The DC Two-cylinder problem PhL 8.6.14

Summary: (8.12.14) We have two infinitely long round conductors (solid cylinders) as in Chapter 6. We are only dealing with the DC situation ($\omega = 0$), and we assume there is some $G \neq 0$, so in the solution, current *does* flow through the dielectric. This is a magnetostatics problem only! What I show here is that in this problem, J_z really is asymmetric around the conductor boundaries and presumably this asymmetry persists inside the conductors, though I don't actually compute things inside other than just below the surface. For me, just below the surface is good enough.

In a different problem with $G = 0$ but $\omega > 0$, we expect a similar asymmetry due to the asymmetric AC current flowing through the capacitor sense of the transmission line. We expect this current to be larger between the two cylinders, just as we expect it to be larger there in this problem.

The big unresolved mystery is of course this: With $G = 0$ and at $\omega = 0$, why does my Appendix D solution still show J_z asymmetry inside the wire?

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1. Statement of J just outside the boundary.

Infinite cylinders, of course. Doing no work, I already know the transverse current density in the dielectric from bipolar doc. I quote for E

$$\mathbf{E} = -\text{grad}\varphi = -(1/h)\,[\,(\partial_\xi \varphi)\,\hat{\boldsymbol{\xi}} + (\partial_u \varphi)\,\hat{\mathbf{u}}\,] \qquad (1/h) = (\text{ch}\xi - \text{cos}u)/a > 0$$

$$= -(1/h)\,[\,(\partial_\xi[-\frac{\xi}{\xi_2-\xi_1} V]\,)\,\hat{\boldsymbol{\xi}} = (V/h)\frac{1}{\xi_2-\xi_1}\hat{\boldsymbol{\xi}} = (V/h)\frac{1}{\xi_2+|\xi_1|}\hat{\boldsymbol{\xi}}$$

$$= (V/a)\,(\text{ch}\xi - \text{cos}u)\frac{1}{\xi_2+|\xi_1|}\hat{\boldsymbol{\xi}}\,. \qquad (10.10)$$

and then $\mathbf{J} = \sigma_d \mathbf{E}$. This current density has no azimuthal component at the round wire surface. I am assuming of course that the longitudinal decay is very smooth so at a given z we have a 2D problem.

Here then is the current density just outside conductor C_1 which has $\xi_1 < 0$:

$$\mathbf{J}(\xi_1,u) = \sigma_d\,(V/a)\,(\text{ch}\xi_1 - \text{cos}u)\frac{1}{\xi_2+|\xi_1|}\hat{\boldsymbol{\xi}}$$

Comment: if you set $\xi = \xi_1$ and move around the circular boundary, E is maximal when cosu is max negative and that is near $u = \pi$, and that is the region between the two conductors at $y = 0$.

2. To show that J is continuous at the boundary.

What happens to **J** at the boundary?

I want to draw a Gaussian box and draw some conclusions from that box, but it is not totally obvious how to do this because we are dealing with two different coordinate systems

(r,θ,z) cylindrical coordinates

(ξ,u,z) bi-cylindrical coordinates

Using the $(ds)^2$ formula, I think one can conclude that

$dr = hd\xi$ = dimension of the box perp to the perimeter

$ds = hdu$ = dimension of the box along the perimeter

$dz = dz$ = dimension of the box in the z direction

The three differential areas of interest are

dS_z = dsdr = area pointing in the z direction $= h^2 du d\xi$

dS_ξ = dsdz = area pointing in the ξ direction (radial) $= hdudz$

dS_u = drdz = a area pointing in the u direction (azimuthal) $= hd\xi dz$

where $h = h(u,\xi)$. Our rule is going to be $\int d\mathbf{S}\bullet\mathbf{J} = 0$, so this then says

$$dS_\xi(\xi+d\xi,u)\, J_\xi(\xi+d\xi,u,z) - dS_\xi(\xi,u)\, J_\xi(\xi,u,z)$$

$$+ dS_u(\xi,u+du)\, J_u(\xi,u+du,z) - dS_u(\xi,u)\, J_u(\xi,u,z)$$

$$+ dS_z(\xi,u)\, J_z(\xi,u,z+dz) - dS_z(\xi,u)\, J_z(\xi,u,z)$$

$$= d\xi\partial_\xi\,[dS_\xi\, J_\xi] + du\partial_u\,[dS_u\, J_u] + dz\partial_z\,[dS_z\, J_z]$$

$$= d\xi\,\partial_\xi\,[hdudz\, J_\xi] + du\,\partial_u\,[hd\xi dz\, J_u] + dz\,\partial_z\,[h^2 du d\xi\, J_z]$$

$$= d\xi du dz\,[\partial_\xi(h\, J_\xi) + \partial_u(h\, J_u) + h^2\partial_z\, J_z\,] = 0$$

So we end up then with

$$\partial_\xi(h\, J_\xi) + \partial_u(h\, J_u) + h^2 \partial_z\, J_z = 0$$

or

$$h^{-2}\partial_\xi(h\, J_\xi) + h^{-2}\partial_u(h\, J_u) + \partial_z\, J_z = 0$$

Compare with bipolar

$$[\text{div } \mathbf{B}](\mathbf{x}) = h^{-2}\, \{\, \partial_\xi\, [h\, B_\xi] + \partial_u\, [hB_u]\, \} \quad . \tag{8.11}$$

So I have done nothing here but rederive div J = 0 in bi-cylindricals. But lets go back and write

Suppose we make the box normal but super thin in the radial direction. Then since nothing is singular, I think this says J_ξ is continuous on both sides of the boundary.

Since we are DC, there is no current feeding the surface charge density. There is no $\partial\rho/dt$, so we basically have div **J** = 0 applied to a Gaussian box straddling the boundary. Since the current is normal to the surface and has no where else to go, I would say that

$$J_\xi(\xi_1+\varepsilon) = J_\xi(\xi_1- \varepsilon)$$

which is to say that the normal current is continuous through the boundary. The E field is not.

Conclusion so far: Just *below* the surface of conductor C_1 we have

$$\mathbf{J}(\xi_1,u) = \sigma_d\, (V/a)\, (ch\xi_1 – cosu)\, \frac{1}{\xi_2+|\xi_1|}\, \hat{\boldsymbol{\xi}}$$

This expression is good on the surface, just above it, or just below it. The surface of course is at $\xi = \xi_1$.

Comment: Since E_z is continuous through the surface, there is some E_z outside the surface which creates some $J_z = \sigma_d E_z$ there. I am neglecting this J_z current (I think) when I treat the dielectric as a 2D problem. I am assuming that $\sigma >> \sigma_d$,

3. Compute J_z just inside the boundary: asymmetry on boundary!

I know that div **J** = 0 inside the round wire, so that is where the answer has to come from. I can write this in bipolar coordinates as follows:

$$[\text{div } \mathbf{J}](\mathbf{x}) = (1/a)\, \{\, (ch\xi – cosu)\, (\partial_\xi J_\xi + \partial_u J_u) - sh\xi\, J_\xi - sinu\, J_u\, \} + \partial_z J_z$$

where I have added the last piece. I can now make the ansatz that $J_u = 0$ everywhere. This is true in the dielectric, and it is true at the round wire boundary, so maybe it is true inside. In this case

$$\text{div } \mathbf{J} = (1/a)\, \{\, (ch\xi – cosu)\, (\partial_\xi J_\xi) - sh\xi\, J_\xi\} + \partial_z J_z$$

This is valid in fact at any ξ,u,z inside C_1. However, I only know J_ξ just below the surface.

Up to this point, I have assumed everything constant in z, but now I have to assume a loss factor in the z direction due to R and G of the transmission line. The decay coefficient is

$$\alpha = \mathrm{Re}[\sqrt{(R+j\omega L)(G+j\omega C)}\ ] = \sqrt{RG} \qquad \text{at } \omega = 0 \qquad \dim \alpha = 1/m$$

Does this make sense? If R = 0, yes it does, there is no voltage drop going down the TL in z.
But suppose G = 0? In this case we have $Z_0 = \infty$ and I = 0, and so no drop, so yes it is true in that case as well though it is less obvious.

What about β?

$$\beta = \mathrm{Im}[\sqrt{(R+j\omega L)(G+j\omega C)}\ ] = 0 \qquad \text{at } \omega = 0$$

We then have

$$\mathbf{J}(z) = \mathbf{J}(0)\, e^{-jkz} = \mathbf{J}(0)\, e^{-\alpha z}\, e^{-j\beta z}$$

$$\partial_z \mathbf{J}(z) = \mathbf{J}(0)\,[-\alpha - j\beta]\, e^{-\alpha z}\, e^{-j\beta z} = \mathbf{J}(0)\,[-\alpha]\, e^{-\alpha z}$$

$$= -\alpha\, \mathbf{J}(z)$$

and **J** this decreases as z increases, as it should. So in particular

$$\partial_z J_z = -\alpha J_z$$

Meanwhile,

$$J_\xi = \sigma_d\, (V/a)\, (\mathrm{ch}\xi - \cos u)\, \frac{1}{\xi_2 + |\xi_1|} \qquad \text{near } \xi = \xi_1$$

$$\partial_\xi J_\xi = \sigma\, (V/a)\, \mathrm{sh}\xi \qquad \text{near } \xi = \xi_1$$

Then looking at div **J** = 0 :

$$(1/a)\ \{\ (\mathrm{ch}\xi - \cos u)\, (\partial_\xi J_\xi)\ - \mathrm{sh}\xi\, J_\xi\} + \partial_z J_z = 0$$

$$(1/a)\ \{\ (\mathrm{ch}\xi - \cos u)\, (\partial_\xi J_\xi)\ - \mathrm{sh}\xi\, J_\xi\} - \alpha\, J_z = 0$$

$$\alpha a J_z = (\mathrm{ch}\xi - \cos u)\, (\partial_\xi J_\xi)\ - \mathrm{sh}\xi\, J_\xi$$

$$\alpha a J_z = (\mathrm{ch}\xi - \cos u)\, [\sigma_d\, (V/a)\, \mathrm{sh}\xi_1]\ - \mathrm{sh}\xi_1\, [\sigma\, (V/a)\, (\mathrm{ch}\xi - \cos u)\, \frac{1}{\xi_2 + |\xi_1|}$$

$$\alpha a J_z = \sigma_d\, (V/a)\ (\mathrm{ch}\xi - \cos u)\, \mathrm{sh}\xi_1\, [\ 1 - \frac{1}{\xi_2 + |\xi_1|}\ ] \qquad \alpha = \sqrt{RG}$$

]

Dimension check:

RHS = mho/m * volts/m = amp/m^2 LHS = m^{-1} * m * amp/m^2 = amp/m^2 OK

Now lets state clearly what the above really says:

$$\alpha a J_z(\xi_1,u) = \sigma_d\,(V/a)\;(ch\xi_1 - cosu)\,sh\xi_1\,[\,1 - \frac{1}{\xi_2+|\xi_1|}\,]$$

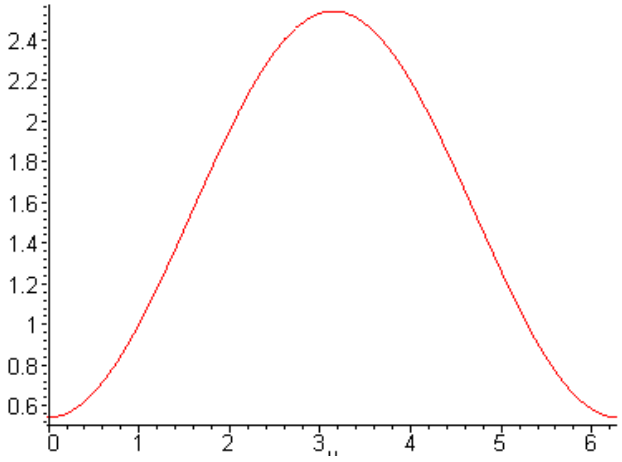
or

$$\alpha a J_z(\xi_1,u) = \sigma_d\,(V/a)\;(ch\xi_1 - cosu)\,sh|\xi_1|\,[\,\frac{1}{\xi_2+|\xi_1|} - 1]$$

This only tells us the value of $J_z(\xi_1,u)$ just below the surface, since that is the J_ξ we inserted.

Fact: J_z is asymmetric around the boundary of C_1! This is so because it is a function of u ! Ignoring overall scale factors we have $J_z = ch\xi_1 - cosu$ and here is what that looks like for $\xi_1 = -1$:

```
xi1 := -1:
xi2 := 1:
Jz := cosh(xi1)-cos(u):
plot(Jz,u=0..2*Pi);
```



Comments :

$$G = C\,(\sigma_d/\varepsilon_d)$$

so we then have

$$\sqrt{RG}\;a\,J_z = \sigma_d\,(V/a)\;(ch\xi_1 - cosu)\,sh|\xi_1|\,[\,\frac{1}{\xi_2+|\xi_1|} - 1]$$

or

$$\sqrt{RC(\sigma_d/\varepsilon_d)}\;a\,J_z = \sigma_d\,(V/a)\;(ch\xi_1 - cosu)\,sh|\xi_1|\,[\,\frac{1}{\xi_2+|\xi_1|} - 1]$$

or

$$\sqrt{RC/(\sigma_d \varepsilon_d)}\, a\, J_z = (V/a)\ (ch\xi_1 - cosu)\, sh|\xi_1|\, [\ \frac{1}{\xi_2+|\xi_1|} - 1]$$

This says that as $\sigma_d \to 0$ (meaning $G \to 0$), we just have $J_z \to 0$ as well. This is then the case with $Z_0 = \infty$ and we expect no current at all. So result is somewhat consistent.

4. Compute J_z everywhere inside the conductor

How would we obtain J_z everywhere inside the conductor C_1? I know that

I have shown that: // scratch and then maple for r > a only

$$cosu = |\, x^2 + y^2 - a^2\, | \,/\, \sqrt{(x^2 + y^2 - a^2)^2 + 4a^2y^2} \quad = \frac{|\, x^2 + y^2 - a^2\, |}{\sqrt{(x^2 + y^2 - a^2)^2 + 4a^2y^2}}$$

Then we have

$$\alpha a J_z(\xi_1,x,y) = \sigma\, (V/a)\ (ch\xi_1 - \frac{|\, x^2 + y^2 - a^2\, |}{\sqrt{(x^2 + y^2 - a^2)^2 + 4a^2y^2}})\, sh|\xi_1|\, [\ \frac{1}{\xi_2+|\xi_1|} - 1]$$

only for x,y located on the boundary of C_1. I could then use this as the boundary value for a Dirichlet problem $\nabla^2 J_z = 0$ and then I would know J_z at all points within the disk. But (x,y) are not quite the coordinates I want for a circle centered at the origin. I have to use

$x' = x - x_c \qquad x_c = a/th(\xi_1) = a\, cth(\xi_1) < 0$

$y' = y$

Then we have this mess

$$\frac{|\, (x'+acth\xi_1)^2 + y'^2 - a^2\, |}{\sqrt{((x'+acth\xi_1)^2 + y'^2 - a^2)^2 + 4a^2y'^2}}$$

and then

$$\alpha a J_z(\xi_1,x',y') = \sigma\, (V/a)\ (ch\xi_1 - \frac{|\, (x'+acth\xi_1)^2 + y'^2 - a^2\, |}{\sqrt{((x'+acth\xi_1)^2 + y'^2 - a^2)^2 + 4a^2y'^2}})\, sh|\xi_1|\, [\ \frac{1}{\xi_2+|\xi_1|} - 1]$$

This is then the value of J_z at any point on the circle $x'^2 + y'^2 = R_1^2$ where $R_1 = a/\, sh|\xi_1$. You could then convert this to polar coordinates and then use Stakgold's general formula for the interior solution.

It would be easier to do this all directly in bipolar coordinates I suspect. But I really don't want to go off and do that problem. I could make it an exercise though!

Conclusion: I have shown that J_z is asymmetric around the boundary, and surely then it is asymmetric in the interior with a shape similar to my lines doc results for $\omega > 0$.