tall and so we expect fields away from the edge effect regions to be independent of y. Then we can simplify the above to read

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_z(x) \qquad (D.1.15)$$

Conversion Rules**:** $\partial_z \rightarrow -j\beta_d$ (D.1.16)

2. The x equation.

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_x(x) = 0 \qquad (D.1.17)$$

3. The y equation.

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_y(x) = 0 \qquad (D.1.18)$$

4. The z equation

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_z(x) = 0 \qquad (D.1.15)$$

5. The divE = 0 equation

$$\partial_x E_x(x) + \partial_y E_y(x) - j\beta_d E_z(x) = 0$$

or

$$\partial_x E_x(x) - j\beta_d E_z(x) = 0 \qquad (D.1.19)$$

The Three Helmholtz Equations and the div E = 0 equation (in partial waves) (D.1.20)

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_z(x) = 0 \qquad \textit{(D.1.15)}$$

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_x(x) = 0 \qquad \textit{(D.1.17)}$$

$$[\partial_x^2 + (\beta^2 - \beta_d^2)]\, E_y(x) = 0 \qquad \textit{(D.1.18)}$$

$$\partial_x E_x(x) - j\beta_d E_z(x) = 0 \qquad \textit{(D.1.19)}$$

D.2 Solutions for E_z, E_x and E_y

(a) The E_z Solution

Equation of interest is

$$[\partial_x^2 + \beta'^2]\, E_z(x) = 0 \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad \textit{(D.1.15)}$$

Still claim $\left|\frac{\beta^2}{\beta_d^2}\right| >> 1$ as before so we then have

$$\beta' \approx \beta = \beta = (j-1)\sqrt{\omega\mu\sigma/2} = [(j-1)/\sqrt{2}\,]\ \sqrt{\omega\mu\sigma} = [(j-1)/\sqrt{2}\,]\ (\sqrt{2}/\delta) = (j-1)\,(1/\delta)$$

The solution to this equation we take to be

$$E_z(x) = c_z \exp(j\,\beta' x) = c_z \exp(j\ [(j-1)/\sqrt{2}\,]\ (\sqrt{2}/\delta)\, x)$$

$$= c_z \exp(j\, j/\sqrt{2}\ (\sqrt{2}/\delta)\, x)\exp(j\,(-1)/\sqrt{2}\ (\sqrt{2}/\delta)\, x)$$

$$= c_z \exp(-x/\delta)\ \exp(-j\, x/\delta) \qquad = c_z \exp[-(1+j)\, x/\delta]$$

We select this sign for the right conductor in order to get exponential decay going into the conductor. This then is our skin effect, very simple. Constant c_z is TBD.

(b) The E_x Solution

Equation of interest is

$$[\partial_x^2 + (\beta^2-\beta_d^2)]\, E_x(x) = 0 \qquad \textit{(D.1.17)}$$

Solution is

$$E_x(x) = c_x \exp(-x/\delta)\ \exp(-j\, x/\delta)$$

Same solution, different constant.

(c) The E_y Solution

Equation of interest is

$$[\partial_x^2 + (\beta^2-\beta_d^2)]\, E_y(x) = 0 \qquad \textit{(D.1.17)}$$

Solution is

$$E_y(x) = c_y \exp(-x/\delta)\ \exp(-j\, x/\delta)$$

Same solution, different constant.

Now what does div E = 0 add to the story?

$$\partial_x E_x(x) - j\beta_d E_z(x) = 0 \qquad (D.1.19)$$

so

$$\partial_x\{ c_x \exp[-(1+j)\, x/\delta] \} - j\beta_d\{ c_z \exp[-(1+j)\, x/\delta] \} = 0$$

or

$$[-(1+j)/\delta * c_x \exp[-(1+j)\, x/\delta] - j\beta_d\{ c_z \exp[-(1+j)\, x/\delta] \} = 0$$

or

$$[-(1+j)/\delta * c_x - j\beta_d\{ c_z \} = 0$$

or

$$-(1+j)\, c_x /\delta * - j\beta_d c_z = 0$$

or

$$-(1+j)\, c_x = j(\beta_d \delta) c_z$$

which relates c_x and c_z, leaving us with only two independent constants, same as with K_m and a_m

First summary of the E field solutions (D.2.21)

$$E_z(x) = c_z \exp(- x/\delta) \exp(-j\, x/\delta)$$

$$E_x(x) = c_x \exp(- x/\delta) \exp(-j\, x/\delta)$$

$$E_y(x) = c_y \exp(- x/\delta) \exp(-j\, x/\delta)$$

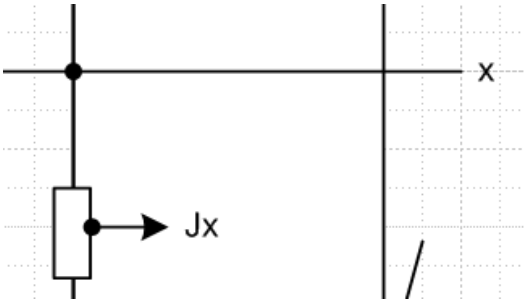
$$-(1+j)\, c_x = j(\beta_d \delta) c_z \qquad \text{// from div } \mathbf{E} = 0$$

Notice that last time we had two constants K_m and a_m

(d) The Charge Pumping Boundary Condition

$$\text{div } \mathbf{J} = - j\omega\rho \quad \Leftrightarrow \quad -j\omega[\int_V \rho\, dV] = \int_S \mathbf{J} \bullet d\mathbf{S}\,. \qquad (D.2.22)$$

Here is our picture, piece of the above



$$\int_S \mathbf{J} \bullet d\mathbf{S} = J_x \, dS$$

$$-j\omega[\int_V \rho \, dV] = -j\omega \, n(y) \, dS$$

Therefore

$$J_x(0+\varepsilon) = -j\omega \, n(y)$$

where J_x is just below the surface. But we know n(y) does not vary with y so we have

$$J_x(x=0+\varepsilon) = -j\omega n$$

and this is the charge pumping boundary condition.

The other boundary condition is that $E_y(x=0) = 0$. This is because the E field is perp to the surface. So:

We have two boundary conditions to impose:

$$E_x(x=0+\varepsilon) = -(j\omega/\sigma)n \qquad \text{// just inside conductor} \qquad \textit{(D.2.26)}$$

$$E_y(x=0) = 0 \qquad \textit{(D.2.27)}$$

We know E_y is continuous through the boundary like E_θ in Appendix D, so no $\pm\varepsilon$ needed.

(e) Application of the Boundary Conditions

The second boundary condition when applied says:

$$E_y(x) = c_y \exp(-0/\delta) \exp(-j\,0/\delta) = 0$$

or

$$c_y = 0$$

That was not too hard! Next we have

$$E_x(x=0+\varepsilon) = c_x \exp(-0/\delta) \exp(-j\,0/\delta) = -(j\omega/\sigma)n$$

Therefore

$$c_x = -(j\omega/\sigma)n$$

Then we bring in our div E = 0 condition,

$$-(1+j)\, c_x = j(\beta_d\delta)c_z$$

or

$$-(1+j)\,\{-(j\omega/\sigma)n\} = j(\beta_d\delta)c_z$$

or

$$(1+j)\,\{(j\omega/\sigma)n\} = j(\beta_d\delta)c_z$$

or

$$(j-1)\,(\omega/\sigma)n = j(\beta_d\delta)c_z$$

or

$$(1+j)\,(\omega/\sigma)n = (\beta_d\delta)c_z$$

$$c_z = (\beta_d\delta)^{-1}\,(1+j)\,(\omega/\sigma)n$$

To summarize:

$$c_y = 0 \qquad \text{(D.2.28)}$$
$$c_x = -(j\omega/\sigma)n$$
$$c_z = (\beta_d\delta)^{-1}\,(1+j)\,(\omega/\sigma)n$$

Installing these constants into (D.2.21) above we get

Second summary of the E field solutions : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ (D.2.33)

$$E_z(x) = (\beta_d\delta)^{-1}\,(1+j)\,(\omega/\sigma)\, n \exp(-x/\delta)\, \exp(-j\, x/\delta) \qquad \delta \equiv \sqrt{2/(\omega\mu\sigma)}$$

$$E_x(x) = -(j\omega/\sigma)n\, \exp(-x/\delta)\, \exp(-j\, x/\delta)$$

$$E_y(x) = 0$$

So far, things have gone well with this parallel plate case, or so it would seem.

Examination of the solution found for $E_z(x)$

To follow Appendix D, the next step would be to show that these solutions solve the three Helm equations and solve the BC's. I will do that in a moment in a new Maple file. But for the moment, let's examine E_z as $\omega \to 0$ which is my main interest right now. to do this note first that

$$(\beta_d\delta)^{-1}\,(\omega/\sigma) = (1/\beta_d)\,(\omega/\sigma)(1/\delta) = (v_d/\omega)\,(\omega/\sigma)(1/\delta) = (v_d/1)\,(1/\sigma)(1/\delta) = (v_d/\sigma\delta)$$

so then

$$E_z(x) = (1+j)\,(v_d/\sigma\delta)\, n \exp(-x/\delta)\, \exp(-j\, x/\delta)$$

$$J_z(x) = (1+j)\,(v_d/\delta)\, n \exp(-x/\delta)\, \exp(-j\, x/\delta)$$

Now let's assume the right conductor is infinitely wide and has some large height Y. Then

$$I = Y * \int_0^\infty J_z(x)\, dx = Y\,(1+j)\,(v_d/\delta)\, n \int_0^\infty \exp[\,-(1+j)\,x/\delta\,]\, dx$$

But we know that the integral $\int_0^\infty dx\, e^{-ax} = 1/a$, and in our case we have $a = (1+j)/\delta$ so then

$$I = Y\,(1+j)\,(v_d/\delta)\, n\, \delta/(1+j) = Y\,(v_d/1)\, n = Y\, n/v_d = \text{total volume current} = \text{finite}$$

Now look at our main result here, which integrates to the above finite value for I,

$$J_z(x) = (1+j)\,(v_d/\delta)\, n \exp(-x/\delta) \exp(-j\,x/\delta)$$

As $\omega \rightarrow 0$, we know that $\delta \rightarrow \infty$. In this limit we get

$$J_z(x) = (1+j)\,(v_d/\delta)\, n \qquad \text{but } \delta \rightarrow \infty$$

We have to have $J_z(x) \rightarrow 0$ because it is going to a constant value and we have an infinite x integral! But here is the point.

Conclusion: For ω very small but not zero, $J_z(x)$ has a very slight decay going into the right conductor from its surface at x = 0. But as $\omega \rightarrow 0$, $J_z(x)$ becomes a *constant value*, spread over the entire conductor. And each conductor carries $I = Y\, n/v_d$ which is a finite current.

So we don't get any residual J_z "asymmetry" in the limit $\omega \rightarrow 0$.

This seems to conflict my conclusion with the two cylinders!

In the parallel plate case we have several advantages:

(1) no need for partial waves
(2) Cartesian coordinates make equations simpler
(3) the solutions are just exponentials, not special functions, so don't need fancy limits.

Verify the E field solutions in Maple

Now finally let's check that our solutions satisfy the three Helm and the two BC's :

Maple file: par plates 1.mws

Here are the four expressions we ant to be 0:

```
restart;
alias(I=I, j= sqrt(-1), J=BesselJ):
```

Enter the left sides of the four equations in box (D.1.20):

```
e1 := Diff(Ez,x,x) + bp^2*Ez;
```

$$e1 := \left(\frac{\partial^2}{\partial x^2} Ez\right) + bp^2\, Ez$$

```
e2 := Diff(Ex,x,x) + bp^2*Ex;
```

$$e2 := \left(\frac{\partial^2}{\partial x^2} Ex\right) + bp^2\, Ex$$

```
e3 := Diff(Ey,x,x) + bp^2*Ey;
```

$$e3 := \left(\frac{\partial^2}{\partial x^2} Ey\right) + bp^2\, Ey$$

```
e4 := Diff(Ex,x) - j*bd*Ez;
```

$$e4 := \left(\frac{\partial}{\partial x} Ex\right) - j\, bd\, Ez$$

Here are the solution fields in terms of their constants

```
Ez := cz*exp(-(1+j)*x/delta);
```

$$Ez := cz\, \mathbf{e}^{\left(\frac{(-1-j)\,x}{\delta}\right)}$$

```
Ex := cx*exp(-(1+j)*x/delta);
```

$$Ex := cx\, \mathbf{e}^{\left(\frac{(-1-j)\,x}{\delta}\right)}$$

```
Ey := cy*exp(-(1+j)*x/delta);
```

$$Ey := cy\, \mathbf{e}^{\left(\frac{(-1-j)\,x}{\delta}\right)}$$

and here are the constants as found above in the main body,

```
cy := 0;
```

$$cy := 0$$

```
cx := -(j*omega/sigma)*n;
```

$$cx := -\frac{j\,\omega\,n}{\sigma}$$

```
cz := (bd*delta)^(-1)*(1+j)*(omega/sigma)*n;
```

$$cz := \frac{(1+j)\,\omega\,n}{bd\,\delta\,\sigma}$$

```
bp := (j-1)/delta;
```

We now show that the four expressions above all evaluate to 0 (they were inspected)

Show that each left side evaluates to zero as box (D.1.20) requires

```
e1:value(e1): simplify(%);
```

$$0$$

```
e2:value(e2): simplify(%);
```

$$0$$

```
e3:value(e3): simplify(%);
```

$$0$$

```
e4:value(e4): simplify(%);
```

$$0$$

And finally we show that the boundary conditions are met:

Now check the boundary conditions

```
> x := 0;
```

$$x := 0$$

```
> Ex;
```

$$-\frac{j\,\omega\,n}{\sigma}$$

```
> Ey;
```

$$0$$

So everything checks out.