Plan C Revised PhL 8.8.14

At the end of "why superposition doesn't work" I realized that you need to add an overall coefficient that is a function of k. So let's step through Plan C of "standing wave version of Appendix D".

In this valiant document, I construct a pretty good "reflection scenario" model which I think correctly handles the transmission line with a shorting bar. Lots of things come out right, but the J_z asymmetry in the DC limit does not go away, now for finite DC current i(z). Basically, as I always suspected and as I now have confirmed, the J_z asymmetry present in the Appendix D solution simply propagates into the reflection scenario. If I really want to solve this problem, I have to solve it in Appendix D. There is another problem with the reflection solution at DC, which is that curl E ≠ 0 and I don't know why [see Section III.10 below), but it is perhaps related to the J_z asymmetry problem of Appendix D. So any further work on this subject should I think be done with the Appendix D single-wave solution and not with this derived reflection scenario solution.

Note added 10.8.14. Notice the comment above about curl E ≠ 0 at DC. Another way to say this is that B is blowing up at ω = 0 since curl E = -jωB, and that is exactly what I much later found as a problem in Chapter 7. Now I know why I was having this curl E problem back in August!

[converted from β_d to k on 8.18.14]

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Part I: General transmission line quantities (outside the conductors)

• Let's now try this variation of Plan B. Let's have the reflected -z wave have opposite sign instead of the same sign, and at the same time replace z → z-L where L will be the "reflection point". Then we get

$$E_{i,tot}(r,m,z,t) = A_i(k)\,[\, e^{j(\omega t - k[z-L])}\, E_i(r,m;k) \;-\; e^{j(\omega t + k[z-L])}\, E_i(r,m;-k)]$$

I have added an arbitrary scaling function $A_i(k)$ but we shall not use the above for a while.

[I am assuming G = 0 here so there is no confusion about the meaning of n(θ), and also **y** = jωC. It is in fact with G = 0 that I have my must mysterious results!]

I expect these two waves to have n(θ) and -n(θ), so I expect to get

$$n(\theta,z)\, e^{j\omega t} = B(k)\, \{e^{j(\omega t - k[z-L])}\, n(\theta) \;+ e^{j(\omega t + k[z-L])}\, [-n(\theta)]\}$$ // ansatz

$$= \; - e^{j\omega t}\, 2j\, B(k)\, \sin(k[z-L])\, n(\theta)$$

where n(θ) is the surface charge for the +z going infinite line, and where I have added a TBD scaling constant which is a function of k. Then

$$n(\theta,z) = - 2j\, B(k)\, \sin(k[z-L])\, n(\theta)$$

and THIS is a charge density that will always be 0 at z = L, as might be correct for a shorting bar at that location!

Now let's assume at at z = 0 we have some n(θ,z=0). Then we can write

$$n(\theta,z=0) = - 2j\, B(k)\, \sin(k[-L])\, n(\theta)$$

or

$$n(\theta,z=0) = 2j\, B(k)\, \sin(kL)\, n(\theta)$$

and we can solve for B:

$$B(k) = [n(\theta,z=0)/\, n(\theta)] \,/\, [2j\, \sin(kL)]$$

Installing this then gives

$$n(\theta,z) = - 2j\, B(k)\, \sin(k[z-L])\, n(\theta)$$

$$= - 2j\, \{\, [n(\theta,z=0)/\, n(\theta)] \,/\, [2j\, \sin(kL)]\}\, \sin(k[z-L])\, n(\theta)$$

$$= - \; \{\, [n(\theta,z=0)] \,/\, [\, \sin(kL)]\}\, \sin(k[z-L])$$

$$= - n(\theta,z=0)\, \frac{\sin(k[z-L])}{\sin(kL)}$$

$$= + \; n(\theta,z=0)\; \frac{\sin(k[L-z])}{\sin(kL)} \qquad (*)$$

This is then a standing wave pattern for n(θ,z) which has a maximum for example at z = 0. For large k there might be several peaks with this same max value, and in between nodes where n = 0. If we take the limit k → 0 we get

$$n(\theta,z) = \; n(\theta,z=0) \; \{ (L-z)/L \} \; = n(\theta,z=0) \{ 1 - z/L \}$$

which I would call a reasonable limit. It just scales down linearly from the driving end to the shorting bar and z = L.

Comment: Everything seems reasonable about this n(θ,z) function (*)

1) it vanishes at z = L where the capacitor is shorted
2) it is maximal at the driving point
3) it exhibits the expected standing waves
4) it gives the right DC limit!

If we were to integrate this over θ, we end up with q(z), the charge density per unit length. Then

$$q(z) = q(0)\, \frac{\sin(k[L-z])}{\sin(kL)} \qquad \text{// general form}$$

$$q(z) = q(0) \{ 1 - z/L \} \qquad \text{// limit } k \rightarrow 0$$

Now since q(z) = CV(z), we have V(z) = q(z)/C and V(0) = q(0)/C, so we then have shown that

$$V(z) = V(0)\, \frac{\sin(k[L-z])}{\sin(kL)} \qquad \text{// general form}$$

$$V(z) = V(0) \{ 1 - z/L \} \qquad \text{// limit } k \rightarrow 0$$

This is a very reasonable form since it does all the things I want!

What about the current i(z)? If we go to some general location z on our line (say to the left of the shorting bar), we have this TL equation,

$$\frac{dV(z)}{dz} = - \mathbf{z}\, i(z) \qquad \mathbf{z} = R + j\omega L \qquad \text{// assumption that TL equations are valid!}$$

We can solve this for i(z) to get general result

$$i(z) = (-1/\mathbf{z})\, \partial_{\mathbf{z}} V(z) = \; (-1/\mathbf{z})\; V(0)\; [\; 1/\sin(kL)]\; \partial_{\mathbf{z}} \sin(k[L-z])$$

$$= (+1/\mathbf{z})\; V(0)\; [\; 1/\sin(kL)]\; \partial_{\mathbf{z}} \sin(k[z-L])$$

$= (+1/\mathbf{z})\ V(0)\ [\ 1/\sin(kL)]\ k \cos(k[z-L])$

$= (k/\mathbf{z})\ V(0)\ \dfrac{\cos(k[z-L])}{\sin(kL)}$

Then as $k \rightarrow 0$ we get

$i(z) = (k/\mathbf{z})\ V(0) / (kL) = V(0)/ (\ L\mathbf{z})$

which is a constant, independent of z. But at $\omega = 0$, we can write $\mathbf{z} = R$ where R is the total resistance of both conductors of the line per unit length. Then we get

$i(z) = V(0)/ (\ LR)$ $\qquad\qquad R = R_1 + R_2$

So here is a summary:

$i(z) = (k/\mathbf{z})\ V(0)\ \dfrac{\cos(k[z-L])}{\sin(kL)}$ // general form

$i(z) = V(0)/ (\ LR)$ // limit $k \rightarrow 0$

This also does just what I want: It is a max at z = L, and perhaps has other maxima for large k of the same size, with nodes in between. We can look at our two special locations:

$i(0) = (k/\mathbf{z})\ V(0) \cot(kL)$ // general form

$i(L) = (k/\mathbf{z})\ V(0)\ \dfrac{1}{\sin(kL)}$ // general form

Thus we can also write

$i(z) = i(L) \cos(k[z-L])$ // general form

$i(0) = i(L) \cos(kL)$

If we wanted, we could go on to define the impedance Z(z) in this way

$Z(z) = V(z)/i(z) = V(0)\ \dfrac{\sin(k[L-z])}{\sin(kL)}\ /\ [\ (k/\mathbf{z})\ V(0)\ \dfrac{\cos(k[z-L])}{\sin(kL)}\ ]$

$= (\mathbf{z}/k) \tan(k[L-z])$ // DC limit here is $Z(z) = \mathbf{z}\ (L-z) = R(L-z)$

$Z(L) = 0$ //which is correct for our shorting bar

$Z(0) = (\mathbf{z}/k) \tan(kL)$

Just for fun, let's check the other TL equation which says this:

$$\frac{di(z)}{dz} = - \mathbf{y}\, V(z)$$

so that

$$V(z) = (-1/\mathbf{y})\, \partial_{\mathbf{z}} i(z) = (-1/\mathbf{y})\, \partial_{\mathbf{z}} [i(L)\ \cos(k[z-L])] = (-1/\mathbf{y})\, i(L)\, \partial_{\mathbf{z}} \cos(k[z-L])$$

$$= (+k/\mathbf{y})\, i(L) \sin(k[z-L])$$

$$= (+k/\mathbf{y})\, (k/\mathbf{z})\, V(0)\ \frac{1}{\sin(kL)}\ \sin(k[z-L])$$

$$= - V(0)\ \frac{1}{\sin(kL)}\ \sin(k[z-L]) \qquad \text{// since } k^2 = k^2 = [\ -j\sqrt{\mathbf{yz}}\ ]^2 = - \mathbf{yz}$$

$$= V(0)\, \frac{\sin(k[L-z])}{\sin(kL)}$$

and yes, this does agree with the result found earlier!

Status: At this point, I think I know all these objects for my reflection superposition model:

V(z), q(z), i(z), Z(z)

$$\int d\theta\, n(\theta,z) = q(z) \quad = a \int d\theta\, n(\theta,z) \qquad\qquad n(\theta,z) = a\, n(\theta,z)$$

At z = 0, we imagine driving with some V(0) and there is then some q(0) = CV(0) per unit length. I know at this point z = 0 that

$$n(\theta,0) = (1/a)\ \frac{q(0)}{2\pi}\ \frac{|sh\xi_1|}{ch\xi_1 + \cos\theta}$$

where a is the radius of my conductor of interest and where ξ_1 is its bipolar coordinate label. So I guess I know n(θ,z) as well as all the other quantities listed.

So the quantities I claim to "know about" are those associated "in general" with a transmission line.

Summary of Transmission Line Quantities

<u>General form for any ω:</u>	<u>DC limit where ω=0</u>	$k = k(\omega) = -j\sqrt{\mathbf{zy}}$
$n(\theta,z) = n(\theta,z=0)\ \frac{\sin(k[L-z])}{\sin(kL)}$	$n(\theta,z) = n(\theta,z=0)\ \{\ 1 - z/L\ \}$	

$$q(z) = q(0)\,\frac{\sin(k[L-z])}{\sin(kL)} \qquad q(z) = q(0)\,\{\,1 - z/L\,\}$$

$$V(z) = V(0)\,\frac{\sin(k[L-z])}{\sin(kL)} \qquad V(z) = V(0)\,\{\,1 - z/L\,\}$$

$$i(z) = (k/\mathbf{z})\,V(0)\,\frac{\cos(k[z-L])}{\sin(kL)} \qquad i(z) = V(0)/\,(\,LR)$$

$$Z(z) = (\mathbf{z}/k)\tan(k[L-z]) \qquad Z(z) = R(L-z)$$

I have only assumed the transmission line equations for the line, and G = 0. The DC limits are all exactly what I expect them to be, so this is certainly encouraging and suggests that a reflection model might actually truly model the situation.

I next want to try to find out what is going on INSIDE one of the round conductors.

Part II: Superposition inside one of the conductors.

Please don't start off on the wrong foot here! At the start of Part I I wrote,

$$E_{i,tot}(r,m,z,t) = A_i(k)\; e^{j(\omega t - k[z-L])}\, E_i(r,m;k) + C_i(k)\; e^{j(\omega t + k[z-L])}\, E_i(r,m;-k)]$$

where the idea is that $E_i(r,m;k)$ is precisely a regular Appendix D E field solution. I am not really sure about the sign between the two terms so I just make two separate constants (functions of k) for now.

Claim: *Each* of the two terms shown above separately satisfies the following equations:

the three Helmholtz equations for E	yes
the div E = 0 equation	defer
the two boundary conditions	defer

Proof: We know the Helms are true for the first term for any $A_i(k)$ because that is what App D shows. It is then true as well for the second term since we just take $k \rightarrow -k$ everywhere and call upon "symmetry" for a wave going to the left instead of to the right.

We shall deal with the last two items below.

I think that I want $E_r = 0$ and $E_\theta = 0$ at z = L. I am now imagining that the "shorting bar" is in fact a reflecting plane or "brick" of something like solid silver that conducts really well, I assume it is a perfect conductor. Thus, just inside this brick, I expect to have $E_r = 0$ and $E_\theta = 0$. Since these are both tangential to the shorting plane, they are continuous through the boundary so then $E_r = 0$ and $E_\theta = 0$ at $z = L-\varepsilon$.

In contrast, I want E_z to be maximal at z = L because $J_z = \sigma E_z$ and i(z) is maximal at z = L.

Now consider the above equation with index i = r (radial),

$$E_{r,tot}(r,m,z,t) = A_r(k)\ e^{j(\omega t - k[z-L])}\ E_r(r,m;k) + C_r(k)\ e^{j(\omega t + k[z-L])}\ E_r(r,m;-k)]$$

Recall now from Appendix D that

$$E_z(r,m;k) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m;k) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m;k) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \quad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where $\beta'^2 = \beta^2 - k^2$. I can see then that g_m is invariant under negation of k. And I can see that (ak) negates. But what is the meanings of I in this expression?

I am going to take a shot in the dark here (based on looking at the E_z solution). I will say I = I(k) is the current going to the right in the infinite line solution, and this negates if you instead make the wave go to the left. This will then cause E_z to have the opposite sign in the left going wave. So therefore I will assume that I(k) negates when you negate k. Therefore, E_r (and also E_θ) do not change sign. We then have

$$E_{r,tot}(r,m,z,t) = A_r(k)\ e^{j(\omega t - k[z-L])}\ E_r(r,m;k) + C_r(k)\ e^{j(\omega t + k[z-L])}\ E_r(r,m;+k)]$$

Now, if we want to have $E_{r,tot}(r,m,z=L,t) = 0$ at the surface of the silver plane/brick terminator, we have to take this form

$$E_{r,tot}(r,m,z,t) = A_r(k)\ \{\ e^{j(\omega t - k[z-L])}\ E_r(r,m;k) - e^{j(\omega t + k[z-L])}\ E_r(r,m;+k)]$$

$$= A_r(k)\ e^{j\omega t}\ E_r(r,m;k)\ \{\ 2j\ \sin(k[L-z])\ \} \qquad \text{ie} \qquad C_r(k) = -A_r(k)$$

The exact same argument applies to E_θ, so we will also require that $C_\theta(k) = -A_\theta(k)$

What about E_z? Using the logic above, I argue that $E_z(r,m;-k) = -E_z(r,m;k)$ and therefore

$$E_{z,tot}(r,m,z,t) = A_z(k)\ e^{j(\omega t - k[z-L])}\ E_z(r,m;k) - C_z(k)\ e^{j(\omega t + k[z-L])}\ E_z(r,m;+k)]$$

$$= e^{j\omega t}\ E_z(r,m;k)\ \{\ A_z(k)\ e^{j(-k[z-L])} - C_z(k)\ e^{j(+k[z-L])}\ \}$$

I want this to be maximal at z = L. As another shot in the dark, I will assume

$$C_z(k) = -A_z(k)$$

and then I get

$$E_{z,tot}(r,m,z,t) = e^{j\omega t} A_z(k) E_z(r,m;k) [2 \cos(k[L-z])]$$

$$J_{z,tot}(r,m,z,t) = e^{j\omega t} \sigma A_z(k) E_z(r,m;k) [2 \cos(k[L-z])]$$

or

$$E_{z,tot}(r,m,z) = A_z(k) E_z(r,m;k) [2 \cos(k[L-z])]$$

$$J_{z,tot}(r,m,z) = \sigma A_z(k) E_z(r,m;k) [2 \cos(k[L-z])]$$

At least this has the property that it is maximal at z = L and so I then expect i(z) to be maximal at z = L since this is essentially an integral of E_z. Recall from lines doc that

$$J_{z,tot}(r,\theta) = \sum_{m=-\infty}^{\infty} J_{z,tot}(r,m) e^{jm\theta} \qquad \text{// expansion} \qquad \text{(D.1.3a)}$$

When we integrate this over the cross section area of our round wire, the dθ integral kills off all but the m = 0 term, so we have

$$\int d\theta J_{z,tot}(r,\theta) = \int d\theta J_{z,tot}(r,m=0) = 2\pi J_{z,tot}(r,m=0)$$

We then get

$$i(z) = \int dA J_{z,tot}(r,\theta) = \int r\, dr\; 2\pi J_{ztot}(r,m=0)$$

$$= \int r\, dr\; 2\pi \sigma A_z(k) E_z(r,0;k) [2 \cos(k[L-z])]$$

$$= 2\pi \sigma A_z(k) [2 \cos(k[L-z])] \int r\, dr\; E_z(r,m=0;k)$$

This same integral appears in lines doc from which I quote [see below (D.2.30)],

$$2\pi \sigma \int_0^a r\, dr\, E_z(r,m=0) = 2\pi\omega (a/k) N_0$$

Thus I get

$$i(z) = A_z(k) [2 \cos(k[L-z])] \{ \; 2\pi \sigma \int_0^a r\, dr\, E_z(r,m=0) \; \}$$

$$= A_z(k) [2 \cos(k[L-z])] \{ \; 2\pi\omega (a/k) N_0 \; \}$$

But in my work from Part I I found that

$$i(z) = (k/\mathbf{z})\, V(0)\, \frac{\cos(k[z-L])}{\sin(kL)}$$

Comparison then says

$$(k/\mathbf{z})\, V(0)\, \frac{\cos(k[z-L])}{\sin(kL)} = A_{\mathbf{z}}(k)\, [\, 2\cos(k[L-z])\,]\ \{\ 2\pi\omega\, (a/k)\, N_0\ \}$$

or

$$(k/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)} = A_{\mathbf{z}}(k)\, [\, 2\,]\ \{\ 2\pi\omega\, (a/k)\, N_0\ \}$$

or

$$(k^2/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)} = A_{\mathbf{z}}(k)\, 4\pi\omega a N_0$$

and we can then solve to get

$$A_{\mathbf{z}}(k) = (k^2/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)}\ \frac{1}{4\pi\omega a N_0}$$

But the Appendix D symbols I and N_0 are related by

$$N_0 = (k/2\pi\omega a)\, I \qquad (D.2.31)$$

or

$$N_0/2 = (k/4\pi\omega a)\, I$$

or

$$1/2 = (k/4\pi\omega a N_0)\, I$$

or

$$1/(2Ik) = (1/4\pi\omega a N_0) = \frac{1}{4\pi\omega a N_0}$$

and then we can write

$$A_{\mathbf{z}}(k) = (k^2/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)}\ \frac{1}{2Ik} = (k/\mathbf{z})\, V(0)\, \frac{1}{2I\sin(kL)}$$

and we have then evaluated our coefficient. We can then insert it into our expression above to get

$$E_{\mathbf{z},\mathrm{tot}}(r,m,z) = A_{\mathbf{z}}(k)\, E_{\mathbf{z}}(r,m;k)\, [\, 2\cos(k[L-z])\,]$$

$$= 2\cos(k[L-z])\ \{\ (k/\mathbf{z})\, V(0)\, \frac{1}{2I\sin(kL)}\ \}\ \{\ (1/4)\, \eta_{\mathrm{m}}\, I\, R_{\mathbf{dc}}\ (a\beta')\, f_{\mathrm{m}}\ \}$$

$$= \cos(k[L-z])\ \{\ (k/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)}\ \}\ \{\ (1/4)\, \eta_{\mathrm{m}}\ R_{\mathbf{dc}}\ (a\beta')\, f_{\mathrm{m}}(\beta' r)\ \}$$

and this then is our final expression for $E_{z,tot}(r,m,z)$ in our superposition model.

Status to this point:

$$E_{z,tot}(r,m,z) = \{ (k/z)\, V(0)\, \frac{\cos(k[L-z])}{\sin(kL)} \} \{ (1/4)\, \eta_m\, R_{dc}\, (a\beta')\, f_m(\beta' r) \}$$

$$E_{r,tot}(r,m,z) = A_r(k)\; E_r(r,m;k) \{ 2j \sin(k[L-z]) \}$$

$$E_{\theta,tot}(r,m,z) = A_\theta(k)\; E_\theta(r,m;k) \{ 2j \sin(k[L-z]) \}$$

Now what about the Appendix D boundary conditions? This one is obvious:

$$E_{\theta,tot}(r=a,m,z) = A_\theta(k)\; E_\theta(r=a,m;k) \{ 2j \sin(k[L-z]) \} = 0 \text{ since } E_\theta(r=a,m;k) = 0$$

The other boundary condition would be this:

$$E_{r,tot}(r=a-\varepsilon,\theta,z) = (j\omega/\sigma)\, n(\theta,z)$$

In m-space this says

$$E_{r,tot}(r=a,m,z) = (j\omega/\sigma)\, N_m(z)$$

From Part I we have that

$$n(\theta,z) = n(\theta,z=0)\, \frac{\sin(k[L-z])}{\sin(kL)}$$

$$N_m(z) = N_m(0)\, \frac{\sin(k[L-z])}{\sin(kL)}$$

where

$$n(\theta,z=0) = (1/a)\, \frac{q(0)}{2\pi}\, \frac{|sh\xi_1|}{ch\xi_1+\cos\theta}$$

and $N_m(0)$ is the transform of this which I computed somewhere.

Thus our boundary condition requires that

$$A_r(k)\; E_r(r=a,m;k) \{ 2j \sin(k[L-z]) \} = (j\omega/\sigma)\, N_m(0)\, \frac{\sin(k[L-z])}{\sin(kL)}$$

I am relieved to see that same sin factor on both sides, so we then have

$$A_{\mathbf{r}}(k)\ E_{\mathbf{r}}(r=a,m;k)\ \{\ 2j\ \}\ =\ (j\omega/\sigma)\ N_{\mathbf{m}}(0)\ 1\ /\ \sin(kL)$$

or

$$A_{\mathbf{r}}(k)\ E_{\mathbf{r}}(r=a,m;k)\ =\ (\omega/2\sigma)\ N_{\mathbf{m}}(0)\ 1\ /\ \sin(kL)$$

But recall now that

$$E_{\mathbf{r,tot}}(r,m,z)\ = A_{\mathbf{r}}(k)\ E_{\mathbf{r}}(r,m;k)\ \{\ 2j\ \sin(k[L-z])\ \}$$

Inserting the previous then gives

$$E_{\mathbf{r,tot}}(r,m,z)\ = A_{\mathbf{r}}(k)\ E_{\mathbf{r}}(r,m;k)\ \{\ 2j\ \sin(k[L-z])\ \}$$

$$= [\ E_{\mathbf{r}}(r,m;k)\ /\ E_{\mathbf{r}}(a,m;k)\ ]\ \ A_{\mathbf{r}}(k)\ E_{\mathbf{r}}(a,m;k)\{\ 2j\ \sin(k[L-z])\ \}$$

$$= [\ E_{\mathbf{r}}(r,m;k)\ /\ E_{\mathbf{r}}(a,m;k)\ ]\ (\omega/2\sigma)\ N_{\mathbf{m}}(0)\ 1\ /\ \sin(kL)\ \ *\ \{\ 2j\ \sin(k[L-z])\ \}$$

$$= \frac{E_{\mathbf{r}}(r,m;k)}{E_{\mathbf{r}}(a,m;k)}\ (j\omega/2\sigma)\ N_{\mathbf{m}}(0)\ \frac{\sin(k[L-z])}{\sin(kL)}$$

This expression then satisfies the boundary condition,

$$E_{\mathbf{r,tot}}(r=a-\varepsilon,\theta,z) = (j\omega/\sigma)\ N_{\mathbf{m}}(z)$$

at every point z along the line. Now

$$E_{\mathbf{r}}(r,m;k)\ = (j/4)\ \eta_{\mathbf{m}}\ I\ R_{\mathbf{dc}}\ (ak)\ g_{\mathbf{m}}(r)$$

$$E_{\mathbf{r}}(a,m;k)\ = (j/4)\ \eta_{\mathbf{m}}\ I\ R_{\mathbf{dc}}\ (ak)\ 2$$

and therefore

$$\frac{E_{\mathbf{r}}(r,m;k)}{E_{\mathbf{r}}(a,m;k)}\ = g_{\mathbf{m}}(r)/2$$

and we then have

$$E_{\mathbf{r,tot}}(r,m,z)\ =\ [g_{\mathbf{m}}(r)/2\ ]\ (j\omega/2\sigma)\ N_{\mathbf{m}}(0)\ \frac{\sin(k[L-z])}{\sin(kL)}$$

This is mildly encouraging (though of course I know better). So far we have

$$E_{\mathbf{z,tot}}(r,m,z)\ =\ \{\ (k/\mathbf{z})\ V(0)\ \ \frac{\cos(k[L-z])}{\sin(kL)}\ \}\ \{\ (1/4)\ \eta_{\mathbf{m}}\ \ R_{\mathbf{dc}}\ \ (a\beta')\ f_{\mathbf{m}}(\beta'r)\ \ \}$$

$$E_{r,tot}(r,m,z) = [g_m(r)/2\ ]\ (j\omega/2\sigma)\, N_m(0)\, \frac{\sin(k[L-z])}{\sin(kL)}$$

$$E_{\theta,tot}(r,m,z) = A_\theta(k)\ E_\theta(r,m;k)\ \{\ 2j\ \sin(k[L-z])\ \}$$

I think $A_\theta(k)$ will be determined now by the div $E_{tot} = 0$ equation.

$$\partial_r\ (r\ E_{r,tot}) + \partial_\theta E_{\theta,tot} + r\ \partial_z E_{ztot} = 0\ .$$

In partial waves this says

$$\partial_r\ (r\ E_{r,tot}(r,m,z)) + jm\ E_{\theta,tot}(r,m,z) + r\ \partial_z E_{z,tot}(r,m,z) = 0\ .$$

I think it will be easiest to write each total field in terms of its App D field. From above,

$$E_{z,tot}(r,m,z) = A_z(k)\ E_z(r,m;k)\ [\ 2\ \cos(k[L-z])\ ]$$

$$E_{\theta,tot}(r,m,z) = A_\theta(k)\ E_\theta(r,m;k)\ \{\ 2j\ \sin(k[L-z])\ \}$$

$$E_{r,tot}(r,m,z) = A_r(k)\ E_r(r,m;k)\ \{\ 2j\ \sin(k[L-z])\ \}$$

Then $\mathrm{div}E_{tot} = 0$ says

$$A_r(k)\ \{\ 2j\ \sin(k[L-z])\ \}\ \partial_r\ (r\ E_r(r,m;k)\ )$$

$$+\ A_\theta(k)\ \{\ 2j\ \sin(k[L-z])\ \}\ jm\ E_\theta(r,m;k)$$

$$+\ A_z(k)\ E_z(r,m;k)\ r\ \partial_z\{\ 2\ \cos(k[L-z])\ \} = 0$$

or

$$A_r(k)\ \{\ 2j\ \sin(k[L-z])\ \}\ \partial_r\ (r\ E_r(r,m;k)\ )$$

$$+\ A_\theta(k)\ \{\ 2j\ \sin(k[L-z])\ \}\ jm\ E_\theta(r,m;k)$$

$$+\ A_z(k)\ E_z(r,m;k)\ r\ \{\ 2k\ \sin(k[L-z])\ \} = 0$$

At least we have a chance here, and things simplify to

$$A_r(k)\ \{\ 2j\ \}\ \partial_r\ (r\ E_r(r,m;k)\ )$$

$$+\ A_\theta(k)\ \{\ 2j\ \}\ jm\ E_\theta(r,m;k)$$

$$+\ A_z(k)\ E_z(r,m;k)\ r\ \{\ 2k\ \} = 0$$

Divide through by 2j to get

$$A_r(k)\,\partial_r\,(r\,E_r(r,m;k))$$

$$+\;A_\theta(k)\,jm\,E_\theta(r,m;k)$$

$$-\,j\,k\,A_z(k)\;E_z(r,m;k)\,r = 0$$

I know from App D that

$$\partial_r\,[r\,E_r(r,m;k)] + jmE_\theta(r,m,k) + r\,(-jk)E_z(r,m,k) = 0$$

I can use this to eliminate the messy r derivative so we then have

$$A_r(k)\,\{\,-\,jmE_\theta(r,m,k) + r\,(jk)E_z(r,m,k)\,\}$$

$$+\;A_\theta(k)\,jm\,E_\theta(r,m;k)$$

$$-\,j\,k\,A_z(k)\;E_z(r,m;k)\,r = 0$$

which we regroup to get

$$E_\theta(r,m,k)\,\{\,-jm\,A_r(k) + A_\theta(k)\,jm\,\}$$

$$+\,E_z(r,m,k)\,\{\,r\,(jk)\,A_r(k)\;-\,r\,j\,k\,A_z(k)\,\} = 0.$$

Rewrite this requirement once again

$$E_\theta(r,m,k)\,jm\,\{\,-\,A_r(k) + A_\theta(k)\;\}$$

$$+\,E_z(r,m,k)\,(jkr)\{\,A_r(k)\;-\,A_z(k)\,\} = 0.$$

Each term has to separately be zero for this to work. This then requires that

$$A_r(k) = A_\theta(k) = A_z(k)$$

We can then select $A_\theta(k) = A_r(k)$ to get the first term to vanish. But what about the second term? We already determined the coefficients and I doubt they are going to "work out". We found that

$$A_r(k)\;E_r(r=a,m;k) \;=\; (\omega/2\sigma)\,N_m(0)\,/\,\sin(kL)$$ correctly copied from above

and

$$A_z(k) = (k^2/z)\,V(0)\;\frac{1}{\sin(kL)}\;\frac{1}{4\pi\omega a N_0}$$

I would like to show that

$A_r(k) - A_z(k) = 0$?

$A_r(k)\, E_r(r=a,m;k) - A_z(k)\, E_r(r=a,m;k) = 0$?

$(\omega/2\sigma)\, N_m(0)\, 1 / \sin(kL) - (k^2/z)\, V(0)\, \frac{1}{\sin(kL)}\, \frac{1}{4\pi\omega a N_0}\, E_r(r=a,m;k) = 0$?

$(\omega/2\sigma)\, N_m(0) - (k^2/z)\, V(0)\, \frac{1}{4\pi\omega a N_0}\, E_r(r=a,m;k) = 0$?

Now I know that

$E_r(r=a,m;k) = (j/4)\, \eta_m\, I\, R_{dc}(ak)\, g_m(a) = (j/2)\, \eta_m\, I\, R_{dc}(ak)$

So have to show that

$(\omega/2\sigma)\, N_m(0) - (k^2/z)\, V(0)\, \frac{1}{4\pi\omega a N_0}\, (j/2)\, \eta_m\, I\, R_{dc}(ak) = 0$?

This looks very unlikely! But recall that $N_0 = (k/2\pi\omega a)\, I$ so that $I/N_0 = 2\pi\omega a/k$ and then

$(\omega/2\sigma)\, N_m(0) - (k^2/z)\, V(0)\, \frac{1}{4\pi\omega a}\, (j/2)\, \eta_m\, 2\pi\omega a/k\, R_{dc}(ak) = 0$?

$(\omega/2\sigma)\, N_m(0) - (k/z)\, V(0)\, (j/4)\, \eta_m\, R_{dc}(ak) = 0$?

$(\omega/2\sigma)\, N_m(0) - (k/z)\, V(0)\, (j/4)\, \eta_m\, (1/\pi a^2\sigma)\, (ak) = 0$?

$\omega\, N_m(0) - (k^2/z)\, V(0)\, \eta_m\, (j/2\pi a) = 0$?

How can this possibly be true?? $N_m(0)$ corresponds to $n(\theta,z=0) = (1/a)\, \frac{q(0)}{2\pi}\, \frac{|sh\xi_1|}{ch\xi_1+\cos\theta}$ which I guess means that $N_m(0)$ is N_m at z = 0, so assume that and continue

$\omega\, N_0\, \eta_m - (k^2/z)\, V(0)\, \eta_m\, (j/2\pi a) = 0$?

$\omega\, N_0 - (k^2/z)\, V(0)\, (j/2\pi a) = 0$?

Now we know the $k^2 = k^2 = -zy$ so we then have

$\omega\, N_0 - (-y)\, V(0)\, (j/2\pi a) = 0$?

$\omega\, N_0 + (j\omega C)\, V(0)\, (j/2\pi a) = 0$? // assume G = 0

$$N_0 + (jC)\, V(0)\, (j/2\pi a) = 0 \quad ?$$

$$N_0 - C\, V(0)\, (1/2\pi a) = 0 \quad ?$$

$$N_0 - q(0)\, (1/2\pi a) = 0 \quad ?$$

But recall that

$$N_0 = (1/2\pi a)\, q(0) = \langle n(\theta) \rangle \tag{D.1.8}$$

so amazingly it works! Maybe this is obvious from the start. Recall that

$$E_{i,tot}(r,m,z,t) = A_i(k)\, e^{j(\omega t - k[z-L])}\, E_i(r,m;k) + C_i(k)\, e^{j(\omega t + k[z-L])}\, E_i(r,m;-k)$$

We then want to show that

$$\partial_r (r\, E_{r,tot}(r,m,z)) + jm\, E_{\theta,tot}(r,m,z) + r\, \partial_z E_{z,tot}(r,m,z) = 0 .$$

Write this out as

$$\partial_r (r \{ A_r(k)\, e^{j(\omega t - k[z-L])}\, E_r(r,m;k) + C_r(k)\, e^{j(\omega t + k[z-L])}\, E_r(r,m;-k)\})$$

$$+ jm \{ A_\theta(k)\, e^{j(\omega t - k[z-L])}\, E_\theta(r,m;k) + C_\theta(k)\, e^{j(\omega t + k[z-L])}\, E_\theta(r,m;-k) \}$$

$$+ r\, \partial_z \{ A_z(k)\, e^{j(\omega t - k[z-L])}\, E_z(r,m;k) + C_z(k)\, e^{j(\omega t + k[z-L])}\, E_z(r,m;-k)\} = 0$$

Now do the z derivatives in the last line , and simplify the first line, to get

$$A_r(k)\, e^{j(\omega t - k[z-L])} \partial_r (r\, E_r(r,m;k) + C_r(k)\, e^{j(\omega t + k[z-L])}\, \partial_r (r\, E_r(r,m;-k)) \}$$

$$+ jm \{ A_\theta(k)\, e^{j(\omega t - k[z-L])}\, E_\theta(r,m;k) + C_\theta(k)\, e^{j(\omega t + k[z-L])}\, E_\theta(r,m;-k) \}$$

$$+ r\, jk \{ - A_z(k)\, e^{j(\omega t - k[z-L])}\, E_z(r,m;k) + C_z(k)\, e^{j(\omega t + k[z-L])}\, E_z(r,m;-k)\} = 0$$

Now in the first line make these replacements using App D.

$$\partial_r (r\, E_r(r,m;k) = - jmE_\theta(r,m;k) + r\, (jk) E_z(r,m;k)$$
$$\partial_r (r\, E_r(r,m;-k) = - jmE_\theta(r,m;-k) - r\, (jk) E_z(r,m;-k)$$

Then we get

$$A_r(k)\, e^{j(\omega t - k[z-L])}\, [- jmE_\theta(r,m;k) + r\, (jk) E_z(r,m;k)]$$

$$+ C_r(k)\, e^{j(\omega t + k[z-L])} [- jmE_\theta(r,m;-k) - r\, (jk) E_z(r,m;-k)]$$

$$+ jm \{ A_\theta(k)\ e^{j(\omega t - k[z-L])} E_\theta(r,m;k) + C_\theta(k)\ e^{j(\omega t + k[z-L])} E_\theta(r,m;-k) \}$$

$$+ r\,jk\{ - A_z(k)\ e^{j(\omega t - k[z-L])} E_z(r,m;k) + C_z(k)\ e^{j(\omega t + k[z-L])} E_z(r,m;-k)\} = 0$$

Now the two functions of z have to have the same coefficients, so this then says

$$A_r(k)\ e^{j(\omega t - k[z-L])} [- jmE_\theta(r,m;k) + r\,(jk)E_z(r,m;k)]$$
$$+ jm \{ A_\theta(k)\ e^{j(\omega t - k[z-L])} E_\theta(r,m;k)$$
$$- r\,jk\{ A_z(k)\ e^{j(\omega t - k[z-L])} E_z(r,m;k) = 0$$

and

$$C_r(k)\ e^{j(\omega t + k[z-L])} [- jmE_\theta(r,m;-k) - r\,(jk)E_z(r,m;-k)]$$
$$+ jm \{ C_\theta(k)\ e^{j(\omega t + k[z-L])} E_\theta(r,m;-k)\}$$
$$+ r\,jk \{ C_z(k)\ e^{j(\omega t + k[z-L])} E_z(r,m;-k) \} = 0$$

In each of these we then cancel the z function to get instead these two simpler equations

$$A_r(k) [- jmE_\theta(r,m;k) + r\,(jk)E_z(r,m;k)]$$
$$+ jm \{ A_\theta(k)\ E_\theta(r,m;k)$$
$$- r\,jk\{ A_z(k)\ E_z(r,m;k) = 0$$

$$C_r(k) [- jmE_\theta(r,m;-k) - r\,(jk)E_z(r,m;-k)]$$
$$+ jm \{ C_\theta(k)\ E_\theta(r,m;-k)\}$$
$$+ r\,jk \{ C_z(k)\ E_z(r,m;-k) \} = 0$$

Now in each of these, we regroup in terms of E_θ and E_θ :

$$E_\theta(r,m;k)\ jm \{ - A_r(k) + A_\theta(k) \} + E_z(r,m;k)\ (jrk)\{ A_r(k) - A_z(k)\} = 0$$

$$E_\theta(r,m;-k)\ jm \{ - C_r(k) + C_\theta(k) \} + E_z(r,m;-k)\ (jrk)\{ -C_r(k) + C_z(k)\} = 0$$

OK, now having done this, you only get it to work if the following four equations are true:

$$A_r(k) = A_\theta(k)$$
$$A_r(k) = A_z(k)$$

$$C_r(k) = C_\theta(k)$$
$$C_r(k) = C_z(k)$$

But we can reduce this two the following

$$A_r(k) = A_\theta(k) = A_z(k) \equiv A(k)$$
$$C_r(k) = C_\theta(k) = C_z(k) \equiv C(k)$$

Now in my previous long calculation above, I did find that the three A_i were equal. I also found that

$C_z(k) = - A_z(k)$
$C_r(k) = - A_r(k)$
$C_\theta(k) = - A_\theta(k)$

So **in retrospect**, I could have just started with this form

$$E_{i,tot}(r,m,z,t) = A(k) \{ e^{j(\omega t - k[z-L])} E_i(r,m;k) - e^{j(\omega t + k[z-L])} E_i(r,m;-k) \}$$

or

$$\mathbf{E}_{tot}(r,m,z,t) = A(k) \{ e^{j(\omega t - k[z-L])} \mathbf{E}(r,m;k) - e^{j(\omega t + k[z-L])} \mathbf{E}(r,m;-k) \}$$

I would then want to show that

$$\text{div } [e^{-jkz} \mathbf{E}(r,m;k)] = 0$$
$$\text{div } [e^{+jkz} \mathbf{E}(r,m;-k)] = 0 \qquad (*)$$

Recall again that we know that

$$r \text{ div } \mathbf{F}(r,\theta,z) = \partial_r (r F_r(r,m,z)) + jm F_\theta(r,m,z) + r \partial_z F_z(r,m,z)$$

Then

$$r \text{ div } [e^{-j\beta d z} \mathbf{E}(r,m;k)]$$

$$= \partial_r (r e^{-jkz} E_r(r,m;k)) + jm e^{-jkz} E_\theta(r,m;k) + r \partial_z e^{-jkz} E_z(r,m;k)$$

$$= e^{-jkz} \{ \partial_r (r E_r(r,m;k)) + jm E_\theta(r,m;k) + r(-jk) E_z(r,m;k) \}$$

But recall from App D that

$$r \text{ div } \mathbf{E}(r,m;k) = \partial_r [r E_r(r,m;k)] + jm E_\theta(r,m;k) + r(-jk) E_z(r,m;k) = 0$$

Thus we have shown that

$$r \text{ div } [e^{-jk d z} \mathbf{E}(r,m;k)] = e^{-jkz} \, r \text{ div } \mathbf{E}(r,m;k) = 0$$

The verification for the second line of (*) is then obvious.

Summary of results for Part II: For r = r,θ,z we have these E fields:

$$E_{i,tot}(r,m,z,t) = A(k) \{ e^{j(\omega t - k[z-L])} E_i(r,m;k) - e^{j(\omega t + k[z-L])} E_i(r,m;-k) \}$$

$$A(k) = (k^2/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)}\, \frac{1}{4\pi\omega a N_0} = (k/\mathbf{z})\, V(0)\, \frac{1}{2I\sin(kL)}$$

$$= (-\mathbf{y})\, V(0)\, \frac{1}{\sin(kL)}\, \frac{1}{4\pi\omega a N_0} = (-\mathbf{y})\, V(0)\, \frac{1}{2Ik\sin(kL)} \qquad \text{// dimensionless}$$

I had these three symmetry relations,

$$E_{\mathbf{r}}(r,m;-k) = +E_{\mathbf{r}}(r,m;k)$$
$$E_{\boldsymbol{\theta}}(r,m;-k) = +E_{\boldsymbol{\theta}}(r,m;k)$$
$$E_{\mathbf{z}}(r,m;-k) = -E_{\mathbf{z}}(r,m;k)$$

We then obtain

$$E_{\mathbf{r},tot}(r,m,z,t) = A(k)\,\{\, e^{j(\omega t-k[z-L])}\, E_{\mathbf{r}}(r,m;k) - e^{j(\omega t+k[z-L])}\, E_{\mathbf{r}}(r,m;k)\,\}$$
$$E_{\boldsymbol{\theta},tot}(r,m,z,t) = A(k)\,\{\, e^{j(\omega t-k[z-L])}\, E_{\boldsymbol{\theta}}(r,m;k) - e^{j(\omega t+k[z-L])}\, E_{\boldsymbol{\theta}}(r,m;k)\,\}$$
$$E_{\mathbf{z},tot}(r,m,z,t) = A(k)\,\{\, e^{j(\omega t-k[z-L])}\, E_{\mathbf{z}}(r,m;k) + e^{j(\omega t+k[z-L])}\, E_{\mathbf{z}}(r,m;k)\,\}$$

which we can write as

$$E_{\mathbf{r},tot}(r,m,z,t) = A(k)\, E_{\mathbf{r}}(r,m;k)\, e^{j\omega t}\{\, e^{-jk[z-L])} - e^{+jk[z-L])}\,\}$$
$$E_{\boldsymbol{\theta},tot}(r,m,z,t) = A(k)\, E_{\boldsymbol{\theta}}(r,m;k)\, e^{j\omega t}\{\, e^{-jk[z-L])} - e^{+jk[z-L])}\,\}$$
$$E_{\mathbf{z},tot}(r,m,z,t) = A(k)\, E_{\mathbf{z}}(r,m;k)\, e^{j\omega t}\{\, e^{-jk[z-L])} + e^{+jk[z-L])}\,\}$$

and again as

$$E_{\mathbf{r},tot}(r,m,z,t) = A(k)\, E_{\mathbf{r}}(r,m;k)\, e^{j\omega t}\{\, -2j\sin(k[z-L])\,\}$$
$$E_{\boldsymbol{\theta},tot}(r,m,z,t) = A(k)\, E_{\boldsymbol{\theta}}(r,m;k)\, e^{j\omega t}\{\, -2j\sin(k[z-L])\,\}$$
$$E_{\mathbf{z},tot}(r,m,z,t) = A(k)\, E_{\mathbf{z}}(r,m;k)\, e^{j\omega t}\{\, 2\cos(k[z-L])\,\}$$

We can then insert the factor A(k) in the form

$$A(k) = (-\mathbf{y})\, V(0)\, \frac{1}{2Ik\sin(kL)} = -(\mathbf{y}V(0)/2Ik)\, 1/\sin(kL)$$

to get

$$E_{\mathbf{r},tot}(r,m,z,t) = -(\mathbf{y}V(0)/2Ik)\, E_{\mathbf{r}}(r,m;k)\, e^{j\omega t}\{\, -2j\sin(k[z-L])\,\}/\sin(kL)$$
$$E_{\boldsymbol{\theta},tot}(r,m,z,t) = -(\mathbf{y}V(0)/2Ik)\, E_{\boldsymbol{\theta}}(r,m;k)\, e^{j\omega t}\{\, -2j\sin(k[z-L])\,\}/\sin(kL)$$
$$E_{\mathbf{z},tot}(r,m,z,t) = -(\mathbf{y}V(0)/2Ik)\, E_{\mathbf{z}}(r,m;k)\, e^{j\omega t}\{\, 2\cos(k[z-L])\,\}/\sin(kL)$$

or

$$E_{\mathbf{r},tot}(r,m,z,t) = +j\,(\mathbf{y}V(0)/Ik)\, E_{\mathbf{r}}(r,m;k)\, e^{j\omega t}\{\, \sin(k[z-L])\,\}/\sin(kL)$$

$$E_{\theta,tot}(r,m,z,t) = +j\,(\mathbf{y}V(0)/Ik)\,E_{\theta}(r,m;k)\;e^{j\omega t}\{\sin(k[z-L])\}/\sin(kL)$$
$$E_{z,tot}(r,m,z,t) = -\,(\mathbf{y}V(0)/Ik)\,E_{z}(r,m;k)\;e^{j\omega t}\{\cos(k[z-L])\}/\sin(kL)$$

or

$$E_{r,tot}(r,m,z) = +j\,(\mathbf{y}V(0)/Ik)\,E_{r}(r,m;k)\,\{\sin(k[z-L])\}/\sin(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\,(\mathbf{y}V(0)/Ik)\,E_{\theta}(r,m;k)\,\{\sin(k[z-L])\}/\sin(kL)$$
$$E_{z,tot}(r,m,z) = -\,(\mathbf{y}V(0)/Ik)\,E_{z}(r,m;k)\,\{\cos(k[z-L])\}/\sin(kL)$$

where

$$E_z(r,m;k) = (1/4)\,\eta_m\,I\,R_{dc}\,(a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m;k) = (j/4)\,\eta_m\,I\,R_{dc}\,(ak)\,g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m;k) = (1/4)\,\eta_m\,I\,R_{dc}\,(ak)\,h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

So now install these fields to get

$$E_{r,tot}(r,m,z) = +j\,(\mathbf{y}V(0)/Ik)\,(j/4)\,\eta_m\,I\,R_{dc}\,(ak)\,g_m\,\{\sin(k[z-L])\}/\sin(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\,(\mathbf{y}V(0)/Ik)\,(1/4)\,\eta_m\,I\,R_{dc}\,(ak)\,h_m\,\{\sin(k[z-L])\}/\sin(kL)$$
$$E_{z,tot}(r,m,z) = -\,(\mathbf{y}V(0)/Ik)\,(1/4)\,\eta_m\,I\,R_{dc}\,(a\beta')\,f_m\,\{\cos(k[z-L])\}/\sin(kL)$$

We then get cancellation of the I factors, and so

$$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,R_{dc}\,(ak)\,g_m\sin(k[z-L])/k\sin(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,\eta_m\,R_{dc}\,(ak)\;h_m\sin(k[z-L])/k\sin(kL)$$
$$E_{z,tot}(r,m,z) = -\,\mathbf{y}V(0)\,(1/4)\;\eta_m\;R_{dc}\,(a\beta')\,f_m\cos(k[z-L])/k\sin(kL)$$

So, in my model for the reflection scenario with a shorting plane at z = L, the above are my predictions for the E fields inside a round wire of radius a which is one of the two conductors for a transmission line that is made from two round wires. We could set $\mathbf{y} = j\omega C$ in the case that G = 0.

Question: do we still have asymmetry for J_z as $\omega \to 0$? Start with this:

$$E_{z,tot}(r,m,z) = -\,\mathbf{y}V(0)\,(1/4)\;\eta_m\;R_{dc}\,(a\beta')\,f_m\cos(k[z-L])/k\sin(kL)$$

so that

$$J_{z,tot}(r,m,z) = -\,\sigma\,\mathbf{y}V(0)\,(1/4)\;\eta_m\;R_{dc}\,(a\beta')\,f_m\cos(k[z-L])/k\sin(kL)$$

As $\omega \to 0$, we know that $k \to 0$ and $\beta' \to 0$. This last fact allows us to use,

$f_m = (r/a)^{|m|} (|m|+1) (2/\beta' a) \qquad f_0 = 4/(a\beta')$

We then have

$$J_{z,tot}(r,m,z) = -\sigma y V(0) (1/4) \eta_m R_{dc}(a\beta') (r/a)^{|m|} (|m|+1) (2/\beta' a)\cos(k[z-L]) / k\sin(kL) \quad m \neq 0$$

$$J_{z,tot}(r,0,z) = -\sigma y V(0) (1/4) \quad R_{dc}(a\beta') 4/(a\beta') \cos(k[z-L]) / k\sin(kL) \quad m = 0$$

or

$$J_{z,tot}(r,m,z) = -\sigma y V(0) (1/4) \eta_m R_{dc} (r/a)^{|m|} (|m|+1) (2)\cos(k[z-L]) / k\sin(kL) \quad m \neq 0$$

$$J_{z,tot}(r,0,z) = -\sigma y V(0) (1/4) \quad R_{dc} 4 \cos(k[z-L]) / k\sin(kL) \qquad m = 0$$

or

$$J_{z,tot}(r,m,z) = -\sigma y V(0) (1/2) \eta_m R_{dc} (r/a)^{|m|} (|m|+1) \cos(k[z-L]) / k\sin(kL) \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = -\sigma y V(0) \ R_{dc} \ \cos(k[z-L]) / k\sin(kL) \qquad m = 0$$

or

$$J_{z,tot}(r,m,z) = -y V(0) (1/2) \eta_m (1/\pi a^2) (r/a)^{|m|} (|m|+1) \cos(k[z-L]) / \sin(kL) \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = -y V(0) \ (1/\pi a^2) \ \cos(k[z-L]) / \sin(kL) \qquad m = 0$$

or

$$J_{z,tot}(r,m,z) = -j\omega C V(0) (1/2) \eta_m (1/\pi a^2) (r/a)^{|m|} (|m|+1) \cos(k[z-L]) / k\sin(kL) \quad m \neq 0$$

$$J_{z,tot}(r,0,z) = -j\omega C V(0) \ (1/\pi a^2) \ \cos(k[z-L]) / k\sin(kL) \qquad m = 0$$

Now what happens when we take the limit $k \rightarrow 0$ corresponding to $\omega \rightarrow 0$?

$$J_{z,tot}(r,m,z) = -j\omega C V(0) (1/2) \eta_m (1/\pi a^2) (r/a)^{|m|} (|m|+1) \ 1 / (k^2 L) \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = -j\omega C V(0) \ (1/\pi a^2) \ 1 / (k^2 L) \qquad m = 0$$

We need to know the *way* in which $k \rightarrow 0$. From Section D.11, for G = 0,

$$k^2 = RC\omega(-j) = -j\omega RC$$

In this case

$$\omega/k^2 = \omega/[\ -j\omega RC] \ = j/(RC)$$

We then get

$$J_{z,tot}(r,m,z) = -j[j/(RC)]C V(0) (1/2) \eta_m (1/\pi a^2) (r/a)^{|m|} (|m|+1) \ 1 / L \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = -j[j/(RC)]C\,V(0)\,(1/\pi a^2)\,1/L \qquad m = 0$$

or

$$J_{z,tot}(r,m,z) = [1/(R)]\,V(0)\,(1/2)\,\eta_m\,(1/\pi a^2)\,(r/a)^{|m|}\,(|m|+1)\,1/L \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = [1/(R)]\,V(0)\,(1/\pi a^2)\,1/L \qquad m = 0$$

or

$$J_{z,tot}(r,m,z) = V(0)\,(1/2)\,\eta_m\,(1/\pi a^2)\,(r/a)^{|m|}\,(|m|+1)\,1/(RL) \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = V(0)\,(1/\pi a^2)\,1/(RL) \qquad m = 0$$

Recall now from Part I that

$$i(z) = V(0)/(LR) \qquad R = R_1 + R_2$$

So the rewrite again as

$$J_{z,tot}(r,m,z) = i(z)\,(1/2)\,\eta_m\,(1/\pi a^2)\,(r/a)^{|m|}\,(|m|+1) \qquad m \neq 0$$

$$J_{z,tot}(r,0,z) = i(z)\,(1/\pi a^2) \qquad m = 0$$

For m = 0, this is exactly the result I expect.

But the asymmetry continues to persist at DC!! I think this is the first time I have obtained this asymmetry in a situation where the current i(z) is finite as $\omega \rightarrow 0$. So this then is a more serious "issue" !

Is it really true that $J_i = \sigma E_i$ since B fields are present? My magnetic Ohm's law discussion concludes that you can in general ignore the magnetic distortion of ohm's law as long as B << 569 T. But you cannot ignore it in the Hall effect, as I point out at the very end. So maybe you cannot ignore it here??

What is going on with the other two field components? For E_r we start with

$$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\,\eta_m\,R_{dc}\,(ak)\,g_m\,\sin(k[z-L])/k\sin(kL)$$

Then write this out for the two cases:

$$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\,\eta_m\,R_{dc}\,(ak)\,g_m\,\sin(k[z-L])/k\sin(kL) \qquad m\neq 0$$
$$E_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\,R_{dc}\,(ak)\,g_0\,\sin(k[z-L])/k\sin(kL) \qquad m=0$$

Now go to the $\omega\rightarrow 0$ limit where β' and k go to zero. We then get from App D.11,

$$g_m = (r/a)^{|m|+1} + (r/a)^{|m|-1} \qquad g_0 = 2\,(r/a)$$

So then

$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,R_{dc}\,(ak)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,\sin(k[z-L])\,/\,k\sin(kL)$ $m\neq 0$

$E_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\;R_{dc}\,(ak)\,[2\,(r/a)]\,\sin(k[z-L])\,/\,k\sin(kL)$ $m=0$

But at the same time we have to do the small k limit on other things, so we have

$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,R_{dc}\,(ak)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,(k[z-L])\,/\,k(kL)$ $m\neq 0$

$E_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\;R_{dc}\,(ak)\,[2\,(r/a)]\,(k[z-L])\,/\,k(kL)$ $m=0$

or

$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,R_{dc}\,(ak)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,([z-L])\,/\,k(L)$ $m\neq 0$

$E_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\;R_{dc}\,(ak)\,[2\,(r/a)]\,([z-L])\,/\,k(L)$ $m=0$

or

$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,R_{dc}\,(a)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,([z-L])\,/\,(L)$ $m\neq 0$

$E_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\;R_{dc}\,(a)\,[2\,(r/a)]\,([z-L])\,/\,(L)$ $m=0$

or

$E_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,R_{dc}\,(a)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,([z-L])\,/\,(L)$ $m\neq 0$

$E_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\;R_{dc}\,(a)\,[2\,(r/a)]\,([z-L])\,/\,(L)$ $m=0$

Assuming Ohm's law is valid, this becomes

$J_{r,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(j/4)\;\eta_m\,(1/\pi a)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,([z-L])\,/\,(L)$ $m\neq 0$

$J_{r,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(j/4)\;(1/\pi a)\;[2\,(r/a)]\,([z-L])\,/\,(L)$ $m=0$

But now use $\mathbf{y} = j\omega C$

$J_{r,tot}(r,m,z) = -\omega C\,V(0)\,(j/4)\;\eta_m\,(1/\pi a)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]\,([z-L])\,/\,(L)$ $m\neq 0$

$J_{r,tot}(r,0,z) = -\omega C\,V(0)\,(j/4)\;(1/\pi a)\;[2\,(r/a)]\,([z-L])\,/\,(L)$ $m=0$

Now since we are doing $\omega \rightarrow 0$, we end up with

$J_{r,tot}(r,m,z) = 0$ $m\neq 0$

$J_{r,tot}(r,0,z) = 0$ $m=0$

so there is then no radial current. This seems consistent with the fact that the surface charge is not changing in this DC limit.

We might as well do E_θ here as well: For E_θ we start with

$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,\eta_m\,R_{dc}\,(ak)\;h_m\,\sin(k[z-L])\,/\,k\sin(kL)$

which we then write as

$$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,\eta_m\,R_{dc}\,(ak)\,h_m \sin(k[z-L])\,/\,k\sin(kL) \qquad m \neq 0$$
$$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,R_{dc}\,(ak)\,h_0 \sin(k[z-L])\,/\,k\sin(kL) \qquad m = 0$$

We then use App D.11 results that for small β',

$$h_m = (r/a)^{|m|+1} - (r/a)^{|m|-1} \qquad h_0 = 0$$

Then we get

$$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,\eta_m\,R_{dc}\,(ak)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,\sin(k[z-L])\,/\,k\sin(kL) \qquad m \neq 0$$
$$E_{\theta,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(1/4)\,R_{dc}\,(ak)\,h_0 \sin(k[z-L])\,/\,k\sin(kL) \qquad m = 0$$

Now do the other limits to get

$$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,\eta_m\,R_{dc}\,(a)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,([z-L])\,/\,(L) \qquad m \neq 0$$
$$E_{\theta,tot}(r,0,z) = +j\,\mathbf{y}V(0)\,(1/4)\,R_{dc}\,(a)\,h_0\,([z-L])\,/\,(L) \qquad m = 0$$

Now we safely set $h_0 = 0$ to get

$$E_{\theta,tot}(r,m,z) = +j\,\mathbf{y}V(0)\,(1/4)\,\eta_m\,R_{dc}\,(a)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]\,([z-L])\,/\,(L) \qquad m \neq 0$$
$$E_{\theta,tot}(r,0,z) = 0 \qquad m = 0$$

Finally use again $\mathbf{y} = j\omega C$ to get

$$E_{\theta,tot}(r,m,z) = 0 \qquad m \neq 0$$
$$E_{\theta,tot}(r,0,z) = 0 \qquad m = 0$$

and then the same for J_θ.

So, assuming Ohm's law is valid, here is our DC limit for all field results inside the round wire:

$$J_{r,tot}(r,m,z) = 0 \qquad m \neq 0$$
$$J_{r,tot}(r,0,z) = 0 \qquad m = 0$$

$$J_{\theta,tot}(r,m,z) = 0 \qquad m \neq 0$$
$$J_{\theta,tot}(r,0,z) = 0 \qquad m = 0$$

$$J_{z,tot}(r,m,z) = i(z)\,(1/2)\,\eta_m\,(1/\pi a^2)\,(r/a)^{|m|}\,(|m|+1) \qquad m \neq 0$$
$$J_{z,tot}(r,0,z) = i(z)\,(1/\pi a^2) \qquad m = 0$$

So what IS IT that is making J_z be asymmetric like this?

Do we still have div E = 0 inside the round wire?? It seems that , after doing $\omega \rightarrow 0$ and then doing div E,

$$\text{div } E = \partial_z E_z = \text{stuff} * \partial_z [i(z)] = 0$$

so div E = 0 is OK even with this asymmetry.

How on earth can J_z be asymmetric here? What mechanism could make that happen? There is no radial current, there is no azimuthal current, so the mechanisms I am used to based on η_m seem to no longer apply. I think the two App D boundary conditions are met.

Part III. Redo Part II in a simpler manner.

We start off with this *ansatz* for the total field E_{tot} in the reflection scenario which matches the scenario of Part I above which was mainly for "outside" the conductors;

$$E_{i,tot}(r,m,z,t) = A(k) \{ e^{j(\omega t - k[z-L])} E_i(r,m;k) - e^{j(\omega t + k[z-L])} E_i(r,m;-k) \}$$

or

$$\mathbf{E}_{tot}(r,m,z,t) = A(k) \{ e^{j(\omega t - k[z-L])} \mathbf{E}(r,m;k) - e^{j(\omega t + k[z-L])} \mathbf{E}(r,m;-k) \}$$

This is the superposition of right and left going waves, where $E_i(r,m;k)$ are the solutions of Appendix D found for a wave going in the +z direction on an infinite transmission line.

1. Show that div $\mathbf{E}_{tot}(r,\theta,z) = 0$ and that the three Helmholtz equations are satisfied

Consider:

$$\mathbf{E}_{tot}(r,\theta,z,t) = A(k) \{ e^{j(\omega t - k[z-L])} \mathbf{E}(r,\theta;k) - e^{j(\omega t + k[z-L])} \mathbf{E}(r,\theta;-k) \}$$

$$= A(k) \{ e^{jkL} [e^{j(\omega t - kz)} \mathbf{E}(r,\theta;k)] - e^{-jkL} [e^{j(\omega t + kz)} \mathbf{E}(r,\theta;-k)] \} \qquad (*)$$

Recall now from App D [where $\mathbf{E}(r,\theta)$ are the specific Appendix D solutions]

$$\mathbf{E}(r,\theta,z,t;k) = e^{j(\omega t - kz)} \mathbf{E}(r,\theta) \ . \qquad (D.1.1)$$

so we then have

$$\mathbf{E}_{tot}(r,\theta,z,t) = A(k) \{ e^{jkL} \mathbf{E}(r,\theta,z,t;k) - e^{-jkL} \mathbf{E}(r,\theta,z,t;-k) \} \qquad (**)$$

We showed in App D that div $\mathbf{E}(r,\theta,z,t;k) = 0$, and this is also true for -k , so each term in (**) has zero divergence, so the sum of the two terms then has zero divergence, QED.

We also showed in App D that $(\nabla^2 + \beta^2)\, \mathbf{E}(r,\theta,z,t;k) = 0$ where β is for the conductor. Looking at (**) we see that each term separately satisfies the three Helmholtz equations, so the sum does as well.

2. Show that (**) satisfies the $E_{\theta,tot}(r=a,m) = 0$ boundary condition:

Our ansatz expression above for i = θ reads:

$$E_{\theta,tot}(r,m,z,t) = A(k)\ \{\ e^{j(\omega t - k[z-L])}\ E_\theta(r,m;k) - e^{j(\omega t + k[z-L])}\ E_\theta(r,m;-k)\ \}$$

Therefore

$$E_{\theta,tot}(a,m,z,t) = A(k)\ \{\ e^{j(\omega t - k[z-L])}\ E_\theta(a,m;k) - e^{j(\omega t + k[z-L])}\ E_\theta(a,m;-k)\ \}$$

But we know from App D that $E_\theta(a,m;k) = 0$, and thus $E_\theta(a,m;-k) = 0$, so $E_{\theta,tot}(a,m,z,t) = 0$, QED.

3. The Reflection Boundary Conditions at z = L

First, if we just look at the Appendix D solutions for **E**(r,θ;k),

$$E_z(r,m;k) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m;k) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m;k) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \quad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

we know that $E_z = \sigma J_z$ must change sign if we make the right-moving wave move to the left. None of the factors shown above in E_z changes sign under k→ - k except I. Obviously the current changes sign if you change the wave direction. Formally, I(-k) = - I(k) = - I. Looking then at E_r and E_θ, the extra visible factor k causes E_r and E_z *not* to change sign. We then get these symmetry relations,

$$E_r(r,m;-k) = +E_r(r,m;k)$$
$$E_\theta(r,m;-k) = +E_\theta(r,m;k)$$
$$E_z(r,m;-k) = -E_z(r,m;k)$$

When there are used in (**) we find

$$E_{r,tot}(r,m,z,t) = A(k)\ \{\ e^{j(\omega t - k[z-L])}\ E_r(r,m;k) - e^{j(\omega t + k[z-L])}\ E_r(r,m;k)\ \}$$
$$E_{\theta,tot}(r,m,z,t) = A(k)\ \{\ e^{j(\omega t - k[z-L])}\ E_\theta(r,m;k) - e^{j(\omega t + k[z-L])}\ E_\theta(r,m;k)\ \}$$
$$E_{z,tot}(r,m,z,t) = A(k)\ \{\ e^{j(\omega t - k[z-L])}\ E_z(r,m;k) + e^{j(\omega t + k[z-L])}\ E_z(r,m;k)\ \}$$

which then simplifies to

$$E_{r,tot}(r,m,z,t) = A(k)\ E_r(r,m;k)\ e^{j\omega t}\{\ -2j\ \sin(k[z-L])\ \}$$
$$E_{\theta,tot}(r,m,z,t) = A(k)\ E_\theta(r,m;k)\ e^{j\omega t}\{\ -2j\ \sin(k[z-L])\ \}$$
$$E_{z,tot}(r,m,z,t) = A(k)\ E_z(r,m;k)\ e^{j\omega t}\{\ 2\ \cos(k[z-L])\ \}$$

Notice these facts at z = L, the reflection point:

$$E_{r,tot}(r,m,z,t) = 0$$
$$E_{\theta,tot}(r,m,z,t) = 0$$
$$E_{z,tot}(r,m,z,t) = A(k)\, E_z(r,m;k)\; e^{j\omega t}\{\, 2 \,\}$$

We are then happy to see that E_r and E_θ vanish at z = L because these are transverse fields at the shorting block, and inside the shorting block (perfect conductor) these fields must vanish. We see also that E_z takes its max value at z = L so we expect current J_z to be max at z = L where voltage is minimum 0. These results just fall out of our Appendix D solutions for the single wave.

4. Determination of the constant A(k)

Assuming Ohm's law is valid, we know from 4" above that

$$J_{z,tot}(r,m,z,t) = A(k)\, J_z(r,m;k)\; e^{j\omega t}\{\; 2\cos(k[z-L]) \,\}$$

or in the θ domain

$$J_{z,tot}(r,\theta,z,t) = A(k)\, J_z(r,\theta;k)\; e^{j\omega t}\{\; 2\cos(k[z-L]) \,\}$$

If we integrate over the wire cross section, the above says

$$i(z) = A(k)\, I\, e^{j\omega t}\{\; 2\cos(k[z-L]) \,\}$$

where we use the fact below D.2.30 which says $J_z(r,\theta;k)$ integrates to I. We could write $i(z) = I_{tot}(z)$. But we found in Part I that

$$i(z) = (k/\mathbf{z})\, V(0)\, e^{j\omega t}\, \frac{\cos(k[z-L])}{\sin(kL)} \qquad \text{// assumption!}$$

where I have reinserted the time factor. We can compare these two expressions for i(z) to obtain coefficient A(k),

$$A(k)\, I\, e^{j\omega t}\{\; 2\cos(k[z-L]) \,\} = (k/\mathbf{z})\, V(0)\, e^{j\omega t}\, \frac{\cos(k[z-L])}{\sin(kL)}$$
$$A(k)\, I\, \{\, 2 \,\} = (k/\mathbf{z})\, V(0)\, \frac{1}{\sin(kL)}$$
$$A(k) = (k/\mathbf{z})\, V(0)\, \frac{1}{2I\sin(kL)}$$

There it is.

5. Final form for the reflection scenario E fields inside the conductor

If we now install this coefficient into our "tot" field results and at the same time insert the E solutions of Appendix D quoted just above, here is what we get

$$E_{r,tot}(r,m,z,t) = (k/z)\, V(0)\, \frac{1}{2I\sin(kL)}\, (j/4)\, \eta_m\, I\, R_{dc}\, (ak)\, g_m\, e^{j\omega t}\{ -2j \sin(k[z-L]) \}$$

$$E_{\theta,tot}(r,m,z,t) = (k/z)\, V(0)\, \frac{1}{2I\sin(kL)}\, (1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, h_m\, e^{j\omega t}\{ -2j \sin(k[z-L]) \}$$

$$E_{z,tot}(r,m,z,t) = (k/z)\, V(0)\, \frac{1}{2I\sin(kL)}\, (1/4)\, \eta_m\, I\, R_{dc}\, (a\beta')\, f_m\, e^{j\omega t}\{ 2 \cos(k[z-L]) \}$$

Notice that the factors of I cancel on each line and we end up with [note k/k multiplied by]

$$E_{r,tot}(r,m,z) = +j\, yV(0)\, (j/4)\, \eta_m\, R_{dc}\, (ak)\, g_m \sin(k[z-L]) / k\sin(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\, yV(0)\, (1/4)\, \eta_m\, R_{dc}\, (ak)\, h_m \sin(k[z-L]) / k\sin(kL)$$
$$E_{z,tot}(r,m,z) = -\, yV(0)\, (1/4)\, \eta_m\, R_{dc}\, (a\beta')\, f_m \cos(k[z-L]) / k\sin(kL)$$

Here we added a k in the denominator on each line in order to get $(k^2/z) = -y$ since $k^2 = -yz$. Since we have assumed that G = 0, we can write this as

$$E_{r,tot}(r,m,z) = +j\, (j\omega C)V(0)\, (j/4)\, \eta_m\, R_{dc}\, (ak)\, g_m \sin(k[z-L]) / k\sin(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\, (j\omega C)V(0)\, (1/4)\, \eta_m\, R_{dc}\, (ak)\, h_m \sin(k[z-L]) / k\sin(kL)$$
$$E_{z,tot}(r,m,z) = -\, (j\omega C)V(0)\, (1/4)\, \eta_m\, R_{dc}\, (a\beta')\, f_m \cos(k[z-L]) / k\sin(kL) \qquad (*)$$

6. Show that (*) satisfies the charge pump boundary condition at z.

That boundary condition (recall G = 0) says

$$E_{r,tot}(r=a-\varepsilon,\theta,z) = (j\omega/\sigma)\, N_m(z)$$

Now recall from Part I that

$$n(\theta,z) = n(\theta,z=0)\, \frac{\sin(k[L-z])}{\sin(kL)} \qquad \text{// assumption}$$

which in m space says

$$N_m(z) = N_m(0)\, \frac{\sin(k[L-z])}{\sin(kL)}$$

where $N_m(0)$ is the charge density moment at z = 0, the start of the line. We can define relative moments for this as

$$\eta_m(0) = N_m(0)/N_0(0)$$

But these moments are really determined by the "capacitor problem" and are independent of z and are the same as the moments which appear in Appendix D for the single wave. So write $\eta_m(0) = \eta_m$ and then

$$N_m(z) = N_0(0)\,\eta_m \frac{\sin(k[L-z])}{\sin(kL)}$$

Now recall (D.1.8),

$$N_0 = (1/2\pi a)\, q(0) \quad = \langle n(\theta) \rangle \qquad \text{(D.1.8)}$$

If we apply this to our reflection scenario quantities, we get

$$N_0(0) = (1/2\pi a)\, q(0) = \langle n_{tot}(\theta,0) \rangle$$

Thus (using only the left equality above) we may write

$$N_m(z) = (1/2\pi a)\, q(0)\, \eta_m \frac{\sin(k[L-z])}{\sin(kL)} .$$

Our charge pump boundary condition requires that

$$E_{r,tot}(r=a-\varepsilon,\theta,z) = (j\omega/\sigma)\, N_m(z)$$

or inserting on both sides,

$$+j\,(j\omega C)V(0)\,(j/4)\;\eta_m\, R_{dc}\;(ak)\; g_m(a)\,\sin(k[z-L])\,/\,k\sin(kL) \;=\; (j\omega/\sigma)\,(1/2\pi a)\,q(0)\,\eta_m \frac{\sin(k[L-z])}{\sin(kL)} \;\; ?$$

Is this condition met? Looking again at our App D solution box quoted above, we see that $g_m(r=a) = 2$. Then cancelling obvious factors the above becomes [note that the z-L order is different in the two sines]

$$-j\,(C)V(0)\,(j/4)\;\; R_{dc}\;\;(ak)\;2\,/\,k \;=\; (1/\sigma)\,(1/2\pi a)\,q(0) \;\; ?$$

Now use $R_{dc} = 1/(\sigma\pi a^2)$ and $C\,V(0) = q(0)$ to get

$$-j\;\;(j/4)\;\;1/(\sigma\pi a^2)\;\;(ak)\;2\,/\,k \;=\; (1/\sigma)\,(1/2\pi a) \quad ?$$

$$-j\;\;(j/4)\;\;2 \;=\; (1/2) \quad ?$$

$$(1/2) = \;(1/2) \quad ? \qquad\qquad \text{yes!}$$

Thus, we have shown that our reflection scenario solutions satisfy the charge pump boundary condition, QED.

7. Summary for the reflection scenario fields inside a round wire conductor of a TL with G = 0:

$E_{r,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(j/4)\;\eta_m\,R_{dc}\;(ak)\;g_m(r)\,\sin(k[z-L])\,/\,k\sin(kL)$
$E_{\theta,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(1/4)\;\eta_m\,R_{dc}\;(ak)\;h_m(r)\,\sin(k[z-L])\,/\,k\sin(kL)$
$E_{z,tot}(r,m,z) = -\,(j\omega C)V(0)\,(1/4)\;\eta_m\;R_{dc}\;(a\beta')\,f_m(r)\,\cos(k[z-L])\,/\,k\sin(kL)$

These solutions satisfy all of the following:
the Helmholtz equation $(\nabla^2 + \beta^2)\mathbf{E} = 0$
div $\mathbf{E} = 0$
the $E_\theta = 0$ boundary condition
the charge pump boundary condition
$E_r = E_\theta = 0$ at z = L, the shorting block location
E_z = maximum at the shorting block, so J_z = max and i(z) = max there
V(z) = 0 at z = L

8. Current J_z is asymmetric in the general solution and then also at DC

Without going any further, we can see that

$$E_{z,tot}(r,m,z) \,/\, E_{z,tot}(r,0,z) = \eta_m\; f_m(r)/f_0(r)$$

$$f_m(r) = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad f_0(r) = 2\,\frac{J_0(x)}{J_1(x_a)} \qquad x = \beta' r \qquad x_a = \beta' a$$

For general values of r and k = k, it is clear that the ratio $f_m(r)/f_0(r)$ is non-vanishing and that therefore the current density J_z is asymmetric in θ inside the wire.

What is very surprising is that this asymmetric continues to exist in the DC limit ω → 0. To see this, we recall from App D.11 that

$$f_m(r) = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0(r) = 4/(a\beta')$$

and then in this limit we have

$$E_{z,tot}(r,m,z) \,/\, E_{z,tot}(r,0,z) = \eta_m\; f_m(r)/f_0(r) = (r/a)^m\,(m+1)\,(2/\beta' a)\;/\;4/(a\beta')$$

$$= (r/a)^m\,(m+1)/2$$

and there is your DC current asymmetry! I now have this problematic situation in the shorted-bar situation where the current i(z) does NOT vanish at DC.

Here then is the Big Question: What physically is causing this asymmetry at DC ? It is not eddy currents since eddy currents go away at ω = 0.

9. What is the limit as ω → 0 of the E fields?

As a first step, take k very small, and I will do this in phases so you can see the details:

$$E_{r,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(j/4)\;\eta_m\,R_{dc}\;(ak)\;g_m(r)\,\sin(k[z-L])\,/\,k\sin(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(1/4)\,\eta_m\,R_{dc}\;(ak)\;h_m(r)\,\sin(k[z-L])\,/\,k\sin(kL)$$
$$E_{z,tot}(r,m,z) = -\,(j\omega C)V(0)\,(1/4)\;\eta_m\;R_{dc}\;(a\beta')\,f_m(r)\,\cos(k[z-L])\,/\,k\sin(kL)$$

$$E_{r,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(j/4)\;\eta_m\,R_{dc}\;(ak)\;g_m(r)\,(k[z-L])\,/\,k(kL)$$
$$E_{\theta,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(1/4)\,\eta_m\,R_{dc}\;(ak)\;h_m(r)\,(k[z-L])\,/\,k(kL)$$
$$E_{z,tot}(r,m,z) = -\,\mathbf{y}V(0)\,(1/4)\;\eta_m\;R_{dc}\;(a\beta')\,f_m(r)\quad 1\qquad /\,k(kL)$$

$$E_{r,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(j/4)\;\eta_m\,R_{dc}\;(a)\;g_m(r)\,([z-L])\,/\,(L)$$
$$E_{\theta,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(1/4)\,\eta_m\,R_{dc}\;(a)\;h_m(r)\,([z-L])\,/\,(L)$$
$$E_{z,tot}(r,m,z) = -\,(\mathbf{y}/k^2)\,V(0)\,(1/4)\;\eta_m\;R_{dc}\;(a\beta')\,f_m(r)\quad 1\quad /\,(L)$$

$$E_{r,tot}(r,m,z) = +j\,(\mathbf{y})V(0)\,(j/4)\;\eta_m\,(1/\sigma\pi a^2)\;(a)\;g_m(r)\,(z-L)/L$$
$$E_{\theta,tot}(r,m,z) = +j\,(\mathbf{y})V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a^2)\;(a)\;h_m(r)\,(z-L)/L$$
$$E_{z,tot}(r,m,z) = \quad(1/\mathbf{z})V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a^2)\;(a\beta')\,f_m(r)/L$$

$$E_{r,tot}(r,m,z) = +j\,(\mathbf{y})V(0)\,(j/4)\;\eta_m\,(1/\sigma\pi a)\quad[g_m(r)]\,(z-L)/L$$
$$E_{\theta,tot}(r,m,z) = +j\,(\mathbf{y})V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a)\quad[h_m(r)]\,(z-L)/L$$
$$E_{z,tot}(r,m,z) = \quad(1/\mathbf{z})V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a^2)\;[f_m(r)]\,(a\beta')\,/L$$

Next, here are limits for the three f_m type factors from App D.11 for $m \geq 0$

$f_m = (r/a)^m\,(m+1)\,(2/\beta' a)$ $\qquad$ $f_0 = 4/(a\beta')$

$g_m = (r/a)^{m+1} + (r/a)^{m-1}$ $\qquad$ $g_0 = 2\,(r/a)$

$h_m = (r/a)^{m+1} - (r/a)^{m-1}$ $\qquad$ $h_0 = 0$

We then have to break things into two groups.

$$E_{r,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(j/4)\;\eta_m\,(1/\sigma\pi a)\quad[(r/a)^{m+1} + (r/a)^{m-1}]\,(z-L)/L$$
$$E_{\theta,tot}(r,m,z) = +j\,(j\omega C)V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a)\quad[(r/a)^{m+1} - (r/a)^{m-1}]\,(z-L)/L$$
$$E_{z,tot}(r,m,z) = \quad(1/\mathbf{z})V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a^2)\;[(r/a)^m\,(m+1)\,(2)]\,/L \qquad m > 0$$

$$E_{r,tot}(r,0,z) = +j\,(j\omega C)V(0)\,(j/2)\;(1/\sigma\pi a)\,(r/a)\,(z-L)/L$$
$$E_{\theta,tot}(r,0,z) = \quad 0$$
$$E_{z,tot}(r,0,z) = \quad(1/\mathbf{z})V(0)\,(1/\sigma\pi a^2)\;/L \qquad m = 0$$

The final step is the explicit ω = 0 so we get

$$E_{r,tot}(r,m,z) = 0$$
$$E_{\theta,tot}(r,m,z) = 0$$

$E_{z,tot}(r,m,z) = (1/R)V(0)\,(1/4)\,\eta_m\,(1/\sigma\pi a^2)\,[(r/a)^m\,(m+1)\,(2)]\,/L$ $\quad m > 0$

$E_{r,tot}(r,0,z) = 0$
$E_{\theta,tot}(r,0,z) = 0$
$E_{z,tot}(r,0,z) = (1/R)V(0)\,(1/\sigma\pi a^2)\,/L$ $\quad m = 0$

10. Does the DC solution satisfy curl E = 0 ? No !!!

Start with general result in cyl coordinates

$$\text{curl } \mathbf{E}(r,\theta,z) = \hat{\mathbf{r}}\,[\,r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$

Then go into partial waves to get

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\,r^{-1}jmE_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,]$$

At DC, none of the fields varies with z so $\partial_z = 0$ (I think this is legal). [Or sayAlso $E_r = E_\theta = 0$ so get

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\,r^{-1}jmE_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r E_z]$$

But this is clearly NOT 0, so something is wrong in my machinery!!! After many months I finally have an indication that something is wrong!

(a) Review Appendix D solutions for curl E = 0 at DC.

In Appendix D, I use the curl E equation to compute B when $\omega \neq 0$: curl $\mathbf{E} = -j\omega\mathbf{B}$. This reminds me of a problem I once thought I had, which was that B must then $\to\infty$ as $\omega \to 0$ which is physically wrong and would be repaired by curl $E \to 0$. So maybe let's look at this entirely in App D terms and forget the reflection scenario for a while.

Question 1: What happens if I look at the E field solutions before BC's are added. Those solutions are

First summary of the E field solutions (D.2.21)

$E_z(r,m) = -j\,(\beta'/k)\,\frac{K_m}{2}\,J_m(x)$ $\quad x = \beta' r$ *(D.1.27)*

$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)$ $\quad \beta'^2 = \beta^2 - k^2$ *(D.2.11)*

$jE_\theta(r,m) = -a_m\,x^{-1}\,J_m(x) + (\,\frac{K_m}{2} + \frac{a_m}{m}\,)\,J_{m+1}(x)$ *(D.2.15)*

Now from above we have, where I replace $\partial_z \to -jk$ as apropos for App D,

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}}\,[\,r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\,[+jk\,E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,]$$

The three component equations are then

$$r^{-1}jmE_z+jkE_\theta = 0$$
$$+jk\,E_r - \partial_r E_z = 0$$
$$r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r = 0 \qquad \text{at DC ?}$$

I presume the E fields are all finite as the limit is approached. I know that $k \to 0$ at DC so we then have

$$mE_z = 0$$
$$\partial_r E_z = 0$$
$$\partial_r(rE_\theta) - jmE_r = 0 \qquad \text{at DC ?}$$

Assuming K_m and a_m are non-singular in this DC limit [can check from (D.2.28)] I see from (D.2.21) [also assuming that all Bessel combinations are finite] that

$$\beta'/k = \sqrt{-j\omega(\mu\sigma - RC)}\ /\ [\sqrt{RC}\ \omega^{1/2}\ e^{-j\pi/4}] = \sqrt{-j(\mu\sigma - RC)}\ /\ [\sqrt{RC}\ e^{-j\pi/4}]$$

for G = 0 and this result is a constant. On the other hand, I know that

$$J_m(x) = (x/2)^m / m! \qquad x = \beta' r$$

so that as $\beta' \to 0$, $x \to 0$ and then $J_m(x) \to 0$ except for m = 0. Thus, for general K_m constant, we really do have $mE_z = 0$ for all m in the DC limit, so the first equation above is OK. Maybe curl E = 0 for all coefficients!

Question 2: Let's repeat the above with our full solutions where BC's are included.

Let's now look at the solutions with BC's and in the DC approaching limit.

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\ (r/a)^{|m|}\ (|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\, R_{dc}$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\ (ak)\ (r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

Go back to our three test equations to see if curl E = 0:

$$mE_z = 0$$
$$\partial_r E_z = 0$$
$$\partial_r(rE_\theta) - jmE_r = 0 \qquad \text{at DC ?}$$

The first one is OK for m = 0. For $m \neq 0$ we need $(1/2)\ \eta_m\ I\ R_{dc}\ (r/a)^{|m|}\ (|m|+1) \rightarrow 0$ but this only works in the case $I \rightarrow 0$! One might use $I = 2\pi\omega\ (a/k)\ N_0$ of (D.2.31) to replace I. Then we have

$$I = 2\pi\omega\ (a/k)\ N_0 \ = \ 2\pi\omega\ (a/\ [\ \sqrt{RC}\ \omega^{1/2}\ e^{-j\pi/4}])\ N_0 \ = \ 2\pi\omega^{1/2}\ (a/\ [\ \sqrt{RC}\ \ e^{-j\pi/4}])\ N_0$$

and then as $\omega \rightarrow 0$, we really do have $I \rightarrow 0$ and then $mE_z = 0$ is rescued. By the same argument, we will find that $\partial_r E_z = 0$. The ∂_r does not create anything singular, and then I = 0. Same argument for the last line.

Conclusion: The solutions of App D do respect curl E = 0 in the DC limit! It just took me a little while to verify that above.

(b) Study curl E = 0 for DC again in the reflection scenario.

Somehow maybe my reflection scenario wrecks this fact by having a finite resulting current? Let's look at the general form of the reflection solution,

$$\mathbf{E}_{tot}(r,m,z,t) \ = A(k)\ \{\ \ e^{j(\omega t-\beta_d[z-L])}\ \mathbf{E}(r,m;k)\ -\ e^{j(\omega t+\beta_d[z-L])}\ \mathbf{E}(r,m;-k)\ \}$$

where

$$A(k) = (k/\mathbf{z})\ V(0)\ \frac{1}{2I\sin(kL)}$$

In the DC limit we see that

$$A(k) \rightarrow\ (V(0)/\ \mathbf{z})\ (1/2IL)$$

so this is immediately suspicious since it appears to blow up at DC! But let's for now take the curl of {} and see what happens. Maybe it is better to work with these forms instead,

$$E_{r,tot}(r,m,z) \ = A(k)\ E_r(r,m;k)\ \ \{\ -2j\ \sin(k[z-L])\ \}$$
$$E_{\theta,tot}(r,m,z) \ = A(k)\ E_\theta(r,m;k)\ \ \{\ -2j\ \sin(k[z-L])\ \}$$
$$E_{z,tot}(r,m,z) \ = A(k)\ E_z(r,m;k)\ \ \{\ \ 2\ \cos(k[z-L])\ \}\ .$$

Then

$$\text{curl}\ \mathbf{E}\ (r,\theta,z) = \hat{\mathbf{r}}\ [\ r^{-1}\partial_\theta E_z(r,\theta,z) - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\ [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta\ E_r\ ]$$

so

$$\text{curl}\ \mathbf{E}\ (r,m,z) = \hat{\mathbf{r}}\ [\ r^{-1}jmE_z(r,m,z) - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\ [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}jm\ E_r\ ]$$

where imagine "tot" in each subscript. Our three equations are

$$r^{-1}jmE_{z,tot} - \partial_z E_{\theta,tot} = 0$$
$$\partial_z E_{r,tot} - \partial_r E_{z,tot} = 0$$
$$r^{-1}\partial_r(rE_{\theta,tot}) - r^{-1}jmE_{r,tot} \qquad = 0 \qquad \text{for DC ?}$$

Let's study just the first r component equation :

$r^{-1}jm\{E_{z,tot}\} = \partial_z\{E_{\theta,tot}\}$?

$r^{-1}jm\{ A(k) E_z(r,m;k) \{ 2\cos(k[z-L]) \} = \partial_z\{ A(k) E_\theta(r,m;k) \{ -2j \sin(k[z-L]) \}$?

$r^{-1}jm\{ A(k) E_z(r,m;k) \{ 2\cos(k[z-L]) \} = \{ A(k) E_\theta(r,m;k) \{ -2j\, k\cos(k[z-L]) \}$?

$A(k)\, r^{-1}jm\{ E_z(r,m;k) \{ 1 \} = A(k)\{ E_\theta(r,m;k) \{ -j\, k\}$?

$A(k)\, r^{-1}m\{ E_z(r,m;k) \} = - A(k)k \{E_\theta(r,m;k)\}$?

Let's now install the App D solutions:

$A(k)\, r^{-1}m\{ (1/4)\, \eta_m\, I\, R_{dc}\, (a\beta')\, f_m \} = - A(k)k \{(1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, h_m \}$?

Now we showed above that

$$A(k)\, I = (k/\mathbf{z})\, V(0)\, \frac{1}{2\sin(kL)}$$

so the above becomes

$$(k/\mathbf{z})\, V(0)\, \frac{1}{2\sin(kL)}\, r^{-1}m\{ (1/4)\, \eta_m\, R_{dc}\, (a\beta')\, f_m \} = - (k/\mathbf{z})\, V(0)\, \frac{1}{2\sin(kL)} \{(1/4)\, \eta_m\, R_{dc}\, (ak)\, h_m \}$$

The ratio k/sin is finite even at $\omega = 0$, so divide it out and other things to get

$(1/\mathbf{z})\, r^{-1}m\{ (1/4) (\beta')\, f_m \} = - (1/\mathbf{z}) \{(1/4) (k)\, h_m \}$?

z is also finite, so get rid of it as well

$r^{-1}m\{ (\beta')\, f_m \} = - \{ (k)\, h_m \}$?

Now look at the DC limits of f_m and h_m :

$$\begin{aligned} f_m &= (r/a)^{|m|}\, (|m|+1)\, (2/\beta' a) & f_0 &= 4/(a\beta') \\ g_m &= (r/a)^{|m|+1} + (r/a)^{|m|-1} & g_0 &= 2\, (r/a) \\ h_m &= (r/a)^{|m|+1} - (r/a)^{|m|-1} & h_0 &= 0 \quad . \end{aligned} \qquad \text{(D.11.6)}$$

For $m > 0$ we then want to check

$r^{-1}m\{ (\beta')\, (r/a)^{|m|}\, (|m|+1)\, (2/\beta' a)\} = - \{ (k)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\}$?

$r^{-1}m\{ (r/a)^{|m|} (|m|+1) (2/a)\} = - \{ (k) [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\}$?

$r^{-1}m\{ (r/a)^{|m|} (|m|+1) (2/a)\} = 0$?

This is clearly NOT true, and therefore we do not have curl $\mathbf{E}_{tot} = 0$ at DC!

Let's now *repeat* what we just did this time using the DC limits earlier on:

$A(k)\, r^{-1}m\{ E_z(r,m;k) \} = - A(k)k \{E_\theta(r,m;k)\}$?

$A(k)r^{-1}m\{ (1/2)\, \eta_m\, I\, R_{dc}\, (r/a)^{|m|} (|m|+1)\} = - A(k)k \{ (1/4)\, \eta_m\, I\, R_{dc}\, (ak) [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\}$

$A(k)r^{-1}m\{ I (r/a)^{|m|} (|m|+1)\} = - A(k)\, k \{ (1/2)\, I\, (ak) [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\}$

Now use the fact that

$A(k)I \rightarrow (V(0)/\mathbf{z})\, (1/2L)$

so we want to test

$(V(0)/\mathbf{z})\, (1/2L)r^{-1}m\{ (r/a)^{|m|} (|m|+1)\} = - (V(0)/\mathbf{z})\, (1/2L)\, k \{ (1/2)\, (ak) [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\}$

$r^{-1}m\{ (r/a)^{|m|} (|m|+1)\} = - k \{ (1/2)\, (ak) [(r/a)^{|m|+1} - (r/a)^{|m|-1}]\}$?

$r^{-1}m\{ (r/a)^{|m|} (|m|+1)\} = 0$?

Same result: This is NOT an equality, so curl E = 0 does not obtain in the DC limit for the reflection scenario. The problem is caused by the fact that A(k)I → constant.

Thus, I have found a FLAW in my little constructed reflection scenario!

For ω > 0, it seems OK because then as usual you compute B from curl E, and then you have div B = div curl E = 0 and then you have all 4 Maxwell's satisfied!

- div $\mathbf{E}_{tot} = 0$ by direct computation I did above in (1).
- the Helmholtz equation $(\nabla^2+\beta^2)\mathbf{E} = 0$ as verified in (1)
- the two boundary conditions as verified in (2) and (6)

Here is a verification of the curl B equation: (imagine all have tot subscripts)

$\mathbf{B} = (-1/j\omega)$ curl $\mathbf{E}$ // this is how I would compute B from E

curl $\mathbf{B} = (-1/j\omega)$ curl curl $\mathbf{E} = (-1/j\omega) [\, \nabla (\text{div } \mathbf{E}) - \nabla^2\mathbf{E} \,] = (+1/j\omega)\, \nabla^2\mathbf{E}$

$= (+1/j\omega)\, (-\beta^2\, \mathbf{E})$ // using Helmholtz

$= (j\beta^2/\omega)\, \mathbf{E}$

But this IS the correct curl B equation since (assuming Ohm's law)

curl $\mathbf{B} = \mu \mathbf{J} + \mu j\omega\varepsilon\mathbf{E} = \mu(\sigma + j\omega\varepsilon)\mathbf{E} = \mu(j\omega)(\varepsilon - j\sigma/\omega)\mathbf{E} = j\omega\mu\xi\mathbf{E}$
$= j(\beta^2/\omega)\mathbf{E}$. // see (1.5.1c)

Notice also that then, computing B as above,

div $\mathbf{B}$ = div $(-1/j\omega)$ curl $\mathbf{E} = (-1/j\omega)$ div curl $\mathbf{E} = 0$

How about the other Helmholtz equation? Can we show it is also respected?

curl $\mathbf{B} = j(\beta^2/\omega)\mathbf{E}$ / / known true from above

curl curl $\mathbf{B} = j(\beta^2/\omega)$ curl $\mathbf{E}$ // apply curl to both sides

$[\nabla(\text{div } \mathbf{B}) - \nabla^2\mathbf{B}] = j(\beta^2/\omega)$ curl $\mathbf{E}$ // vector ident

$[-\nabla^2\mathbf{B}] = j(\beta^2/\omega)[-j\omega\mathbf{B}]$ // div $\mathbf{B} = 0$ from above.

$[-\nabla^2\mathbf{B}] = (\beta^2)[\mathbf{B}]$

$(\nabla^2 + \beta^2)\mathbf{B} = 0$

Thus, for $\omega > 0$ I can extend the above list to

- div $\mathbf{E}_{\mathbf{tot}} = 0$ by direct computation I did above in (1).
- the Helmholtz equations $(\nabla^2+\beta^2)\mathbf{E}_{\mathbf{tot}} = 0$ as verified in (1)
- the two boundary conditions as verified in (2) and (6)
- curl $\mathbf{B}_{\mathbf{tot}} = \mu \mathbf{J} + \mu j\omega\varepsilon \mathbf{E}_{\mathbf{tot}}$
- div $\mathbf{B}_{\mathbf{tot}} = 0$
- curl $\mathbf{E}_{\mathbf{tot}} = -j\omega\mathbf{B}_{\mathbf{tot}}$
- the Helmholtz equations $(\nabla^2+\beta^2)\mathbf{B}_{\mathbf{tot}} = 0$

How about continuity? Inside we have that div E = 0 so therefore div J = 0. At the boundary, the CPBC assures that div J = $-j\omega\rho$ in the interaction with the surface charge and G = 0.

Thus, for $\omega > 0$ my round wire transmission line reflection scenario solution satisfies :

- all four Maxwell equations
- both E and B Helmholtz equations
- the two boundary conditions
- continuity inside and at the surface

I would have thought that uniqueness would imply that this then is *the* solution to the problem!!!

It just seems very unlikely that this solution would fail in the limit $\omega \to 0$ if it is OK for any $\omega > 0$! It really makes me think I have made a mistake somewhere.

Let's just recap the calculation above. I get

$$[\text{curl } \mathbf{E}_{tot}(r,m,z)]_r = r^{-1} jm E_{z,tot} - \partial_z E_{\theta,tot}$$

$$= r^{-1} jm \{ A(k) E_z(r,m;k) \{ 2\cos(k[z-L]) \} - \{ A(k) E_\theta(r,m;k) \{ -2j\, k\cos(k[z-L]) \}$$

$$= 2\, A(k) \cos(k[z-L]) \{ r^{-1} jm\, E_z(r,m;k) - E_\theta(r,m;k)(-jk) \}$$

$$= 2\, A(k) \cos(k[z-L]) \{ r^{-1} jm\, (1/4)\, \eta_m I R_{dc} (a\beta') f_m - (1/4)\, \eta_m I R_{dc} (ak) h_m (-jk) \}$$

$$= 2 [\, A(k) I \,] \cos(k[z-L])\, \eta_m R_{dc} (1/4) \{ r^{-1} jm (a\beta') f_m + (ak) h_m jk) \}$$

$$= 2 [\, A(k) I \,] \cos(k[z-L])\, \eta_m R_{dc} (j/4) \{ r^{-1} m (a\beta') f_m + (ak^2) h_m) \}$$

$$= 2 \left[(k/z)\, V(0) \frac{1}{2\sin(kL)}\right] \cos(k[z-L])\, \eta_m R_{dc} (j/4) \{ r^{-1} m (a\beta') f_m + (ak^2) h_m) \}$$

Now we take the DC limit to find

$$= 2 \left[(1/z)\, V(0) \frac{1}{2L}\right] \cos(k[z-L])\, \eta_m R_{dc} (j/4) \{ r^{-1} m (a\beta') f_m + (ak^2) h_m) \}$$

$$= 2 \left[(1/z)\, V(0) \frac{1}{2L}\right] \cos(k[z-L])\, \eta_m R_{dc} (j/4) \{ r^{-1} m (a\beta') (r/a)^{|m|} (|m|+1) (2/\beta' a) + (ak^2)\ 0]) \}$$

$$= 2 \left[(1/z)\, V(0) \frac{1}{2L}\right] \cos(k[z-L])\, \eta_m R_{dc} (j/4) \{ r^{-1} m (a\beta') (r/a)^{|m|} (|m|+1) (2/\beta' a)) \}$$

$$= 2 \left[(1/z)\, V(0) \frac{1}{2L}\right] \eta_m R_{dc} (j/4) \{ r^{-1} m\ (r/a)^{|m|} (|m|+1) (2)) \} \qquad m > 0$$

Thus, we conclude that for $m > 0$

$$[\text{curl } \mathbf{E}_{tot}(r,m,z)]_r = 2 \left[(1/z)\, V(0) \frac{1}{2L}\right] \eta_m R_{dc} (j/4) \{ r^{-1} m\ (r/a)^{|m|} (|m|+1) (2)) \}$$

Note: As shown just below, we must have curl $\mathbf{E} = \omega$ * [stuff] where stuff is finite near $\omega = 0$. But the above expression does NOT have this form! Big trouble here in River City, my friend, yes sir.

This DC result is a finite value which is a function of r and z and m. It is not 0.

One implication is this. Recall that

$$\mathbf{B} = (-1/j\omega)\ \text{curl}\ \mathbf{E}$$

so

$$B_{\mathbf{r}}(r,m) = (-1/j\omega)\ \{2\ [(1/\mathbf{z})\ V(0)\ \frac{1}{2L}\ ]\ \eta_{\mathbf{m}} R_{\mathbf{dc}}\ (j/4)\ \{\ r^{-1}m\ \ (r/a)^{|\mathbf{m}|}\ (|m|+1)\ (2))\ \}\}$$

As $\omega \to 0$. we get $B_{\mathbf{r}} = \infty$ everywhere which is real nonsense! In a magnetostatic situation, one must have curl **E** = 0 as in electrostatics. We know this from curl **E** = jω**B** with a finite **B** field. So what is causing $B_{\mathbf{r,tot}}$ to be infinite at DC ?

Part IV. So what is wrong?

I reread Part III today 8.12.14 and it seems fine, but I was looking for assumptions made. I first checked for the $100^{\mathbf{th}}$ time the limit of $f_{\mathbf{m}}$ and of course it is the same as it was the previous 99 times.

Idea 1. One thing I did notice is that when it comes to determining A(k) in section (4), I have to make use of my Part I expression for i(z) which is a standing wave. But this i(z) was obtained from the TL equations, and I make a big issues of the fact that I think these TL equations are invalid for small ω. They are valid in the network model all the way down to $\omega = 0$, but I don't think they are valid in the true physics model! But I sort of *rescue* the TL equations in Chapter 4 by doing "averaging" of various things. As part of this rescue, I get at one point

$$\partial_{\mathbf{z}} V(z) = -\ [\ Z_{\mathbf{s1}} + Z_{\mathbf{s1}} + j\omega L_{\mathbf{e}}]\ i(z)$$
$$\partial_{\mathbf{z}}\ i(z) = -\ [\ jk^{\mathbf{2}}/(\omega L_{\mathbf{e}})]\ V(z)\ . \qquad (4.11.14a)$$

In these equations which are basically the TL equations, these quantities are all averages:

$$V(z) \qquad Z_{\mathbf{si}} \qquad L_{\mathbf{e}}$$

whereas i(z) is not an average, it is the total current in the conductor. The first equation here is basically what we used above to compute i(z). It just reads

$$\partial_{\mathbf{z}} V(z) = -\ \mathbf{z}\ i(z)$$

and this is the only equation I used in Part I. So this really seems reasonable and is probably NOT the source of my problem!

Idea 2. When all is said and done, the $J_{\mathbf{z}}$ asymmetry present in the Appendix D solutions just passes through into the reflection scenario. In this doc, I think I finally constructed a solid reflection scenario, getting all the details and limits right. But I think "the problem" exists right in that Appendix D solution for the single direction TL wave for an infinite line. The problem is present there, and of course it just moves into the reflection solution pretty much unaltered. The big difference is that whereas I→0 in appendix D, we have i(z) → constant ≠ 0 in the reflection scenario, so now the asymmetry is a more painful violation of our eddy current intuition. I was tempted to ignore the $J_{\mathbf{z}}$ asymmetry present in the

Appendix D solution because I→0. But deep down, I was always expecting to see the J_z asymmetry gradually fade away for small ω, before the ω→0 final limit was reached, but it does not fade away, it approaches a constant value.

In Section D.11 (a) I try to justify this asymmetry failure of the theory. I argue that the TL equations are exact for ω all the way down to 0 in the network model, but they are not exact in the physics model. In retrospect, the averaging was supposed to repair this problem, so the argument at best is "weak". I am just looking for something to explain the asymmetry anomaly!

Perhaps I was hoping that somehow the reflection construction would make the asymmetry go away, but it did not do that! The only way to find out was to do the scenario in detail, and that is what this document has done!

Conclusion: If I really want to solve the J_z asymmetry mystery, I need to do it directly in the App D solution and not worry about the reflection scenario.

****

Appendix. What IS the function that keeps appearing?

[This stuff is not very interesting since I can just plot the series.]

$$E_z(r,\theta) = I R_{dc} \{ 1 + \Sigma_{m=1}^{\infty} (r/a)^m (m+1) \eta_m \cos(m\theta) \} \qquad \eta_m \equiv N_m/N_0 = (-1)^m e^{-|m\xi_1|}$$

$$= I R_{dc} \{ 1 + \Sigma_{m=1}^{\infty} (r/a)^m (m+1) (-1)^m e^{-m|\xi_1|} \cos(m\theta) \}$$

$$\equiv I R_{dc} S$$

For shorthand, lets write

$$\xi \equiv |\xi_1|$$

so then we have

$$S = 1 + \Sigma_{m=1}^{\infty} (r/a)^m (m+1) (-1)^m e^{-m\xi} \cos(m\theta)$$

So I will now attempt to evaluate this sum. But first simplify

$$\cos(m\theta) = (e^{jm\theta} + e^{-jm\theta})/2$$

$$S = 1 + \Sigma_{m=1}^{\infty} (r/a)^m (m+1) (-1)^m e^{-m\xi} (e^{jm\theta} + e^{-jm\theta})/2$$

$$= 1 + \{ (1/2) \Sigma_{m=1}^{\infty} (r/a)^m (m+1) (-1)^m e^{-m\xi} e^{jm\theta} + \text{c.c.} \}$$

Next, write

$(r/a)^m = e^{m\ln(r/a)}$

so we then have

$$S = 1 + \{ (1/2) \Sigma_{m=1}^{\infty} e^{m\ln(r/a)} (m+1) (-1)^m e^{-m\xi} e^{jm\theta} + c.c. \}$$

$$= 1 + \{ (1/2) \Sigma_{m=1}^{\infty} (m+1) (-1)^m e^{-m[\ln(a/r)+\xi-j\theta]} + c.c. \}$$

Let

$$x \equiv \ln(a/r)+\xi-j\theta$$

Then we have'

$$= 1 + \{ (1/2) \Sigma_{m=1}^{\infty} (m+1) (-1)^m e^{-mx} + c.c. \}$$

Now the expo is as simple as it can get. Now define

$$S_1 = \Sigma_{m=1}^{\infty} (-1)^m e^{-mx}$$

$$S_2 \equiv - \partial_x S_1 = \Sigma_{m=1}^{\infty} (-1)^m m e^{-mx}$$

Then we can write

$$S = 1 + (1/2) [(S_1 + S_2) + c.c.]$$

Now call upon GR7 page 27,

1. $$\tanh x = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx} \qquad [x > 0]$$

which says

$$\tanh(x/2) = 1 + 2 \Sigma_{m=1}^{\infty} (-1)^m e^{-mx}$$

or

$$\Sigma_{m=1}^{\infty} (-1)^m e^{-mx} = (1/2)[\tanh(x/2) - 1]$$

Then

$$S_1 = \Sigma_{m=1}^{\infty} (-1)^m e^{-mx} = (1/2)[\tanh(x/2) - 1]$$

$$S_2 = - \partial_x S_1 = -(1/2) (1/2) \operatorname{sech}^2(x/2) = -(1/4) \operatorname{sech}^2(x/2)$$

We then know that

S = 1 + (1/2) { (1/2)[tanh(x/2) - 1] - (1/4) sech^2(x/2) + c.c. }

= 1 + (1/4) { [tanh(x/2) - 1] - (1/2) sech^2(x/2) + c.c. }

= 1 + (1/2) Re { [tanh(x/2) - 1] - (1/2) sech^2(x/2) } = 1 + (1/2) Ref

Now let

y ≡ x/2

Then we have

S = 1 + (1/2) Re { [tanh(y)- 1] - (1/2) sech^2(y) }

where y = x/2 = [ln(a/r)+ξ-jθ]/2 = [(1/2)(ln(a/r) + ξ)] + j [-(1/2)θ] = a + jb

a = (1/2)(ln(a/r) + ξ) b = - θ/2

This is the first time I have ever computed this result. Maybe Maple can simplify this result?

```
> restart;
> assume(a,real,b,real);
> f := (tanh(y) -1) - (1/2)*(sech(y))^2;
```

$$f := \tanh(y) - 1 - \frac{1}{2}\,\text{sech}(y)^2$$

```
> y := a + I*b;
```

$$y := a + I\,b$$

```
> f;
```

$$\tanh(a + I\,b) - 1 - \frac{1}{2}\,\text{sech}(a + I\,b)^2$$

```
> f1 := evalc(f);
```

$$f1 := \frac{\sinh(a)\cosh(a)}{\sinh(a)^2 + \cos(b)^2} - 1 - \frac{1}{2}\,\frac{\cos(b)^2\cosh(a)^2}{(\sinh(a)^2 + \cos(b)^2)^2} + \frac{1}{2}\,\frac{\sin(b)^2\sinh(a)^2}{(\sinh(a)^2 + \cos(b)^2)^2} + I\left(\frac{\sin(b)\cos(b)}{\sinh(a)^2 + \cos(b)^2} + \frac{\cos(b)\cosh(a)\sin(b)\sinh(a)}{(\sinh(a)^2 + \cos(b)^2)^2}\right)$$

```
> Ref := Re(f1);
```

$$Ref := \frac{\sinh(a)\cosh(a)}{\sinh(a)^2 + \cos(b)^2} - 1 - \frac{1}{2}\,\frac{\cos(b)^2\cosh(a)^2}{(\sinh(a)^2 + \cos(b)^2)^2} + \frac{1}{2}\,\frac{\sin(b)^2\sinh(a)^2}{(\sinh(a)^2 + \cos(b)^2)^2}$$

```
> S := unapply(1+(1/2)*Ref,a,b);
```

$$S := (a, b) \to \frac{1}{2} + \frac{1}{2}\,\frac{\sinh(a\sim)\cosh(a\sim)}{\sinh(a\sim)^2 + \cos(b\sim)^2} - \frac{1}{4}\,\frac{\cos(b\sim)^2\cosh(a\sim)^2}{(\sinh(a\sim)^2 + \cos(b\sim)^2)^2} + \frac{1}{4}\,\frac{\sin(b\sim)^2\sinh(a\sim)^2}{(\sinh(a\sim)^2 + \cos(b\sim)^2)^2}$$

```
> S(xi/2 + (1/2)*ln(alpha/r),-theta/2);
>
```

$$\frac{1}{2} + \frac{1}{2}\,\frac{\sinh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)\cosh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)}{\sinh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)^2 + \cos\left(\frac{1}{2}\theta\right)^2} - \frac{1}{4}\,\frac{\cos\left(\frac{1}{2}\theta\right)^2\cosh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)^2}{\left(\sinh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)^2 + \cos\left(\frac{1}{2}\theta\right)^2\right)^2} + \frac{1}{4}\,\frac{\sin\left(\frac{1}{2}\theta\right)^2\sinh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)^2}{\left(\sinh\left(\frac{1}{2}\xi + \frac{1}{2}\ln\left(\frac{\alpha}{r}\right)\right)^2 + \cos\left(\frac{1}{2}\theta\right)^2\right)^2}$$

This result is more complicated than I want to deal with even though the denominators are the same apart for powers. But at least I know there is a closed form. However, for plotting purposes it is a lot easier to just use the series for S up to some finite number m_{max} of terms.

The plot of E_z as this sum S

$$S = 1 + \Sigma_{m=1}^{\infty} (r/a)^m (m+1) (-1)^m e^{-m|\xi_1|} \cos(m\theta)$$

has "the usual form" :

```
restart;
Ez := 1 + sum( (r/a)^m*(m+1)*(-1)^m*exp(-m*xi)*cos(m*theta),m=0..20):
a := 10; xi := 1;
```

$$a := 10$$

$$\xi := 1$$

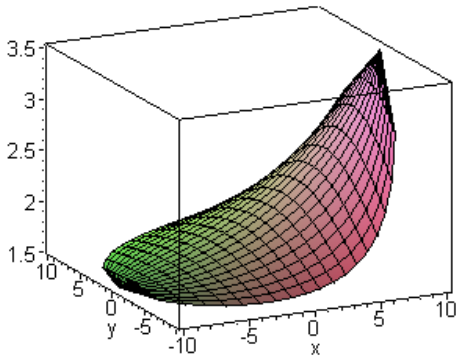
```
r := sqrt(x^2+y^2);
```

$$r := \sqrt{x^2+y^2}$$

```
theta := arctan(y,-x);
```

$$\theta := \arctan(y, -x)$$

```
plot3d(Ez,x=-a..a,y=-sqrt(a^2-x^2)..sqrt(a^2-x^2),axes=boxed, grid = [20,20]);
```



which shows strong asymmetry. What is new is that I am now getting this asymmetry at ω = 0 !!!

End