The Reflection Scenario PhL 5.16.14

This is 30 long pages on this subject. Among other things, I tried rewriting all of Appendix D in the reflection framework. Although I don't see the conclusion here, somewhere I concluded that even when you do this reflection stuff, you *still* end up with J_z asymmetry as $\omega \to 0$. It is embedded in the two waves you superpose, so it does not go away. (10.8.14)

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I started this in "conclusions on low freq" and will start over here.

1. Setup

Transmission line segment runs from z = 0 to z = L. The line is driven at z = 0 and is shorted at z = L. don't ever confuse L with inductance in the following please.

We model the wave action for V(z) on this segment by superposing waves of amplitude A going to the right and B going to the left. Thus,

$$V(z)/V = A\, e^{j(\omega t - \beta_d z)} + B\, e^{j(\omega t + \beta_d z)}$$

The boundary conditions at the two ends are:

$A+B = 1$ at the left driving end

$Ae^{-j\beta_d L} + Be^{+j\beta_d L} = 0$ at the shorting bar point

Maple says

```
eq1 := A + B = V;
```

$$eq1 := A + B = V$$

```
eq2 := A*exp(-I*bd*L) + B*exp(I*bd*L) = 0;
```

$$eq2 := A\,e^{(-I\,bd\,L)} + B\,e^{(I\,bd\,L)} = 0$$

```
S := solve({eq1,eq2},{A,B});
```

$$S := \left\{ B = \frac{e^{(-I\,bd\,L)}\,V}{e^{(-I\,bd\,L)} - e^{(I\,bd\,L)}},\ A = -\frac{V\,e^{(I\,bd\,L)}}{e^{(-I\,bd\,L)} - e^{(I\,bd\,L)}} \right\}$$

So Maple tells us that,

A = $e^{+j\beta_d L}$/ [2jsin(β_dL)]

B = -$e^{-j\beta_d L}$/ [2jsin(β_dL)]

Therefore the solution V(z) is given by

$$V(z) = \frac{V}{2j\sin(\beta_d L)}\left[\, e^{j(\omega t-\beta_d z+\beta_d L)} - e^{j(\omega t+\beta_d z-\beta_d L)} \,\right]$$

$$= \frac{V}{2j\sin(\beta_d L)}\, e^{j\omega t}\ 2j\sin(-\beta_d z + \beta_d L) = \frac{V}{\sin(\beta_d L)}\, e^{j\omega t}\sin(\beta_d (L-z))$$

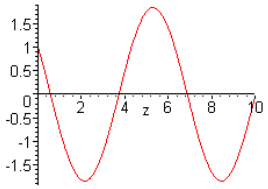
Ignoring the time phasor as we normally do, this says

$$V(z) = V\,\frac{\sin(\beta_d (L-z))}{\sin(\beta_d L)}$$ = standing wave pattern $\beta_d = \omega/v$

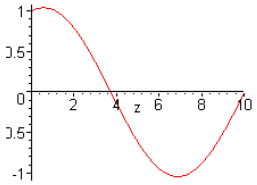
As $\omega \to 0$ we get

$$V(z) = V\,\frac{L-z}{L}$$

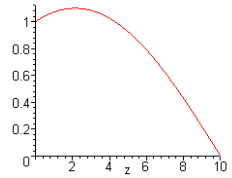
Here are some pictures of V(z) at various values of ω using L = 10 units and v_d = 1 unit so β_d = ω :



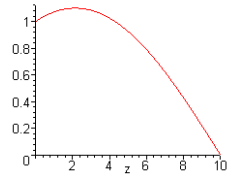
Above is ω = 1



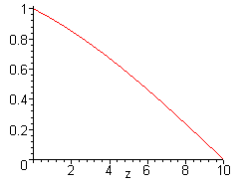
Above is ω = 1/2



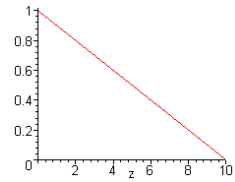
Above is ω = 1/5



Above is ω = 1/5



Above is ω = 1/10



Above is ω = 1/100

2. The meaning of ∂_z when acting on V at low ω

For a normal wave we know that $\partial_z \rightarrow -j\beta_d = -j(\omega/v_d)$.

For our standing wave above, the result is different: [here $v_d = 1$]

```
f := sin(w*(L-z))/sin(w*L);
```

$$f := \frac{\sin(w\,(L-z))}{\sin(w\,L)}$$

```
diff(f,z);
```

$$-\frac{\cos(w\,(L-z))\,w}{\sin(w\,L)}$$

As $\omega \rightarrow 0$ we can see that

$$\partial_z V(z) \rightarrow -(V/L)\cos[\omega(L-z)] = -V/L \qquad \text{or} \qquad \partial_z \rightarrow -1/L$$

and this then is just the negative slope in the last picture above. So we can say $\partial_z \rightarrow -1/L$ for very low values of $\omega > 0$ as well as $\omega = 0$.

Now, certain other quantities have this same behavior. For example

$$n(\theta,z) = n(\theta,0)\,\frac{\sin(\beta_d(L-z))}{\sin(\beta_d L)}$$

so charge sloshes back and forth at ω maximally at each antinode, and not at all at each node. In the limit though the only node is at $z = L$, and the max sloshing is at $z = 0$. ok to here 6/5/14

3. What is the current i(z) in the wire segment?

Go back to V(z) again:

$$V(z) = A\,e^{j(\omega t-\beta_d z)} + B\,e^{j(\omega t+\beta_d z)}$$

where we superpose the two wave potentials. Perhaps we can say

current in right going wave is $i_R(z) = A\,e^{j(\omega t-\beta_d z)}/Z_0$

current in left going wave is $i_L(z) = B\,e^{j(\omega t-\beta_d z)}/Z_0$

This would say $i(z) = V(z)/Z_0$ which I think is wrong because I expect i(z) to be large at $z = L$ because there is a short, whereas $V(z) = 0$ there! On the other and maybe expect $i(z) = 0$ at L since $V(z) = 0$. I am not sure of that.

What about Chapter 4.11 equations? As long as I don't replace ∂_z I think they still apply! Thus, I end up with all these applicable equations

$$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\,i(z) \qquad \frac{\partial i(z)}{\partial z} = -\mathbf{y}\,V(z) \qquad \text{" the classics"}$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C\,. \qquad (4.11.15)$$

In particular, this says that

$$i(z) = (-1/\mathbf{z})\,\partial_{\mathbf{z}} V(z)$$

So then we get

$$V(z) = V\,\frac{\sin(\beta_d(L-z))}{\sin(\beta_d L)}$$

$$i(z) = (-1/\mathbf{z})\left[\,-V\beta_d\,\frac{\cos(\beta_d(L-z))}{\sin(\beta_d L)}\,\right] = (V\beta_d/\mathbf{z})\,\frac{\cos(\beta_d(L-z))}{\sin(\beta_d L)}$$

where derivative is from Maple

```
Vz := V*sin(beta*(L-z))/sin(beta*L);
```

$$Vz := \frac{V\sin(\beta\,(L-z))}{\sin(\beta\,L)}$$

```
diff(Vz,z);
```

$$-\frac{V\cos(\beta\,(L-z))\,\beta}{\sin(\beta\,L)}$$

Now for small β_d this becomes

$$i(z) = (V\beta_d L/L\mathbf{z})\,\frac{\cos(\beta_d(L-z))}{\sin(\beta_d L)} \approx (V/L\mathbf{z})\cos(0) = V/(L\mathbf{z})$$

and for small ω this is a constant, independent of z, as I would expect.

What happens then with the other classic TLE:

$$\frac{\partial i(z)}{\partial z} = -\,\mathbf{y}\,V(z) \quad \Rightarrow \quad V(z) = (-1/\mathbf{y})\,\frac{\partial i(z)}{\partial z}$$

$$i(z) = (-1/\mathbf{z})\,\partial_{\mathbf{z}} V(z)$$

$$V(z) = (-1/\mathbf{y})\,(-1/\mathbf{z})\,\partial_{\mathbf{z}}{}^2 V(z) = (1/\mathbf{yz})\left[\,-V\,\beta_d{}^2\,\frac{\sin(\beta_d(L-z))}{\sin(\beta_d L)}\,\right]$$

according to Maple,

```
diff(Vz,z,z);
```

$$-\frac{V\sin(\beta\,(L-z))\,\beta^2}{\sin(\beta\,L)}$$

So I then get

$$V(z) = (-\beta_d^2/\mathbf{yz})\, V(z)$$

But if I go down to (5.3.4) I find

$$k_\varphi^2 = k_A^2 \equiv k^2 = -\mathbf{zy} = -(R+j\omega L)(G+j\omega C) \qquad (5.3.4)$$

and in the low loss worked we set $k^2 = \beta_d^2$ and then we have

$$\beta_d^2 \approx -\mathbf{zy}$$

and then everything is consistent (whew!!).

So we have answered the question "what is the current in the wire"

$$i(z) = (V\beta_d/\mathbf{z}) \frac{\cos(\beta_d(L-z))}{\sin(\beta_d L)} \qquad \text{// for any } \omega$$

$$= V/(L\mathbf{z}) = \text{constant} \qquad \text{// for small } \omega$$

4. What is the impedance seen from the driving end?

At some point z we have shown that

$$V(z) = V \frac{\sin(\beta_d(L-z))}{\sin(\beta_d L)} \qquad i(z) = (V\beta_d/\mathbf{z}) \frac{\cos(\beta_d(L-z))}{\sin(\beta_d L)}$$

Therefore

$$Z(z) = V(z)/i(z) = (\mathbf{z}/\beta_d) \tan[\, \beta_d(L-z)\,] \text{ ohms} \qquad \text{// any } \omega$$

$$= (\mathbf{z}/\beta_d)[\, \beta_d(L-z)] = \mathbf{z}\,(L-z) \text{ ohms} \qquad \text{// small } \omega$$

Fascinating. It has nodes of 0 (where V(z) = 0) and antinodes of ∞ (where i(z) = 0). In particular

$$Z(0) = V(0)/i(0) = (\mathbf{z}/\beta_d) \tan[\, \beta_d L\,] \qquad \text{// any } \omega$$

$$= (\mathbf{z}/\beta_d)(\beta_d L) = \mathbf{z}L \qquad \text{// small } \omega$$

$$= RL \qquad \text{// } \omega = 0$$

Notice that this is completely different from Z_0 of the transmission line.

$$Z_0 = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \infty \text{ at DC}$$

5. What is the current density J_z in the wire segment?

I can only assume it has the same z dependence as i(z), so then

$$J_{\mathbf{z}}(r,\theta,z) = f(r,\theta)\,(V\beta_{\mathbf{d}}/\mathbf{z})\,\frac{\cos(\beta_{\mathbf{d}}(L-z))}{\sin(\beta_{\mathbf{d}}L)} \qquad \text{// general } \omega$$

$$= f(r,\theta)\,V/(L\mathbf{z}) \qquad \text{// small } \omega$$

$$E_{\mathbf{z}}(r,\theta,z) = f(r,\theta)\,(1/\sigma)\,V/(L\mathbf{z}) \qquad \text{// small } \omega$$

Maybe similar forms for $E_{\mathbf{r}}$ and $E_{\boldsymbol{\theta}}$.

6. What does div E = 0 have to say?

$$\partial_{\mathbf{r}}\,(r\,E_{\mathbf{r}}(r,\theta,z)) + \partial_{\boldsymbol{\theta}}E_{\boldsymbol{\theta}}(r,\theta,z) + r\,\partial_{\mathbf{z}}E_{\mathbf{z}}(r,\theta,z) = 0\ .$$
$$\partial_{\mathbf{r}}\,(r\,E_{\mathbf{r}}(r,\theta,z)) + r\,\partial_{\mathbf{z}}E_{\mathbf{z}}(r,\theta,z) \approx 0 \qquad \text{// near } r = a \qquad \text{" near surface"}$$

Now for small ω I get $\partial_{\mathbf{z}}E_{\mathbf{z}}(r,\theta) \approx 0$ so then

$$\partial_{\mathbf{r}}\,(r\,E_{\mathbf{r}}(r,\theta,z)) = 0$$

$$r\,E_{\mathbf{r}}(r,\theta,z) = g(\theta,z), \text{ some unknown function}$$

$$E_{\mathbf{r}}(r,\theta,z) = g(\theta,z)\,/\,r$$

But this says we must have $g(\theta,z) = 0$, and we conclude that $E_{\mathbf{r}}$ = for small ω.

7. What does the cpbc have to say?

$$E_{\mathbf{r}}(a,\theta,z)) = (j\omega/\sigma)\,n(\theta,z))$$

Both sides go to 0 as $\omega\rightarrow 0$. So here we end up for very small ω with div E saying nothing at all about asymmetry or even $E_{\mathbf{z}}$!

I think in this "scenario", as $\omega\rightarrow 0$ the asymmetry goes away! A detailed analysis would require a whole new Appendix D1 which is geared to this physical experiment.

But maybe there is still some way to use superposition and then make use of existing Appendix D results.

8. What would Appendix D1 look like?

Try this:

$$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t-\beta_{\mathbf{d}}\mathbf{z})}\,\mathbf{E}_{\mathbf{R}}(r,\theta) + e^{j(\omega t+\beta_{\mathbf{d}}\mathbf{z})}\,\mathbf{E}_{\mathbf{L}}(r,\theta) \qquad \text{(D.1.1)'}$$

$$\partial_z \mathbf{E}(r,\theta,z,t) = -j\beta_d\, e^{j(\omega t-\beta_d z)}\, \mathbf{E_R}(r,\theta) + j\beta_d\, e^{j(\omega t+\beta_d z)}\, \mathbf{E_L}(r,\theta)$$

$$= (-j\beta_d)[\, e^{j(\omega t-\beta_d z)}\, \mathbf{E_R}(r,\theta) - e^{j(\omega t+\beta_d z)}\, \mathbf{E_L}(r,\theta)\,]$$

$$\partial_z^2\, \mathbf{E}(r,\theta,z,t) = (-j\beta_d)^2\, \mathbf{E}(r,\theta,z,t) = -\beta_d^2\, \mathbf{E}(r,\theta,z,t)$$

so the second derivative is the same as before, the first derivative is different. The only place the first derivative appears is in the div E = 0 equation section.

$$\partial_r (r\, E_r) + \partial_\theta E_\theta + r\, \partial_z E_z = 0\ .$$

What does this now say?

$$\partial_r (r\, [e^{j(\omega t-\beta_d z)}\, E_{Rr}(r,\theta) + e^{j(\omega t+\beta_d z)}\, E_{Lr}(r,\theta)])$$

$$+ \partial_\theta ([e^{j(\omega t-\beta_d z)}\, E_{R\theta}(r,\theta) + e^{j(\omega t+\beta_d z)}\, E_{L\theta}(r,\theta)])$$

$$+ r\, \partial_z([e^{j(\omega t-\beta_d z)}\, E_{R\theta}(r,\theta) + e^{j(\omega t+\beta_d z)}\, E_{L\theta}(r,\theta)]) = 0$$

or

$$e^{j(\omega t-\beta_d z)}\ \{\, \partial_r(r\, E_{Rr}(r,\theta)) + \partial_\theta\, E_{R\theta}(r,\theta) + r(-j\beta_d)\, E_{R\theta}(r,\theta)\,\}$$

$$+ e^{j(\omega t+\beta_d z)}\ \{\, \partial_r(r\, E_{Lr}(r,\theta)) + \partial_\theta\, E_{L\theta}(r,\theta) + r(+j\beta_d)\, E_{L\theta}(r,\theta)\,\} = 0$$

So now I think we end up with TWO divE = 0 equations to worry about

$$\partial_r(r\, E_{Rr}(r,\theta)) + \partial_\theta\, E_{R\theta}(r,\theta) + r(-j\beta_d)\, E_{R\theta}(r,\theta) = 0$$

$$\partial_r(r\, E_{Lr}(r,\theta)) + \partial_\theta\, E_{L\theta}(r,\theta) + r(+j\beta_d)\, E_{L\theta}(r,\theta) = 0$$

Giving each the usual θ expansion, we then get

$$\partial_r(r\, E_{Rr}(r,\theta)) + jm\, E_{R\theta}(r,\theta) + r(-j\beta_d)\, E_{R\theta}(r,\theta) = 0$$

$$\partial_r(r\, E_{Lr}(r,\theta)) + jm\, E_{L\theta}(r,\theta) + r(+j\beta_d)\, E_{L\theta}(r,\theta) = 0$$

I think we then end up with this summary starting position:

The Three Helmholtz Equations and the div E = 0 equation (in partial waves) (D.1.20)

$$[\nabla^2 \mathbf{E}]_z + \beta^2\, E_z = 0\ :$$

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2\,(\beta^2 - \beta_d^2)]\,E_z(r,m) = 0 \qquad \textit{(D.1.15)}$$

$[\nabla^2\mathbf{E}]_r + \beta^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\,E_r(r,m) - 2jm\,E_\theta(r,m) = 0 \qquad \textit{(D.1.17)}$$

$[\nabla^2\mathbf{E}]_\theta + \beta^2 E_\theta = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\,E_\theta(r,m) + 2jmE_r(r,m) = 0 \qquad \textit{(D.1.18)}$$

div $\mathbf{E} = 0$:

$$\partial_r(r\,E_{Rr}(r,m)) + jm\,E_{R\theta}(r,m) + r(-j\beta_d)\,E_{R\theta}(r,m) = 0 \qquad \textit{(D.1.19)}$$
$$\partial_r(r\,E_{Lr}(r,m)) + jm\,E_{L\theta}(r,m) + r(+j\beta_d)\,E_{L\theta}(r,m) = 0$$

$$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t-\beta_d z)}\,\mathbf{E}_R(r,\theta) + e^{j(\omega t+\beta_d z)}\,\mathbf{E}_L(r,\theta)$$

which is not THAT much more complicated than we had. What we really have is this in fact:

The Three Helmholtz Equations and the div E = 0 equation (in partial waves) (D.1.20)

$[\nabla^2\mathbf{E}]_z + \beta^2 E_z = 0$:

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2\,(\beta^2 - \beta_d^2)]\,E_{Rz}(r,m) = 0 \qquad \textit{(D.1.15)}$$
$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2\,(\beta^2 - \beta_d^2)]\,E_{Lz}(r,m) = 0 \qquad \textit{(D.1.15)}$$

$[\nabla^2\mathbf{E}]_r + \beta^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\,E_{Rr}(r,m) - 2jm\,E_{R\theta}(r,m) = 0 \qquad \textit{(D.1.17)}$$
$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\,E_{Lr}(r,m) - 2jm\,E_{L\theta}(r,m) = 0 \qquad \textit{(D.1.17)}$$

$[\nabla^2\mathbf{E}]_\theta + \beta^2 E_\theta = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\,E_{R\theta}(r,m) + 2jmE_{Rr}(r,m) = 0 \qquad \textit{(D.1.18)}$$
$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\,E_{L\theta}(r,m) + 2jmE_{Lr}(r,m) = 0 \qquad \textit{(D.1.18)}$$

div $\mathbf{E} = 0$:

$$\partial_r(r\,E_{Rr}(r,m)) + jm\,E_{R\theta}(r,m) + r(-j\beta_d)\,E_{R\theta}(r,m) = 0 \qquad \textit{(D.1.19)}$$
$$\partial_r(r\,E_{Lr}(r,m)) + jm\,E_{L\theta}(r,m) + r(+j\beta_d)\,E_{L\theta}(r,m) = 0$$

$$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t-\beta_d z)}\,\mathbf{E}_R(r,\theta) + e^{j(\omega t+\beta_d z)}\,\mathbf{E}_L(r,\theta)$$

This appears to be two completely independent sets of four equations each.

(a) The E_z Solution

The solutions to the E_z equation will be as before:

$$E_z(r,m) = C_{zm}\, J_m(\beta' r) \qquad \text{(D.2.4)}$$

where C_{zm} is at this point unknown. Ah, but I now see that it will be two equations!

$$E_{Rz}(r,m) = C_{Rzm}\, J_m(\beta' r) \qquad \text{(D.2.4)}$$

$$E_{Lz}(r,m) = C_{Lzm}\, J_m(\beta' r) \qquad \text{(D.2.4)}$$

(b) The E_r Solution

The E_r equation is exactly as in Appendix D

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_r(r,m) - 2jm\, E_\theta(r,m) = 0 \qquad \text{(D.2.5)}$$

I am inclined to treat this now as two different equations, matching z dependence

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_{Rr}(r,m) - 2jm\, E_{R\theta}(r,m) = 0 \qquad \text{(D.2.5)}$$
$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_{Lr}(r,m) - 2jm\, E_{L\theta}(r,m) = 0 \qquad \text{(D.2.5)}$$

and now I have two div E equations of the form:

$$-jmE_{R\theta}(r,m) = [1 + r\partial_r]\, E_{Rr}(r,m) - r\,(j\beta_d)E_{Rz}(r,m)\,. \qquad \text{(D.2.6)}$$

$$-jmE_{L\theta}(r,m) = [1 + r\partial_r]\, E_{Lr}(r,m) + r\,(j\beta_d)E_{Lz}(r,m)\,. \qquad \text{(D.2.6)}$$

which I can insert to get these two E_r equations:

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\,\beta'^2]\, E_{Rr}(r,m) = 2r\,(j\beta_d)E_{Rz}(r,m)\,. \qquad \text{(D.2.7)}$$

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\,\beta'^2]\, E_{Lr}(r,m) = -\,2r\,(j\beta_d)E_{Lz}(r,m)\,. \qquad \text{(D.2.7)}$$

I then insert the E_z solutions to get

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\,\beta'^2]\, E_{Rr}(r,m) = 2r\,(j\beta_d)\, C_{Rzm}\, J_m(\beta' r) = K_{Rm}\,\beta' r\, J_m(\beta' r)$$

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\,\beta'^2]\, E_{Lr}(r,m) = -\,2r\,(j\beta_d)\, C_{Lzm}\, J_m(\beta' r) = K_{Lm}\,\beta' r\, J_m(\beta' r)$$

where

$$K_{Rm} \equiv +\,2j\,(\beta_d/\beta')\, C_{Rzm}$$
$$K_{Lm} \equiv -\,2j\,(\beta_d/\beta')\, C_{Lzm}$$

I think we will end up then with the following E_r solutions:

$$E_{rR}(r,m) = a_{Rm}\ x^{-1}\ J_m(x) + \frac{K_{Rm}}{2}\ J_{m+1}(x)\ . \qquad \text{(D.2.15)}$$

$$E_{rL}(r,m) = a_{Lm}\ x^{-1}\ J_m(x) + \frac{K_{Lm}}{2}\ J_{m+1}(x)\ . \qquad \text{(D.2.15)}$$

(c) The E_θ Solution

I think I first get these results

$$jmE_{R\theta}(r,m) = -a_{Rm}\ J_m'(x) + \frac{K_{Rm}}{2}\ [\ -J_{m+1}(x) - x\ J_{m+1}'(x) + x\ J_m(x)\ ]$$

$$jmE_{L\theta}(r,m) = -a_{Lm}\ J_m'(x) + \frac{K_{Lm}}{2}\ [\ -J_{m+1}(x) - x\ J_{m+1}'(x) - x\ J_m(x)\ ] \qquad \text{// note - sign}$$

The sign difference then makes a difference. Use next

$$-xJ_{m+1}' = -xJ_m + (m+1)J_{m+1}$$

$$J_m' = -J_{m+1} + (m/x)J_m$$

and then you get

$$jmE_{R\theta}(r,m) = -a_{Rm}\ [-J_{m+1} + (m/x)J_m] + \frac{K_{Rm}}{2}\ [\ -J_{m+1}(x) + \{\ (m+1)J_{m+1} - xJ_m\ \} + x\ J_m(x)\ ]$$

$$jmE_{L\theta}(r,m) = -a_{Lm}\ [-J_{m+1} + (m/x)J_m] + \frac{K_{Lm}}{2}\ [\ -J_{m+1}(x) + \{\ (m+1)J_{m+1} - xJ_m\ \} - x\ J_m(x)\ ]$$

or

$$jmE_{R\theta}(r,m) = -a_{Rm}\ [-J_{m+1} + (m/x)J_m] + \frac{K_{Rm}}{2}\ [\ -J_{m+1}(x) + (m+1)J_{m+1}\ ]$$

$$jmE_{L\theta}(r,m) = -a_{Lm}\ [-J_{m+1} + (m/x)J_m] + \frac{K_{Lm}}{2}\ [\ -J_{m+1}(x) + (m+1)J_{m+1} - 2x\ J_m(x)\ ]$$

so maybe easier to just keep the original forms.

First summary of E field solutions:

First write

$$E_{Rz}(r,m) = C_{Rzm}\ J_m(\beta' r) \qquad K_{Rm}/2 \equiv +j\ (\beta_d/\beta')\ C_{Rzm}$$

$$E_{Lz}(r,m) = C_{Lzm}\ J_m(\beta' r) \qquad K_{Lm}/2 \equiv -2j\ (\beta_d/\beta')\ C_{Lzm}$$

so that

$$E_{Rz}(r,m) = -\,j\,(\beta'/\beta_d)\,\frac{K_{Rm}}{2}\,J_m(x)$$

$$E_{Lz}(r,m) = +\,j\,(\beta'/\beta_d)\,\frac{K_{Lm}}{2}\,J_m(x)$$

$$E_{rR}(r,m) = a_{Rm}\,x^{-1}\,J_m(x) + \frac{K_{Rm}}{2}\,J_{m+1}(x)\ . \qquad \text{(D.2.15)}$$

$$E_{rL}(r,m) = a_{Lm}\,x^{-1}\,J_m(x) + \frac{K_{Lm}}{2}\,J_{m+1}(x)\ . \qquad \text{(D.2.15)}$$

$$jmE_{R\theta}(r,m) = -\,a_{Rm}\,J_m'(x) + \frac{K_{Rm}}{2}\,[\,-J_{m+1}(x) - x\,J_{m+1}'(x) + x\,J_m(x)\,]$$

$$jmE_{L\theta}(r,m) = -\,a_{Lm}\,J_m'(x) + \frac{K_{Lm}}{2}\,[\,-J_{m+1}(x) - x\,J_{m+1}'(x) - x\,J_m(x)\,] \qquad \text{// note - sign}$$

Now of course there are four unknown constants instead of 2. If I were doing App D1, I would then check that these solutions really solve the four equations claimed!

The cpbc ?

What is it?

$$E_r(a-\varepsilon,\theta) = (j\omega/\sigma)\,n(\theta)\ . \qquad \text{(D.2.24)}$$

But I think I have to state this as

$$E_{Rr}(a-\varepsilon,\theta) = (j\omega/\sigma)\,n_R(\theta)\ . \qquad \text{(D.2.24)}$$
$$E_{Lr}(a-\varepsilon,\theta) = (j\omega/\sigma)\,n_L(\theta)\ . \qquad \text{(D.2.24)}$$

where now each wave is associated with its own charge density, and it must then be that

$$n(\theta,z) = e^{j(\omega t-\beta_d z)}\,n_R(\theta) + e^{j(\omega t+\beta_d z)}\,n_L(\theta)$$

I think n_R and n_L have the same shape, but different amplitudes.

I have pushed far enough into this Appendix D1 wilderness. Basically you cannot just superpose two Appendix D solutions, because more info is needed, such as V(z) which has no place in the discussion of a single round wire in an Appendix D. I gave it a valiant try, don't think it is meaningful. But the reflection scenario I think is reasonable.

Resume this development on 6.9.14.

I basically have two sets of four equations each. The sets are independent, but they will be linked together by the boundary conditions and by the reflection conditions I assume. The two boundary conditions ought to be these:

$$E_{Rr}(a-\varepsilon,\theta,z) + E_{Lr}(a-\varepsilon,\theta,z) = (j\omega/\sigma)\,[n_R(\theta,z) + n_L(\theta,z)] \qquad \text{(D.2.24)}$$

$$E_{R\theta}(a,\theta,z) + E_{L\theta}(a,\theta,z) = 0$$

Suppose I imagine a right and left directed n_i component. Then we really have two separate conditions.

$$E_{Rr}(a-\varepsilon,\theta) = (j\omega/\sigma)\, n_R(\theta)$$
$$E_{Lr}(a-\varepsilon,\theta) = (j\omega/\sigma)\, n_L(\theta)$$

If we meet these two separate conditions, they we satisfy the first equation above. I think the same would have to apply to the E_θ condition as well, so

$$E_{R\theta}(a,\theta,z) = 0$$
$$E_{L\theta}(a,\theta,z) = 0$$

But then each directional wave has the overall general solution I already know:

$$E_{Rz}(r,m) = (1/4)\,\eta_{Rm}\, I_R\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Rr}(r,m) = (j/4)\,\eta_{Rm}\, I_R\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{R\theta}(r,m) = (1/4)\,\eta_{Rm}\, I_R\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_{Lz}(r,m) = (1/4)\,\eta_{Lm}\, I_L\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Lr}(r,m) = -(j/4)\,\eta_{Lm}\, I_L\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{L\theta}(r,m) = -(1/4)\,\eta_{Lm}\, I_L\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Back up. Here is a wave going to the right:

$$E_{Rz}(r,m) = (1/4)\,\eta_{Rm}\, I_R\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Rr}(r,m) = (j/4)\,\eta_{Rm}\, I_R\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{R\theta}(r,m) = (1/4)\,\eta_{Rm}\, I_R\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where

n_{Rm} are moments of $n_R(\theta)$, and I_R is the total current in the wire, integral of J_{Rz} or E_{rz}.

What happens if I do $I_R \rightarrow -I_R$? Well, E_z changes sign. We still have $\exp(-j\beta_d z)$ so I think β_d has to also change sign to represent current going the opposite direction. . Then E_{Rr} gets a double sign change and so stays the same, and same for $E_{R\theta}$.

Resume this development on 6.10.14.

I think for either of the two waves, the η_m are the same and are determined by the capacitance of the line and the electrostatic n(θ) which must be independent of direction. Even if the waves have different amplitude in the two directions, η_m is normalized to the current. So we then have this slightly simpler form

$$E_{Rz}(r,m) = (1/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Rr}(r,m) = (j/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{R\theta}(r,m) = (1/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_{Lz}(r,m) = (1/4)\,\eta_m\, I_L\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Lr}(r,m) = -(j/4)\,\eta_m\, I_L\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{L\theta}(r,m) = -(1/4)\,\eta_m\, I_L\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Notice the two minus signs in the last two lines, since β_d is negated for the L wave.

Example: Suppose the reflected traveling wave has $I_L = -I_R$? Then:

$$E_{Rz}(r,m) = (1/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Rr}(r,m) = (j/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{R\theta}(r,m) = (1/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_{Lz}(r,m) = -(1/4)\eta_m\, I_R\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Lr}(r,m) = +(j/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{L\theta}(r,m) = + (1/4)\,\eta_m\, I_R\, R_{dc}\ (a\beta_d)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Fact: The exponential $e^{j(\omega t-\beta_d z)}$ or $e^{j(\omega t+\beta_d z)}$ which you tack onto $E_i(r,\theta)$ is the same as what you should tack onto $E_i(r,m)$. Therefore in this example we have

$$E_z(r,m,z,t) = (1/4)\,\eta_m\, I_R\, R_{dc}\ (a\beta')\, f_m\ \left[\ e^{j(\omega t-\beta_d z)} - e^{j(\omega t+\beta_d z)}\ \right]$$

$$= (1/4)\,\eta_m\, I_R\, R_{dc}\ (a\beta')\, f_m\, e^{j\omega t}\ [\ 2j \sin(-\beta_d z)]$$

and there (finally!) is our standing wave situation with full nodes and maxima. If we picked some other ratio of I_R to I_L, we would get some kind of SWR situation where the maxima are not 100%.

```
A := 1;
```

$$A := 1$$

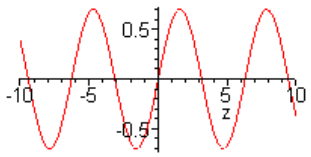
```
B := 0.3;
```

$$B := .3$$

```
f := A*exp(I*z) + B*exp(-I*z);
```

$$f := e^{(Iz)} + .3\,e^{(-Iz)}$$

```
plot(Im(f),z);
```



Fact: No matter *what* ratio you take for I_R and I_L, you are going to get $E_z(r,m,z,t)$ NOT vanishing for m = 1,2,3... because neither η_m nor f_m vanish. Thus, you will never get uniform J_z. So I don't think that the notion of "reflection" resolves this uniform J_z issue. Recall that for low ω we have

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$$

$$= (1/2)\,\eta_m\, I\, R_{dc}\ \ (r/a)^m\,(m+1) \qquad \text{for } m \neq 0$$

and thus in the above example

$$E_z(r,m,z,t) = (1/2)\,\eta_m\, I\, R_{dc}\ \ (r/a)^m\,(m+1)\ [\ 2j \sin(-\beta_d z)] \qquad m \neq 0$$

Question: Is this two-wave superposition idea non-viable due to the BC's?

Since Max's are linear, we should be able to superpose solutions to get new solutions. But I must admit the BC situation is very fuzzy to me. Go back to

$$n(\theta,z) = e^{j(\omega t-\beta_d z)}\ n_R(\theta) + e^{j(\omega t+\beta_d z)}\ n_L(\theta)$$

Perhaps since this is something like

$$n(\theta,z) = n(\theta)\,[\; Ae^{j(\omega t-\beta_d z)} + Be^{j(\omega t+\beta_d z)}\;]$$

Perhaps even

$$n(\theta,z) = n(\theta)\,[\,2j\,\sin(-\beta_d z)]$$

and there is your standing wave n(θ) deal. I wrote above the BC this way (cpbc)

$$E_{Rr}(a-\varepsilon,\theta,z) + E_{Lr}(a-\varepsilon,\theta,z) = (j\omega/\sigma)\,[n_R(\theta,z) + n_L(\theta,z)] \qquad \text{(D.2.24)}$$

so maybe this just says, after you cancel the $[\,2j\,\sin(-\beta_d z)]$ factor on both sides,

$$E_{Rr}(a-\varepsilon,\theta) + E_{Lr}(a-\varepsilon,\theta) = (j\omega/\sigma)\,[An(\theta) + Bn(\theta)] \qquad \text{(D.2.24)}$$

where A and B are some constants.

$$\beta^2 \approx -j\omega\mu\sigma \qquad \beta = e^{j3\pi/4}\,(\sqrt{2}/\delta) \qquad \delta \equiv \sqrt{2/\omega\mu\sigma}$$

New Starting Point:

Resume this development on 6.19.14.

Had a new idea this AM for resolving the DC limit mystery for the shorted transmission line of length L.

Idea of superposing AC and DC solutions. But I cannot find a way top justify doing this!

Idea: Perhaps the solution should be a <u>superposition of these two solutions</u>:

(a) the reflected wave situation as described above (asym J_z)
(b) a DC solution determined by R and G (sym J_z)

Each solution seems to have its own world and is computable. As $\omega\to 0$, the solution (a) gradually goes away to nothing and you are then left with just solution (b). Then you can see the total J_z gradually go from asym to uniform! [that is the motivation for doing this]

With this as motivation, let's reconsider the reflection solution allowing as its J_z will always be asym. To this end, let's start with this assumption for the two waves of the reflection Let's also assume that the right-going wave has factor $e^{j(\omega t-\beta_d[z-L])}$. This just adds a factor $e^{+j\beta_d L}$ to the initial App D ansatz. It does not change ∂_z or ∂_t so I don't think it changes the solutions above AT ALL. Only when you assemble the full solution physical will you get this factor. Assume that for the moment. Then if we add the two waves shown above E_z we get

$$E_z(r,m,z,t) = (1/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta')\, f_m \{ e^{j(\omega t-\beta_d[z-L])} - e^{j(\omega t+\beta_d[z-L])} \}$$

$$= (1/4)\,\eta_m\, I_R\, R_{dc}\, (a\beta')\, f_m\, e^{j\omega t} \{ 2j \sin(-\beta_d[z-L]) \} \qquad \text{// wrong choice}$$

and this is our standing wave solution which is the AC part of the problem. It vanishes at z = L. I am not sure this is the component we want to vanish however. If the termination is a "gold block", then in that black the transverse fields will be 0 for example, and this will then force the transverse fields just to the left of the block to be 0. So maybe start over with this:

$$E_{Rz}(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Rr}(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{R\theta}(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_{Lz}(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_{Lr}(r,m) = -(j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \; g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_{L\theta}(r,m) = -(1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m\, h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where the minus signs in the last two equations arise because β_d changes sign for the left going wave, but we are writing β_d as if it were positive everywhere. THEN suppose we add the r and θ components which are the transverse components to get

$$E_r(r,m,z,t) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \{ e^{j(\omega t-\beta_d[z-L])} - e^{j(\omega t+\beta_d[z-L])} \}$$

$$= (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \{ 2j \sin(-\beta_d[z-L]) \}\, e^{j(\omega t)}$$

So get rid of the time to define

$$E_r(r,m,z) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \{ 2j \sin(-\beta_d[z-L]) \}$$

So then we have

$$E_z(r,m,z) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \{ 2 \cos(-\beta_d[z-L]) \}$$
$$E_r(r,m,z) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \{ 2j \sin(-\beta_d[z-L]) \}$$
$$E_\theta(r,m,z) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \{ 2j \sin(-\beta_d[z-L]) \} \qquad \text{// this is the AC solution!}$$

E_z is a max at the gold block surface and this can be associated with surface charges on that surface which affect E_z only. The transverse fields vanish at z = L as desired. So this is then the "reflection AC solution" for our shorted pair of cylindrical conductors! We also have

$I = V/Z_0$

and if G = 0, we get I→0 as ω→0 as we well know, since $Z_0 \rightarrow \infty$. One more line:

$$J_z(r,m,z) = \sigma\,(1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \{ 2\cos(-\beta_d[z-L]) \} \equiv J_{zAC}$$

Now what is the DC solution? Pretty simple

$I_{DC} = V/R_{DC}$ $\qquad$ $R_{DC} = R_{dc}*L*2$

since two conductors, each length L, and R_{dc} is resistance per unit length. So

$I_{DC} = V/(2LR_{dc})$ $\qquad$ $R_{dc} = 1/(\sigma\pi a^2)$ $\qquad$ $J_{zDC} = I_{DC} / (\pi a^2)$

$= (V/2L)(\sigma\pi a^2)$

Now

$J_{zDC} = I_{DC} / (\pi a^2)$

so we then have

$J_{zDC} = (V/2L)(\sigma\pi a^2) / (\pi a^2) = \sigma\,(V/2L)$ $\qquad$ $E_{zDC} = (V/2L)$

Of course $E_{zDC} = (V/2)/L$ since the voltage drop in each conductor is V/2. Then we are just getting

$J_{zDC} = \sigma\, E_{zDC}$

How do I express this DC solution in partial waves so I can add apples to apples?

$$J_{zDC}(r,m,z) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, [\sigma\, E_{zDC}]\, e^{-jm\theta} = [\sigma\, E_{zDC}]\ \delta_{m,0} = (\sigma V/2L)\, \delta_{m,0}$$

where it happens there is

So now I can add these two solutions to get

$J_{ztot}(r,m,z) = J_{zAC} + J_{zDC}$

$$= \sigma\,(1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \{ 2\cos(-\beta_d[z-L]) \} + \sigma\,(V/2L)\,\delta_{m,0}$$

$$= \sigma\,[\ (1/4)\,\eta_m\,(V/Z_0)\, R_{dc}\ (a\beta')\, f_m \{ 2\cos(-\beta_d[z-L]) \} + (V/2L)\,\delta_{m,0}\ ]$$

$$= \sigma\, V\,[\ (1/4)\,\eta_m\,(1/Z_0)\, R_{dc}\ (a\beta')\, f_m \{ 2\cos(-\beta_d[z-L]) \} + (1/2L)\,\delta_{m,0}\ ]$$

$$= \sigma V [\ (1/4) \eta_m (R_{dc}/Z_0) \ (a\beta') f_m \{ 2 \cos(-\beta_d[z-L]) \} + (1/2L) \delta_{m,0} \ ]$$

In the above, I make the replacement $I = V/Z_0$ without justification. This was valid for the single wave.

Note that $\dim(R_{dc}/Z_0) = m^{-1}$ since R_{dc} is per length and Z_0 is not.

$$\dim(\sigma V) = \text{ohm}^{-1} \, m^{-1} \text{ volts} = \text{volt/ohm} \ m^{-1} = \text{amp/m}$$

So here is my proposed final solution to the problem of the shorted two cylinders:

$$J_z(r,m,z) = \sigma V [\ (1/4) \eta_m (R_{dc}/Z_0) \ (a\beta') f_m \{ 2 \cos(-\beta_d[z-L]) \} + (1/2L) \delta_{m,0} \ ]$$

$$= \sigma V [\ (1/4) \eta_m \ (1/Z_0) \ 1/(\sigma\pi a^2) \ (a\beta') f_m \{ 2 \cos(-\beta_d[z-L]) \} + (1/2L) \delta_{m,0} \ ]$$

$$= V [\ (1/4) \eta_m \ (1/Z_0) \ 1/(\pi a^2) \ (a\beta') f_m \{ 2 \cos(-\beta_d[z-L]) \} + (\sigma/2L) \delta_{m,0} \ ]$$

$$= V [\ (1/4) \eta_m \ (1/Z_0) \ 1/(\pi a^2) \ (a\beta') f_m \{ 2 \cos(-\beta_d[z-L]) \} + (\sigma/2L) \delta_{m,0} \ ]$$

Now

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad 1/Z_0 = \sqrt{\frac{G + j\omega C}{R + j\omega L}} = \sqrt{\frac{j\omega C}{R + j\omega L}} \quad \text{assuming } G = 0.$$

Then

$$J_z(r,m,z) = V [\ (1/4) \eta_m \sqrt{\frac{j\omega C}{R + j\omega L}} \ 1/(\pi a^2) \ (a\beta') f_m \{ 2 \cos(-\beta_d[z-L]) \} + (\sigma/2L) \delta_{m,0} \ ]$$

Now what about β' ?? First,

$$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu (\ \varepsilon - j\sigma/\omega) = \omega^2\mu \, \xi \qquad \xi \equiv \varepsilon - j\sigma/\omega \ . \qquad (1.5.1)$$

So write

$$\beta^2 = \omega\mu(\varepsilon\omega - j\sigma) \approx -j\sigma\omega\mu \quad \text{for reasonable high } \omega \text{ and also as } \omega \rightarrow 0.$$

Meanwhile,

$$\beta_d^2 = [(\omega - j\omega_c)/v_d]^2 \qquad \text{at general } \omega$$

$$= [(0 - j\omega_c)/v_d]^2 = -(\omega_c/v_d)^2 \qquad \text{at low } \omega$$

Then we get

$$\beta'^2 = \beta^2 - \beta_d^2 = -j\sigma\omega\mu - [(\omega - j\omega_c)/v_d]^2 \qquad \text{at general } \omega$$

$$= -j\sigma\omega\mu + (\omega_c/v_d)^2 \qquad \text{at low } \omega$$

$$= (\omega_c/v_d)^2 \qquad \text{as } \omega \to 0$$

Now as I claimed in App D, β' is going to be small, so we can use

$$f_m = (x/x_a)^m (m+1)(2/x_a) \qquad m > 0$$
$$f_0 = 4/x_a \qquad m = 0$$

But recall that these cancel, so I don't need all the β stuff above. For low ω we have then

$$J_z(r,m,z)$$

$$= V\left[(1/4)\eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\, x_a (x/x_a)^m (m+1)(2/x_a)\{ 2\cos(-\beta_d[z-L]) \} \right] m>0$$

$$= V\left[(1/4)\eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\, x_a\, 4/x_a \{ 2\cos(-\beta_d[z-L]) \} + (\sigma/2L) \right] \qquad m=0$$

$$= V\left[(1/4)\eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\ (x/x_a)^m (m+1)(2)\{ 2\cos(-\beta_d[z-L]) \} \right] \qquad m>0$$

$$= V\left[(1/4)\eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\ 4\ \{ 2\cos(-\beta_d[z-L]) \} + (\sigma/2L) \right] \qquad m=0$$

$$= V\left[\eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\ (x/x_a)^m (m+1)\{\cos(\beta_d[z-L]) \} \right] \qquad m>0$$

$$= V\left[\eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\ \{ 2\cos(\beta_d[z-L]) \} + (\sigma/2L) \right] \qquad m=0$$

$$= V\left[\delta_{m,0}(\sigma/2L) + \eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\ \cos(\beta_d[z-L]) \begin{cases} (r/a)^m (m+1) & m>0 \\ 2 & m=0 \end{cases} \right]$$

$$= V\left[\delta_{m,0}(\sigma/2L) + \eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2)\ \cos(\beta_d[z-L]) \begin{cases} (r/a)^m (m+1) & m>0 \\ 2 & m=0 \end{cases} \right]$$

Now I still have β_d sitting in there. If I ignore the loss model, then $\cos(\beta_d[z-L]) \approx 1$ and then

$$J_z(r,m,z) = V\left[(\sigma/2L)\,\delta_{m,0} + \eta_m \sqrt{\frac{j\omega C}{R + j\omega L}}\ 1/(\pi a^2) \begin{cases} (r/a)^m (m+1) & m>0 \\ 2 & m=0 \end{cases} \right]$$

and then at low ω we further get

$$J_z(r,m,z) = V\left[\ (\sigma/2L)\,\delta_{m,0} + \eta_m\sqrt{\frac{j\omega C}{R}}\ 1/(\pi a^2)\begin{cases}(r/a)^m\,(m+1) & m>0\\ 2 & m=0\end{cases}\ \right]$$ DC + AC super

$$= V\left[\ (\sigma/2L)\,\delta_{m,0} + q_m\ \right] \qquad \text{// assume } q_{-m} = q_m$$

Now (finally after 4 months!) you see that $J_z \to$ constant as $\omega \to 0$. In the θ domain I guess this says

$$J_z(r,\theta,z) = \Sigma_{m=-\infty}^{\infty}\ V\left[\ (\sigma/2L)\,\delta_{m,0} + q_m\ \right] e^{jm\theta}$$

$$= V\left[\ (\sigma/2L) + q_0\ \right] + 2\,\Sigma_{m=1}^{\infty}\,V\,q_m\cos(m\theta)$$

$$= V\left[(\sigma/2L) + \sqrt{\frac{j\omega C}{R}}\ 1/(\pi a^2)\ 2\ \right] + 2\,V\sqrt{\frac{j\omega C}{R}}\ 1/(\pi a^2)\ \Sigma_{m=1}^{\infty}\cos(m\theta)\,\eta_m\,(r/a)^m\,(m+1)$$

So now I have to justify why I think I am allowed to just brute-force add in this DC solution to the AC solution. The entire section above is very poor.

What is the meaning of symbol I ?
What is the total current in the wire with reflection running?

Here I consider only the AC solution.

From above, for the reflecting wire we have

$$E_z(r,m,z) = (1/2)\,\eta_m\,I\,R_{dc}\ (a\beta')\,f_m\cos(\beta_d[z-L])$$

$$J_z(r,m,z) = \sigma\,(1/2)\,\eta_m\,I\,R_{dc}\ (a\beta')\,f_m\ \cos(\beta_d[z-L])$$

The total current in a wire ought to be the integral of $J_z(r,\theta,z)$. So let's first compute $J_z(r,\theta,z)$:

$$J_z(r,\theta,z) = \Sigma_{m=-\infty}^{\infty}\ J_z(r,m,z)e^{jm\theta} = J_z(r,0,z) + 2\Sigma_{m=1}^{\infty}\,J_z(r,m,z)\cos(m\theta)$$

$$= \sigma\,(1/2)\,\eta_0\,I\,R_{dc}\ (a\beta')\,f_0\ \cos(\beta_d[z-L])$$

$$+ 2\,\sigma\,(1/2)\,I\,R_{dc}\ (a\beta')\ \cos(\beta_d[z-L])\,\Sigma_{m=1}^{\infty}\,\eta_m\,f_m\cos(m\theta)$$

Now if I integrate this over θ, all the $m\neq 0$ terms vanish, so in fact

$$\int d\theta\ J_z(r,\theta,z) = 2\pi\,\sigma\,(1/2)\,\eta_0\,I\,R_{dc}\ (a\beta')\,f_0\ \cos(\beta_d[z-L])$$

Finally then, since dA = rdrdθ, we have ($\eta_0 = 1$)

$$I_{tot}(z) = 2\pi\,\sigma\,(1/2)\,I\,R_{dc}\,(a\beta')\cos(\beta_d[z-L])\int_0^a dr\,r\,f_0(r)$$

I already did this integral somewhere, but I will do it again,

$$\int_0^a dr\,r\,f_0(r) = \int_0^a dr\,r\,\{2\frac{J_0(x)}{J_1(x_a)}$$

$$= [\,2/\,J_1(xa)\,]\int_0^a dr\,r\,J_0(\beta' r) \qquad x = \beta' r \qquad dx = \beta' dr$$

$$= [\,2/\,J_1(xa)\,]\int_0^{\beta' a}[dx/\beta'][x/\beta']\,J_0(x) = [\,2/\,J_1(xa)\,](1/\beta')^2\int_0^{x_a} dx\,x\,J_0(x)$$

But quoting from GR7 on this,

2. $$\int x\,Z_0(x)\,dx = x\,Z_1(x)$$

we have

$$\int_0^{x_a} dx\,x\,J_0(x) = x\,J_1(x)\Big|_0^{x_a} = x_a J_1(x_a) - 0\,J_1(0) = x_a J_1(x_a)$$

Therefore

$$\int_0^a dr\,r\,f_0(r) = [\,2/\,J_1(xa)\,](1/\beta')^2 * x_a J_1(x_a) = 2x_a/\beta'^2 = 2(\beta' a)/\beta'^2 = 2\,a/\beta'$$

Then the total current is this:

$$I_{tot}(z) = 2\pi\,\sigma\,(1/2)\,I\,R_{dc}\,(a\beta')\cos(\beta_d[z-L])\int_0^a dr\,r\,f_0(r)$$

$$= 2\pi\,\sigma\,(1/2)\,I\,R_{dc}\,(a\beta')\cos(\beta_d[z-L])\,2\,a/\beta'$$

$$= 2\pi\,\sigma\,I\,R_{dc}\,a^2\cos(\beta_d[z-L])$$

$$= 2\pi\,\sigma\,I\,(1/\pi a^2\sigma)\,a^2\cos(\beta_d[z-L])$$

$$= 2\,I\,\cos(\beta_d[z-L]) \qquad \text{// I think this is correct}$$

which even looks reasonable, it is a standing wave. Then

$I_{tot}(0) = 2\ I\ \cos(\beta_d L)$

This is the total current in the wire at the left end where we are presumably applying $V\ e^{j\omega t}$.

What happens as $\omega \rightarrow 0$? Assuming $\beta_d = \omega / v_d$ (without the loss model), we just get

$I_{tot}(z; \omega) = 2\ I\ \cos(\beta_d [z-L])$

$I_{tot}(z; \omega=0) = 2\ I$

So! In this gold block reflection scenario, there IS current all along the wire and it is 2I when $\omega = 0$. But at DC, I know that the current is going to be

$I_{tot}(z; \omega=0) = V(0)/R_{DC}$ with $R_{DC} = R_{dc} * L * 2$

so

$I_{tot}(z; \omega=0) = V(0)/[2LR_{dc}]$

But since this is 2I, I now have a meaning for the symbol I :

$I = (1/4)\ V(0)/(LR_{dc})$ // OK if you require I_{tot} = DC solution at $\omega = 0$ (**)

and then I know that

$I_{tot}(z; \omega) = 2\ I\ \cos(\beta_d [z-L]) = (1/2)\ V(0)/(LR_{dc})\ \cos(\beta_d [z-L])$

So let's go back now and insert this value for I in previous reflection equations:

$E_z(r,m,z) = (1/2)\ \eta_m\ [(1/4)\ V/(LR_{dc})]\ R_{dc}\ (a\beta')\ f_m \cos(\beta_d [z-L])$

$J_z(r,m,z) = \sigma\ (1/2)\ \eta_m [(1/4)\ V/(LR_{dc})]\ R_{dc}\ (a\beta')\ f_m\ \cos(\beta_d [z-L])$

or

$E_z(r,m,z) = (1/8)\ \eta_m\ (V/L)\ (a\beta')\ f_m \cos(\beta_d [z-L])$

$J_z(r,m,z) = \sigma\ (1/8)\ \eta_m\ (V/L)\ (a\beta')\ f_m(r)\ \cos(\beta_d [z-L])$

$J_z(r,\theta,z) = \sigma\ (1/8)\ \eta_0\ (V/L)\ (a\beta')\ f_0\ \cos(\beta_d [z-L])$

$+ 2\ \sigma\ (1/8)\ (V/L)\ (a\beta')\ \cos(\beta_d [z-L])\ \Sigma_{m=1}^{\infty}\ \eta_m\ f_m \cos(m\theta)$

$= \sigma\ (1/8)\ (V/L) \cos(\beta_d [z-L])\ (a\beta')\ [\ f_0(r) + 2\ \Sigma_{m=1}^{\infty}\ \eta_m\ f_m(r) \cos(m\theta)]$

So here we have an asymmetric J_z with a finite magnitude (no Z_0 anymore) and it is clearly asymmetric. If I now put in the low ω values for the f_m I get

$$J_z(r,\theta,z) = \sigma\,(1/8)\,(V/L)\cos(\beta_d[z-L])\,(a\beta')\,[\,f_0(r) + 2\,\Sigma_{m=1}^{\infty}\,\eta_m\,f_m(r)\cos(m\theta)]$$

$$= \sigma\,(1/8)\,(V/L)\cos(\beta_d[z-L])\,x_a\,[4/x_a + 2\,\Sigma_{m=1}^{\infty}\,\eta_m(x/x_a)^m\,(m+1)\,(2/x_a)\cos(m\theta)]$$

$$= \sigma\,(1/8)\,(V/L)\cos(\beta_d[z-L])\;\;[4 + 4\,\Sigma_{m=1}^{\infty}\,\eta_m(x/x_a)^m\,(m+1)\cos(m\theta)]$$

$$= \sigma\,(1/2)\,(V/L)\cos(\beta_d[z-L])\;\;[1 + \;\Sigma_{m=1}^{\infty}\,\eta_m(r/a)^m\,(m+1)\cos(m\theta)]$$

and there staring you in the face is the asym J_z in the reflection scenario! And there is no Z_0 to give a rescue any more!

In the last lines above, I have replaced symbol I by (**) . I then have my usual asym J_z Sisyphus paradox and at the same time, I no longer have Z_0 so this paradox exists at ω = 0 while current remains finite!

Try to compute the series that appears above. // just an aside.

Recall that for the twin lead we have,

$$\eta_m \equiv N_m/N_0 = (-1)^m\,e^{-|m\xi_1|}\,. \qquad \textit{Bipolar (A.12)} \qquad (6.5.4)$$

so we have then this strange function

$$g(\theta) = 1 + \;\Sigma_{m=1}^{\infty}\,(-1)^m\,e^{-|m\xi_1|}\,(r/a)^m\,(m+1)\cos(m\theta) \;=\; 1 + \;\Sigma_{m=1}^{\infty}\,e^{jmx}\,(m+1)\cos(m\theta)$$

Lets try first this sum

$$S1 = \Sigma_{m=1}^{\infty}\,(-1)^m\,e^{-|m\xi_1|}\,(r/a)^m\cos(m\theta) \;= \Sigma_{m=1}^{\infty}\,(-1)^m\,e^{-m|\xi_1|}\,e^{m\,\ln(r/a)}\cos(m\theta)$$

$$= \Sigma_{m=1}^{\infty}\,e^{jm\pi}\,e^{-m|\xi_1|}\,e^{m\,\ln(r/a)}\cos(m\theta)$$

$$= \Sigma_{m=1}^{\infty}\,e^{jm[\pi-|\xi1|+\ln(r/a)]}\cos(m\theta)$$

$$= \Sigma_{m=1}^{\infty}\,e^{jmx}\cos(m\theta) \qquad\qquad x = \pi-|\xi1|+\ln(r/a)$$

$$= \Sigma_{m=1}^{\infty}\,e^{jmx}\,(1/2)[\,e^{jm\theta} + e^{-jm\theta}\,]$$

$$= \;(1/2)[\;\Sigma_{m=1}^{\infty}\,e^{jm(x+\theta)} \;+ \Sigma_{m=1}^{\infty}\,e^{jm(x-\theta)}\;]$$

Let $y = e^{j(x+\theta)}$ and $z = e^{j(x-\theta)}$ so then we have

$= (1/2)[\ \Sigma_{m=1}^{\infty} y^m + \Sigma_{m=1}^{\infty} z^m\ ]$

$= (1/2)[\ y/(1-y) + z/(1-z)\ ] \equiv s1 = S1$

Now the other sum is

$s2 = \Sigma_{m=1}^{\infty} e^{jmx}\, m \cos(m\theta)$

$= \Sigma_{m=1}^{\infty} e^{jmx}\, d/d\theta\ [\sin(m\theta)]$

$= d/d\theta\ [\Sigma_{m=1}^{\infty} e^{jmx} \sin(m\theta)]$

$= d/d\theta\ [\Sigma_{m=1}^{\infty} e^{jmx}\, (1/2j)[\ e^{jm\theta} - e^{-jm\theta}\ ]]$

$= (1/2j)\ d/d\theta\ [\Sigma_{m=1}^{\infty} e^{jm(x+\theta)} - \Sigma_{m=1}^{\infty} e^{jm(x-\theta)}]$

$= (1/2j)\ d/d\theta\ [\Sigma_{m=1}^{\infty} y^m - \Sigma_{m=1}^{\infty} z^m]$

$= (1/2j)\ d/d\theta\ [y/(1-y) - z/(1-z)]$

$= (1/j)\ d/d\theta\ \{\ (1/2)\ [y/(1-y) - z/(1-z)]\ \} = (1/j)\ d/d\theta\ s_2$

Maple gets an answer, but it ain't purdy, lots of terms.

Does the superposed wave solution meet the charge pumping boundary condition?

I am now looking for reasons to disqualify the reflection solution, since I see it is giving asym J_z

I think the charge density looks like this:

$n(\theta,z,t) = n(\theta)\ [\ e^{j(\omega t - \beta_d[z-L])} - e^{j(\omega t + \beta_d[z-L])}\ ]$

$= n(\theta)\ 2j \sin(-\beta_d(z-L))e^{j\omega t}$

I choose the minus sign because n is associated with J_r and E_r have this minus sign.
Also, I like n = 0 at the shorting point where the cap is shorted out. So

$n(\theta,z) = n(\theta)\ 2j \sin(-\beta_d(z-L))$

which I think says

$$N_m(z) = N_m\, 2j\, \sin(-\beta_d(z-L))$$

Meanwhile from above we had

$$E_r(r,m,z) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \{\, 2j \sin(-\beta_d[z-L]) \,\}$$

Now $g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)} \right]$ and so $g_m(r=a) = 2$. Thus, I get

$$E_r(a,m,z) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, 2\{\, 2j \sin(-\beta_d[z-L]) \,\}$$

Now the CPBC is *supposed to say* (using a Gaussian box!)

$$E_r(r=a,m) = (j\omega/\sigma)\, N_m$$

or I guess at location z,

$$E_r(r=a,m,z) = (j\omega/\sigma)\, N_m(z)$$

This would require, based on the above, that

$$(j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, 2\{\, 2j \sin(-\beta_d[z-L]) \,\} = (j\omega/\sigma)\, N_m\, 2j \sin(-\beta_d(z-L))$$

or

$$(j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, 2 = (j\omega/\sigma)\, N_m$$

or

$$(1/2)\,\eta_m\, I\, R_{dc}\, (a\beta_d) = (\omega/\sigma)\,\eta_m N_0$$

or

$$I\, R_{dc}\, (a\beta_d) = (\omega/\sigma)\, 2N_0$$

If I solve for I, I get

$$I = (\omega/\sigma)\, 2N_0\, 1/R_{dc}\, (1/a\beta_d) = (\omega/\sigma)\, 2N_0\pi a^2\sigma\, (1/a\beta_d)$$

$$= (\omega)\, 2N_0\pi(a/\beta_d) = 2\pi\omega(a/\beta_d)\, N_0$$

which is the same as (D.2.31). Now I found above that, if I require this theory to produce the correct DC limit,

$$I = (1/4)\, V/(LR_{dc})$$

so I would have to be able to show (in order to have the CPBC be respected) that

$$(1/4)\, V/(LR_{dc}) = 2\pi\omega(a/\beta_d)\, N_0$$

or

$$(1/4)\, V/(LR_{dc}) = 2\pi\omega(a/\beta_d)\, N_0 = 2\pi\omega(a/\beta_d)\, (1/2\pi a)\, q(0) = \omega(1/\beta_d)\, q(0) = q(0)\, v_d$$

All would be well if this really determined q(0) to be

$$q(0) = (1/4)\ V/(LR_{dc}v_d)$$

But in fact q(0) is determined by the capacitance, so we really have

$$q(0) = CV$$

so I think this says that the CPBC is NOT respected by my reflection solution!

Conclusion: The Charge Pump Boundary Condition is NOT met by my reflection superposition if I require the model to have the correct DC limit.

In the next section I show that the driving V is related to N_0 by

$$V = (1/2j)\ \sin(\beta_d L)\ N_0$$

and this is definitely not consistent with the strange claim above.

What do we know about V(z) ??

This is all useless with lots of algebra errors.

Let's assume as I did above that

$$n(\theta,z) = n(\theta)\ 2j\ \sin(-\beta_d(z-L))$$

But

$$n(\theta) = \sum_{m=-\infty}^{\infty} N_m\, e^{jm\theta} \qquad \int n(\theta)d\theta = N_0$$

The integral of n(θ) around the surface is going to be (q is per unit length)

$$q(z) = \int n(\theta,z)ad\theta = 2j\ a\ \sin(-\beta_d(z-L)) \int n(\theta)d\theta = 2j\ a\ \sin(-\beta_d(z-L))\ 2\pi\ N_0$$

where this N_0 is not the same as App D since n(θ) is not the same.

So here a statement that q(z) is also a standing wave, based closely on n(θ). If cap is C, we then have

$$V(z) = q(z)/C = 4\pi j\ a\ \sin(-\beta_d(z-L))\ N_0/C$$

So now we have V(z) also doing a standing wave. V(L) = 0 by design. And

$V = V(0) = 4\pi j\, a \sin(-\beta_d(-L))\, N_0/C = 4\pi j\, a \sin(\beta_d L)\, N_0/C$

So this relates the applied V and N_0 ! From above I had (**)

$I_{tot}(z; \omega) = (1/2)\, V(0)/(LR_{dc}) \cos(\beta_d[z-L])$

$I_{tot}(0; \omega) = (1/2)\, V(0)/(LR_{dc}) \cos(\beta_d L)$

This suggests the following <u>impedance looking in at the left end</u>

$1/ Z(0) = I_{tot}(0; \omega)/V(0) = (1/2)\, 1/(LR_{dc}) \cos(\beta_d L)$

or

$Z(0) = 2\, LR_{dc} \sec(\beta_d L)$

In fact,

$1/ Z(z) = I_{tot}(z; \omega)/V(z) = (1/2)\, V(0)/(LR_{dc}) \cos(\beta_d[z-L]) / 2j\, a \sin(-\beta_d(z-L))\, N_0/C$

$= (1/4j)\, V(0)/(LR_{dc}) * C/N_0 * \cos(\beta_d(z-L))/ a \sin(-\beta_d(z-L))$

$= - (1/4j)\, V(0)/(LR_{dc}) * C/N_0 * \cos(\beta_d(z-L))/ a \sin(\beta_d(z-L))$

$= - (1/4j)\, V(0)/(LR_{dc}) * C/N_0 * \cot(\beta_d(z-L))$

$= - (1/4j)\, 2j\, a \sin(\beta_d L)\, N_0/C /(LR_{dc}) * C/N_0 * \cot(\beta_d(z-L))$

$= - (1/2)\, a \sin(\beta_d L)\, 1 /(LR_{dc}) * \cot(\beta_d(z-L))$

$Z(z) = -2LR_{dc} \csc(\beta_d L) \tan(\beta_d(z-L))$

so this thing has poles periodically along the way. As a check, set z = 0 to get

$Z(0) = -2LR_{dc} \csc(\beta_d L) \tan(\beta_d(-L)) = 2LR_{dc} \csc(\beta_d L) \tan(\beta_d L) = 2LR_{dc} \sec(\beta_d L)$

which agrees with above.

Now try to redo this with G present! Recall first from Section D.9,

Second summary of the E field solutions :

$$R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - \beta'_d{}^2 \qquad \text{(D.9.25)}$$

$$I' = V/Z_0 \qquad \xi_d = \varepsilon_d + \sigma_d/j\omega$$

$$\beta_d' = \sqrt{\xi_d/\varepsilon_d}\ \beta_d \quad \text{where} \quad \beta_d = \omega\sqrt{\mu_d\varepsilon_d} \qquad (\ I' = \sqrt{\xi_d/\varepsilon_d}\ I \quad \text{where I was for } \sigma_d = 0\ )$$

$$E'_z(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E'_r(r,m) = (j/4)\,\eta_m\, I'\, R_{dc}\ (a\beta'_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E'_\theta(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta'_d)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Note that β does not change at all (inside conductor). Once I use $I' = V/Z_0$, I' is gone so I can forget about I'. Maybe better to rewrite the above as

Second summary of the E field solutions :

$$R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - \beta'_d{}^2 \qquad \text{(D.9.25)}$$

$$I' = V/Z_0 \qquad \xi_d = \varepsilon_d + \sigma_d/j\omega$$

$$\beta_d' = \sqrt{\xi_d/\varepsilon_d}\ \beta_d \quad \text{where} \quad \beta_d = \omega\sqrt{\mu_d\varepsilon_d} \qquad (\ I' = \sqrt{\xi_d/\varepsilon_d}\ I \quad \text{where I was for } \sigma_d = 0\ )$$

$$E'_z(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E'_r(r,m) = (j/4)\,\eta_m\, I'\, R_{dc}\ \sqrt{\xi_d/\varepsilon_d}\ (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E'_\theta(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ \sqrt{\xi_d/\varepsilon_d}\ (a\beta_d)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

This does alter β' but β' cancels out in my limit. The transverse fields have extra factor $\sqrt{\xi_d/\varepsilon_d}$, but I was only working with the E_z field which really has no difference at all other than that fact that I' got larger! So I think the result for the AC solution is exactly the same except $G \neq 0$.

I think this DC calculation is correct.

What about the DC solution? I still have the shorting bar in place at location L. It is not obvious what the solution is here, so I turn to the Network appendix. // Well this took quite a while. If I set z = 0 at the short and have z increase to the left, I find that

$$R(z) = \sqrt{2R/G}\ \tanh(\sqrt{2RG}\ z)$$

is the resistance looking into the ladder from the left end. If the ladder has length L, then

$$R(L) = \sqrt{2R/G}\ \tanh(\sqrt{2RG}\ L)$$

If $\sqrt{2RG}\ L >> 1$, then the limit is just

$$R(L) = \sqrt{2R/G}$$

Let's check now the limit $G \rightarrow 0$. Then $\tanh x \approx x$ so we get

$$R(L) \approx \sqrt{2R/G}\ (\sqrt{2RG}\ L) = 2RL$$

which agrees with earlier

$$R_{DC} = R_{dc} * L * 2$$

So in the G case we get

$$J_{zDC} = I_{DC} / (\pi a^2) \quad \text{where} \quad I_{DC} = V/R_{DC} \quad \text{where} \quad R_{DC} = \sqrt{2R/G}\ \tanh(\sqrt{2RG}\ L)$$

or

$$J_{zDC} = V/[\ \pi a^2 R_{DC}] = V/[\ \pi a^2 \sqrt{2R/G}\ \tanh(\sqrt{2RG}\ L)]$$