Standing wave version of Appendix D PhL 6.12.14

Reread Aug 7, 2014: here is a summary (red comments added today)

I successfully develop a "standing wave version" of Appendix D in the Plan C section. Both BC's, the divE = 0 and the three Helm equations are all satisfied by this E field solution. Basically the solution is the same as appears in regular Appendix D, except symbol I is no longer the total current in the line. All quantities exhibit reasonable standing waves. The problem is that this new solution, being of the same form as the old solution, has J_z asymmetry, so it does not solve THAT problem. Thus, although I seem to have a successful "reflection solution" with a shorting bar at z = L, I still get J_z asymmetry when G = 0.

The first section Plan A just shows that you cannot assume the same trig z function for all E components. Section Plan B just explains why this does not work and suggests Plan C.

Plan D is wrong. With the new loss model, $\beta' = 0$ when G = 0 and $\omega = 0$, so J_z = constant is in fact a viable solution to the Helm equation, and in fact is just the m = 0 term.

Plan E comes up with an interesting but complicated loss model and I think my $k(\omega)$ model is easier.

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[When this appendix was written, what is now k in Appendix D was called β_d]

Plan A Trying a different App D starting form -- NO GO

If you superpose traveling waves of the same amplitude, you end up with a standing wave as Maple animate well demonstrates. In this case you can maintain V(L) = 0 all the time as you would have a shorted line ending. Basically, to find the E field solutions for this case, you have to redo Appendix D making this replacement right at the very start:

$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t - \beta_d z)}\ \mathbf{E}(r,\theta)$ traveling wave

$\mathbf{E}(r,\theta,z,t) = e^{j\omega t} \sin(\beta_d z)\ \mathbf{E}(r,\theta)$ standing wave

There is a big difference in these two forms. Notice that

$\partial_z^2\ \mathbf{E}(r,\theta,z,t) = -\beta_d^2\ \mathbf{E}(r,\theta,z,t)$ same as before

but

$$\partial_z \mathbf{E}(r,\theta,z,t) = \beta_d \cot(\beta_d z)\, \mathbf{E}(r,\theta,z,t)$$

Proof of this last line:

$$\partial_z \mathbf{E}(r,\theta,z,t) = e^{j\omega t}\, \mathbf{E}(r,\theta)\beta_d \cos(\beta_d z) = e^{j\omega t}\, \mathbf{E}(r,\theta)\beta_d \cos(\beta_d z)/\sin(\beta_d z) * \sin(\beta_d z)$$

$$= \beta_d \cot(\beta_d z)\, \mathbf{E}(r,\theta,z,t)$$

So we then have to replace the rule

$$\partial_z \rightarrow -j\beta_d$$

with the new rule

$$\partial_z \rightarrow \beta_d \cot(\beta_d z)$$

which I am sure is a very significant difference! [agreed]

Comment: I went down this path long ago I think, but don't know right now where it is.

So what then happens in Appendix D?

(1) Since the Helmholtz equations all have ∂_z^2 ONLY, they are unchanged.

(2) The div E = 0 gets the replacement $-j \rightarrow \cot(\beta_d z)$ for the ∂_z term. We then have

$$\partial_r (r E_r) + \partial_\theta E_\theta + r\, \partial_z E_z = 0 .$$

Applying the conversion rules gives

$$\partial_r [r E_r(r,m,z,t)] + jmE_\theta(r,m,z,t) + r\,(\cot(\beta_d z)\beta_d)E_z(r,m,z,t) = 0$$

$$[1 + r\partial_r]\, E_r(r,m,z,t) + jmE_\theta(r,m,z,t) + r\,(\cot(\beta_d z)\beta_d)E_z(r,m,z,t) = 0 \quad . \qquad \text{(D.1.19)}$$

Now install:

$$E_r(r,m,z,t) = e^{j\omega t} \sin(\beta_d z)\, E_r(r,m) \qquad \text{etc}$$

to get

$$\sin(\beta_d z)\partial_r [r E_r(r,m)] + jm \sin(\beta_d z)E_\theta(r,m) + r\,(\cos(\beta_d z)\beta_d)E_z(r,m) = 0$$

But this is a contradiction by saying that that functions are of (r,m) but you see functions of z entering the equation! Again I think this is saying:

"The assumed form

$\mathbf{E}(r,\theta,z,t) = e^{j\omega t} \sin(\beta_d z)\, \mathbf{E}(r,\theta)$
$\mathbf{E}(r,m,z,t) = e^{j\omega t} \sin(\beta_d z)\, \mathbf{E}(r,m)$

cannot solve div **E** = 0 ". [agreed]

Plan B -- Why Plan A is No Go

(a) I think I have an explanation of why the above form fails.

What I have written down in App D as solutions for a +z wave are really these

$E_i(r,m;\beta_d)$

For a -z wave, the solutions are NOT THE SAME. They are in fact

$E_i(r,m; -\beta_d)$

which in general is not the same function. Now when you superpose you get

$e^{j(\omega t-\beta_d z)}\, E_i(r,m;\beta_d) \;+\; e^{j(\omega t+\beta_d z)}\, E_i(r,m; -\beta_d)$

and this is NOT of the simple standing wave form

$E_i(r,m)\, e^{j\omega t}\, 2\cos(\beta_d z)$

and that is because the two functions are DIFFERENT (at least for some components). [agreed]

(b) Suppose you superpose two opposite-directed waves of exactly the same amplitude as I have just done in writing

$e^{j(\omega t-\beta_d z)}\, E_i(r,m;\beta_d) \;+\; e^{j(\omega t+\beta_d z)}\, E_i(r,m; -\beta_d)$

It seems to me that each wave would have the same function n(θ). Then you might get

$n(\theta,z) = e^{j(\omega t-\beta_d z)}\, n(\theta) \;+\; e^{j(\omega t+\beta_d z)}\, n(\theta) \;=\; e^{j\omega t}\, 2\cos(\beta_d z)\, n(\theta)$ [ok]

and then you CAN get a standing wave form for the charge density and thus for the external problem! For example you would then expect to have a standing wave form for V(z) since this is based on n.

Plan C Standing wave version of Appendix D

Let's now try this variation of Plan B. Let's have the reflected -z wave have opposite sign instead of the same sign, and at the same time replace z → z-L where L will be the "reflection point". Then we get

$$E_{i,tot}(r,m,z,t) = e^{j(\omega t-\beta_d[z-L])}\, E_i(r,m;\beta_d) - e^{j(\omega t+\beta_d[z-L])}\, E_i(r,m;-\beta_d)$$

I expect these two waves to have n(θ) and -n(θ), so I expect to get

$$n(\theta,z) = e^{j(\omega t-\beta_d[z-L])}\, n(\theta) + e^{j(\omega t+\beta_d[z-L])}\, [-n(\theta)] = -e^{j\omega t}\, 2j\sin(\beta_d[z-L])\, n(\theta)$$

and THIS is a charge density that I can then force to always be 0 at z = L, as might be correct for a shorting bar at that location! [seems good]

[BUT Fatal Flaw noted at this point: If $\beta_d \to 0$ as $\omega \to 0$, then according to the line above, $n(\theta,z) \to 0$ and thus $q(z) \to 0$. Since q = CV, that implies V(z) = 0 for all z. But I seek a DC solution with some $V(0) \neq 0$ to drive DC current through the shorted line! So Plan C is dead meat right here.]

[Possible rescue: If $\beta_d \to A$ as $\omega \to 0$, where A is small, then we would get $n(\theta,z) = -e^{j\omega t}\, 2jA[z-L]\, n(\theta)$ and in turn $q(z) = -e^{j\omega t}\, 2jA[z-L]\, q_0$ and in turn $V(z) = -e^{j\omega t}\, 2jA[z-L]\, q_0/C$ and this IS in fact a linearly decreasing V(z) which vanishes at z = L, more what I want to see happen. For small $G \neq 0$, this is exactly what happens since then $A = -j\sqrt{RG}$ and then $V(z) = -e^{j\omega t}\, 2\sqrt{RG}\, [z-L]\, q_0/C$. But I am looking for my non-vanishing DC solution V(z) in particular when G = 0.]

Now, how do I "access" the work of Appendix D and make it bear on this standing wave situation? I could first take hold of the "first summary"

$$E_z(r,m;\beta_d) = -j\,(\beta'/\beta_d)\,\frac{K_m(\beta_d)}{2}\, J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m;\beta_d) = a_m(\beta_d)\, x^{-1}\, J_m(x) + \frac{K_m(\beta_d)}{2}\, J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad (D.2.11)$$

$$jE_\theta(r,m;\beta_d) = -a_m(\beta_d)\, x^{-1}\, J_m(x) + \left(\frac{K_m(\beta_d)}{2} + \frac{a_m(\beta_d)}{m}\right) J_{m+1}(x) \qquad (D.2.15)$$

with the idea that K_m and a_m might be functions of β_d and in particular its sign. Then for each component the total field is given by

$$E_{i,tot}(r,m,z,t) = e^{j(\omega t-\beta_d[z-L])}\, E_i(r,m;\beta_d) - e^{j(\omega t+\beta_d[z-L])}\, E_i(r,m;-\beta_d)$$

$$E_{i,tot}(r,\theta,z,t) = e^{j(\omega t-\beta_d[z-L])}\, E_i(r,\theta;\beta_d) - e^{j(\omega t+\beta_d[z-L])}\, E_i(r,\theta;-\beta_d)$$

Next, we must "face up to" the two boundary conditions. This is where I always collapse. How about this idea: do the CPBC Gaussian box at a particular value of z :

$$E_{r,tot}(r=a-\varepsilon,\theta,z,t) = (j\omega/\sigma)\, n(\theta,z) \;=\; -(j\omega/\sigma)\, e^{j\omega t}\, 2j\sin(\beta_d[z-L])\, n(\theta)$$

In order for this to be viable, we must then have this form for the non z,t $E_{r,tot}$:

$$E_{r,tot}(r=a-\varepsilon,\theta,z,t) \;= -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_{r,tot}(r=a-\varepsilon,\theta) \qquad (*)$$

This then in turn says

$$E_{r,tot}(r=a-\varepsilon,\theta) = (j\omega/\sigma)\, n(\theta) \qquad (**)$$

which at least "looks like" my original BC. [so far so good]

Now suppose we assume that

$$E_r(r,m;-\beta_d) \;= +E_r(r,m;\beta_d)$$

for the r component only. Then we get

$$E_{r,tot}(r,\theta,z,t) \;=\; e^{j(\omega t-\beta_d[z-L])}\, E_r(r,\theta;\beta_d) \;-\; e^{j(\omega t+\beta_d[z-L])}\, E_r(r,\theta;-\beta_d)$$

$$= e^{j(\omega t-\beta_d[z-L])}\, E_r(r,\theta;\beta_d) \;-\; e^{j(\omega t+\beta_d[z-L])}\, E_r(r,\theta;\beta_d)$$

$$= [e^{j(\omega t-\beta_d[z-L])} \;-\; e^{j(\omega t+\beta_d[z-L])}]\, E_r(r,\theta;\beta_d)$$

$$= -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_r(r,\theta;\beta_d) \qquad [\text{ good to have } E_r = 0 \text{ at } z = L\text{; gold brick idea }]$$

and this is then consistent with the form I assumed in (*) above if I say

$$E_{r,tot}(r,\theta) \;=\; E_r(r,\theta;\beta_d)$$

Then our CPBC is (**) which looks exactly the same as it did in the single-wave Appendix D, namely:

$$E_r(r=a,\theta;\beta_d) \;= (j\omega/\sigma)\, n(\theta)$$

For the first time EVER, I now at least have some kind of standing wave form for one of the internal E field components of the round wire.

Now, I will make the ansatz that E_θ has this same form, so

$$E_\theta(r,m;-\beta_d) \;= +E_\theta(r,m;\beta_d)$$

$E_{\theta,tot}(r,\theta,z,t) = -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_\theta(r,\theta;\beta_d)$ [same gold brick idea]

$E_{\theta,tot}(r,\theta) = E_\theta(r,\theta;\beta_d)$

In this case, the other BC ALSO looks the same

$E_\theta(r=a,\theta;\beta_d) = 0$

I now make this third ansatz, which is that E_z gets a minus sign, so

$E_z(r,m;-\beta_d) = -E_z(r,m;\beta_d)$

$E_{z,tot}(r,\theta,z,t) = 2\cos(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,\theta;\beta_d)$. [different sign from E_r and E_θ]

Summary of the three $-\beta_d$ equations:

$E_r(r,m;-\beta_d) = +E_r(r,m;\beta_d)$
$E_\theta(r,m;-\beta_d) = +E_\theta(r,m;\beta_d)$
$E_z(r,m;-\beta_d) = -E_z(r,m;\beta_d)$.

Summary of the resulting three total field-component expressions:

$E_{r,tot}(r,\theta,z,t) = -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_r(r,\theta;\beta_d)$
$E_{\theta,tot}(r,\theta,z,t) = -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_\theta(r,\theta;\beta_d)$
$E_{z,tot}(r,\theta,z,t) = 2\cos(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,\theta;\beta_d)$

Reminder: Since $\partial_z^2 = -\beta_d^2$ for each of the new $E_{i,tot}$ components, the three Helm equations are the same for these functions as they were for the old App D functions.

I am hopeful now that the div **E** = 0 equation, which was problematical in Plan A, is now going to be OK. We have the raw cyl coordinates form,

$\partial_r (r\, E_{r,tot}(r,\theta,z,t)) + \partial_\theta E_{\theta,tot}(r,\theta,z,t) + r\, \partial_z E_{z,tot}(r,\theta,z,t) = 0$

which then says

$\partial_r (r\,(-)2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_r(r,\theta;\beta_d))$
$+ \partial_\theta\, 2\,(-)j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_\theta(r,\theta;\beta_d)$
$+ r\, \partial_z 2\cos(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,\theta;\beta_d) = 0$

or

$\partial_r (r\ \ (-)2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_r(r,\theta;\beta_d))$
$+ \partial_\theta\ \ (-)2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_\theta(r,\theta;\beta_d)$
$- r\,\beta_d\ \partial_z 2\sin\,(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,\theta;\beta_d) = 0$

or

$-\partial_r (r\, j)\, e^{j\omega t} E_r(r,\theta;\beta_d))$

$-\partial_\theta\, j\, e^{j\omega t} E_\theta(r,\theta;\beta_d)$

$-\, r\, \beta_d\, e^{j\omega t} E_z(r,\theta;\beta_d) = 0$

or

$-j\, \partial_r(r\, E_r(r,\theta;\beta_d)) - j\, \partial_\theta\, E_\theta(r,\theta;\beta_d) - r\, \beta_d\, E_z(r,\theta;\beta_d) = 0$

or

$\partial_r(r\, E_r(r,\theta;\beta_d)) + \partial_\theta\, E_\theta(r,\theta;\beta_d) - j\, r\, \beta_d\, E_z(r,\theta;\beta_d) = 0$

Now this equation is at least consistent with itself since there are no functions of z appearing (recall in Plan A there was a cot thing). Also, this equation is identical to (D.1.19) !!! [good]

At this point then we have this situation:

The three Helm equations are the same

The div E equation is the same

The two BC's are the same. [seems very promising at this point]

I think this means that our solutions must be the same TO THIS POINT: [agreed]

$$E_z(r,m;\beta_d) = -\, j\, (\beta'/\beta_d)\, \frac{K_m}{2}\, J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m;\beta_d) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad (D.2.11)$$

$$jE_\theta(r,m;\beta_d) = -\, a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad (D.2.15)$$

$$a_m = (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}\,. \qquad a_0 = 0 \qquad (D.2.28)$$

$$\frac{K_m}{2} = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \qquad \frac{K_0}{2} = (j\omega/\sigma)\, N_0 \frac{1}{J_1(x_a)}$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] \qquad \left(\frac{K_0}{2} + \frac{a_0}{0}\right) = 0$$

The next question is this: What happens to the normalization condition relating I and N_0 which appears in Appendix D below (D.2.30) ? It starts off this way:

$$I_{tot} = \int_0^{2\pi} d\theta \int_0^a r\, dr\, J_{z,tot}(r,\theta,z,t) = \int_0^{2\pi} d\theta \int_0^a r\, dr \left\{ \sigma \sum_{m=-\infty}^{\infty} E_{z,tot}(r,m,z,t)\, e^{jm\theta} \right\}$$

$$= \sigma \sum_{m=-\infty}^{\infty} \int_0^a r\, dr\, E_{z,tot}(r,m,z,t) \int_0^{2\pi} d\theta\, e^{jm\theta} = 2\pi\, \sigma \int_0^a r\, dr\, E_{z,tot}(r,0,z,t) \qquad [\text{ ok }]$$

But now we want to use,

$$E_{z,tot}(r,m,z,t) = 2\cos(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,m;\beta_d)$$

and then

$$I_{tot}(z) = 2\pi\sigma \int_0^a r\,dr \;\; 2\cos(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,0;\beta_d)$$

$$= [\, 2\cos(\beta_d[z-L])\, e^{j\omega t}\,]\, 2\pi\sigma \int_0^a r\,dr\, E_z(r,0;\beta_d)$$

Apart from the red factor, things proceed exactly as below (D.2.30) and we get

$$I_{tot}(z,t) = [\, 2\cos(\beta_d[z-L])\, e^{j\omega t}\,]\; 2\pi\omega\, (a/\beta_d)\, N_0$$

We could at this point just define

$$I \equiv 2\pi\omega\, (a/\beta_d)\, N_0 \qquad [\text{ this has the exact same form as above (D.2.31) }]$$

and then

$$I_{tot}(z,t) = [\, 2\cos(\beta_d[z-L])\, e^{j\omega t}\,]\; I \qquad I \equiv 2\pi\omega\, (a/\beta_d)\, N_0$$

So then I is no longer "the magnitude" of the current in the wire, but it is the coefficient you see here of the standing wave of the current.

I think then that this leads to the following solutions:

$$E_z(r,m;\beta_d) = (1/4)\,\eta_m\, I\, R_{dc}\; (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m;\beta_d) = (j/4)\,\eta_m\, I\, R_{dc}\; (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m;\beta_d) = (1/4)\,\eta_m\, I\, R_{dc}\; (a\beta_d)\; h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where I has the interpretation just stated above.

The problem now is that each of these solution fields has the wrong "symmetry" for negation of β_d.

OK, ignoring this detail (which could of course be fatal) I will rewrite the solutions as

$$E_z(r,m;\beta_d) = \text{sign}(\beta_d)\,(1/4)\,\eta_m\, I\, R_{dc}\; (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m;\beta_d) = \text{sign}(\beta_d)\,(j/4)\,\eta_m\,I\,R_{dc}\,(a\beta_d)\,g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m;\beta_d) = \text{sign}(\beta_d)\,(1/4)\,\eta_m\,I\,R_{dc}\,(a\beta_d)\,h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where I have to just assume that β_d is real and ignore the loss business. [so far so good]

So MAYBE I now have a standing wave model for a shorting bar at z = L. The following quantities vanish at z = L: (all the sine quantities)

$n(\theta,z)$ $E_{r,tot}(r,\theta,z,t)$ $E_{\theta,tot}(r,\theta,z,t)$ V (since based on n)

However, the following quantities are a maximum at z = L

$E_{z,tot}(r,\theta,z,t)$ $I_{tot}(z,t)$

This does at least SEEM right for a shorting bar. The transmission line ends in a gold plane which does not allow $n(\theta,z)$ to exist at z = L. The current is always a maximum, as is J_z current density.

So this was my "idea" which motivated the standing wave approach of Plan C.

[it all seems very promising to this point]

So let us then look at the current density $J_{z,tot}$:

$$J_{z,tot}(r,m,z,t) = 2\sigma\cos(\beta_d[z-L])\,e^{j\omega t}\,E_z(r,m;\beta_d)$$

$$= 2\sigma\cos(\beta_d[z-L])\,e^{j\omega t}\,\text{sign}(\beta_d)\,(1/4)\,\eta_m\,I\,R_{dc}\,(a\beta')\,f_m(\beta',r,a)$$

At every point z, we still have our asymmetry problem! The η_m are the same as they have always been, and if we do a low ω limit, we will find moments of J_z for $m \neq 0$ and we will have current asymmetry.

Conclusion: This "shorting bar" model does not seem to "solve" the J_z asymmetry problem.

Nevertheless, I think it was a valiant attempt, perhaps one of my best so far.

[I agree, this was indeed a carefully done and "valiant" attempt, and I agree with the conclusion.]

Plan D (a question) : Does the solution J_z = constant [E_z = constant] satisfy Helmholtz?

This is not so much a "plan" as an "explanation" of what is happening.

1. If you start off with the Helm equation with β^2, things make sense in this regard: If you take $\omega \rightarrow 0$, you find that $\beta \rightarrow 0$ and then Helm is just vector Laplace $\nabla^2 \mathbf{E} = 0$ and then E_z = constant is a viable solution for inside a conductor. E_z = constant also satisfies div $\mathbf{E} = 0$.

2. However, once you make the ANSATZ that field components go like $e^{j(\omega t - \beta_d z)}$, the situation changes. Now the Helm that gets used contains β'^2, for example here from below (D.1.14), [agreed]

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 + \beta'^2\ ]\ e^{j(\omega t - \beta_d z)} E_z(r,\theta) = 0\ . \qquad (1)$$

In order to still allow a DC solution, you would need $\beta' = 0$. If you ignore the loss model, then you do in fact get $\beta \rightarrow 0$ and $\beta_d \rightarrow 0$ which allows $\beta' \rightarrow 0$ and then you could have a DC solution for E_z. But ignoring the loss model means perfect conductors which means $E_z \equiv 0$ and that is not what we want.

[for my "modern" loss model with G = 0, you do in fact get all three $\beta \rightarrow 0$ as $\omega \rightarrow 0$ even when there are losses. See current (D.11.2).]

3. When I use my loss model of Section D.11, I have to assume the transmission line equations, and that in turn means I have to assume $\omega > \omega_0$ for some min ω_0, and that in turn rules out the DC limit.

[but I have disowned that old loss model with its ω_c except perhaps for very small loss.]

4. Thus, once I make the $e^{j(\omega t - \beta_d z)}$ ansatz along with the D.11 loss model, I have barricaded the Appendix D solution from reaching a DC limit! It is an impossible pathway. You can see directly from the equation (1) above that E_z = constant only solves (1) if $\beta' = 0$.

So there is the explanation. No matter what you do, reflections, large Z_0, whatever, the $J_z = 0$ solution will always be illegal in the assumptions context.

[I disagree then with this conclusion. For $\omega = 0$ we have $\beta' = 0$ and constant is a viable solution.]

Plan E: An interesting though complicated Loss Model

Is there **some other loss model** that is valid down to $\omega = 0$? Assuming G = 0 as I always do in Appendix D, all loss is due to σ in the conductor. We know that $\mathbf{J} = \sigma \mathbf{E}$ (at least for our frequency range!), so I ought to be able to compute the loss per unit length of transmission line. I have often argued that J_z is the predominant current, and that is certainly true near DC. So maybe I can just use $J_z = \sigma E_z$ to compute the loss. I sort of do this right now in Appendix P where I write

$$dP = (dI)^2(dR) = [J_z(r,\theta)dA]^2\ \rho\ dz / dA = \rho\ J_z(r,\theta)^2\ dA\ dz\ . \qquad (P.10.3)$$

which is the loss in a "tiny resistor" inside the medium. If I integrate over the conductor cross section

$$P = \int dP = dz \int dA\ \rho\ J_z(r,\theta)^2\ . \qquad (P.10.4)$$

and this is the Ohmic power loss in a coin-shaped resistor of thickness dz. I might write this as

$$dP/dz = (1/\sigma) \int dA \; J_z(r,\theta)^2$$

Before continuing, however, there are several complications! From Chapter 2 and just generally we know that E_z and J_z are complex, not simply real. But imagine a tiny resistor and $P = I^2R$. The phase of the current I does not really affect the real power burn in the resistor. So I guess I should modify the above stuff to read

$$dP = |dI|^2(dR) = |J_z(r,\theta)dA|^2 \, \rho \, dz / dA = \rho \, |J_z(r,\theta)|^2 \, dA \, dz \, . \qquad (P.10.3)$$

$$P = \int dP = dz \int dA \, \rho \, |J_z(r,\theta)|^2 \, . \qquad (P.10.4)$$

$$dP/dz = (1/\sigma) \int dA \; |J_z(r,\theta)|^2$$

Even though tiny resistors at different radii have different phase current, you still just add up the little real power dissipations! For a round wire we then have

$$dP/dz = (1/\sigma) \int_0^a r dr \int_{-\pi}^{\pi} d\theta \, |J_z(r,\theta)|^2$$

I will check later, but I think this leads to (up to a constant I would have to determine)

$$dP/dz = (1/\sigma) \Sigma_m \, {}_{=-\infty}^{\infty} \int_0^a r dr \, |J_z(r,m)|^2$$

and then each partial wave contributes to the dissipation. This can also be written

$$dP/dz = \sigma \Sigma_m \int_0^a r dr \, |E_z(r,m)|^2$$

Now what happens if I go ahead and use the Appendix D solution?

$$E_z(r,m) = (1/4) \, \eta_m \, I \, R_{dc} \, (a\beta') \, f_m(r)$$

Then we have

$$dP/dz = \sigma \, [\, (1/4) \, I \, R_{dc} \, (a|\beta'|)]^2 \; \Sigma_m \; |\eta_m|^2 \; \int_0^a r dr \; |f_m(r)|^2$$

I am hacking a completely new first-time path now, I have never done this before. The integral might be doable !

$$f_m = J_m(x)\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\int_0^a r\,dr\ |f_m|^2 \qquad\qquad x = \beta' r \qquad\qquad dx = \beta'\,dr$$

$$\int_0^a r\,dr\ |f_m|^2 = \int_0^{x_a} (x/\beta')(dx/\beta')\,|f_m(x)|^2 = (1/\beta')^2 \int_0^{x_a} dx\ x\,|f_m(x)|^2$$

$$= \left|\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right|^2 (1/\beta')^2 \int_0^{x_a} dx\ x\,|J_m(x)|^2$$

What does this integral mean? Well consider:

2. $$\int x\,[Z_p(\alpha x)]^2\,dx = \frac{x^2}{2}\left\{[Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x)\,Z_{p+1}(\alpha x)\right\}$$ WA 135(11)

Wow, just what the doctor ordered with $\alpha = 1$ and $p = m$, so

$$\int dx\ x\ |J_m(x)|^2 = (x^2/2)\ \{\ |J_m(x)|^2 - J_{m-1}(x)J_{m+1}(x)\ \}$$

I already know the small argument stuff

$J_m(x) = (x/2)^m / m!$ for $m = 0,1,2,.....$
$J_{-1}(x) = -(x/2)$ for $m = -1$. (D.11.16)

So for m = 1,2,3... I get

$$RHS_m(\text{small } x) = (x^2/2)\ \{\ [(x/2)^m / m!]^2 - [(x/2)^{m-1} / (m-1)!]\ [(x/2)^{m+1} / (m+1)!]\ \}$$

and then as $x \to 0$ this is all zero. For m = 0 we get

$$RHS_0(\text{small } x) = (x^2/2)\ \{\ |J_0(x)|^2 - J_{-1}(x)J_1(x)\ \} = (x^2/2)\ \{\ 1 - (-x/2)(x/2)\ \}$$

and as $x \to 0$ this is also 0. This we may conclude that

$$\int_0^{x_a} dx\ x\,|J_m(x)|^2 = (x_a^2/2)\ \{\ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a)\ \}$$

and so

$$\int_0^a \text{rdr}\ |f_m|^2 = \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 (1/\beta')^2 (x_a^2/2) \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

Although messy, it is in closed form. Now $x_a/\beta' = \beta' a/\beta' = a$, so can simplify

$$\int_0^a \text{rdr}\ |f_m|^2 = (1/2)\, a^2 \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

and then

$$dP/dz = \sigma [(1/4)\, I\, R_{dc}\, (a|\beta'|)]^2 \Sigma_m |\eta_m|^2 * a^2 \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

$$= \sigma\, (1/16)\, I^2\, 1/(\pi^2 a^4 \sigma^2)\, a^2 |\beta'|^2 \Sigma_m |\eta_m|^2 * a^2 \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

$$= (1/16)\, I^2\, 1/(\pi^2 \sigma)\, |\beta'|^2 \Sigma_m |\eta_m|^2 * \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

Dimension check: $I^2 / \sigma * |\beta'|^2 =$ amp^2 [ohm-m] m^{-2} = amp-volt /m = watt/m = correct

Amazingly, I have a closed-form expression for the power loss per meter of this conductor.

[if no math errors, seems OK. Note that J_r and J_θ contributions are ignored.]

Comments: This formula gives the rate at which power is consumed inside a round conductor that is part of a transmission line. The total power at location z on the transmission line is the energy stored both inside and outside the conductors, and I would guess this was mostly outside. Without using the transmission line equations (not valid at low ω), I don't have a simple expression for this power, so I cannot then get a simple expression for the β_d imaginary part or decay exponent. Perhaps I could just say that $P = I^2 Z_0$? Then we have

$$dP/dz = (1/16)\, (P/Z_0)\, 1/(\pi^2 \sigma)\, |\beta'|^2 \Sigma_m |\eta_m|^2 * \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

or

$$dP/P = (1/16)\, (dz/Z_0)\, 1/(\pi^2 \sigma)\, |\beta'|^2 \Sigma_m |\eta_m|^2 * \left| \frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)} \right|^2 \{ |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \}$$

$$\ln P = (1/16)\,(z/Z_0)\,1/(\pi^2\sigma)\,|\beta'|^2\,\Sigma_m\,|\eta_m|^2 * \left|\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right|^2 \{\, |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \,\}$$

$$= -2\alpha\, z$$

$$P(z) = P(0)\, e^{-2\alpha z}$$

So we seem to have

$$-2\alpha = (1/16)\,(1/Z_0)\,1/(\pi^2\sigma)\,|\beta'|^2\,\Sigma_m\,|\eta_m|^2 * \left|\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right|^2 \{\, |J_m(x_a)|^2 - J_{m-1}(x_a)J_{m+1}(x_a) \,\}$$

where $x_a = \beta' a$ with $\beta'^2 = \beta^2 - \beta_d^2 = \beta^2 - (\beta_{d0} - j\alpha)^2$

so we have some kind of complicated equation which determines α. If α is assumed << β_{d0} then we are done and the above is our result for α.

[but I now have a much simpler way to compute α using the k(ω) stuff of Appendix Q]

Plan F: Review fact that J_z asymmetry exists in the Regular App D solution.

Let's go back to lossless β_d and let's go back to this result below (D.1.14):

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 - \beta_d^2 + \beta^2\,]\,E_z(r,\theta) = 0\,. \qquad (2)$$

This is a PDE and I don't know much about general PDE solutions. In the lossless model, as ω→0 we get β→0 and β_d→ 0 so this reads

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2\,]\,E_z(r,\theta) = 0\,. \qquad (2)$$ [agreed]

I think this is the 2D Laplace equation, and it has general solutions of this form

$$r^m \cos(m\theta)$$

$$[\partial_r^2 + (1/r)\,\partial_r]\, r^m \cos(m\theta) = [\, m(m-1)r^{m-2}\cos(m\theta) + mr^{m-2}\cos(m\theta)\,]$$

$$= m^2 r^{m-2}\cos(m\theta)$$

$$(1/r^2)\,\partial_\theta^2\, r^m \cos(m\theta) = -\, r^{m-2}\, m^2 \cos(m\theta)$$

The m = 0 solution form is just a constant, and that too is a viable solution. [yes]

But somehow as ω→0, my low ω solutions are

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, [\, 4 + 4 \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d)\, [2\, (r/a) + 2 \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} + (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta_d)\, [2 \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} - (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,] \ . \qquad \text{(D.11.18)}$$

and then as $\beta_d \rightarrow 0$ this says

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, [\, 4 + 4 \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = 0$$

$$E_\theta(r,\theta) = 0 \qquad \text{(D.11.18)}$$

which just seems immensely strange to me. Yes, it IS a solution of the Helm → Laplace. What about div E = 0 equation $\partial_r (r\, E_r) + \partial_\theta E_\theta + r\, \partial_z E_z = 0$? Each of the three terms is 0 in this limit.

The CPBC in this limit

$$E_r(r=a,m) = (j\omega/\sigma)\, N_m \ . \qquad \text{(D.2.25)}$$

says that

$0 = 0$ // since $E_r = 0$ on the left and since $\omega = 0$ on the right

The other BC just says 0 = 0 as well. But somehow the "ghost" of the CPBC is shaping my solution even in this $\omega \rightarrow 0$ limit.

Plan G So what IS the ω = 0 solution obtained in Plan C ?

First of all, recall this summary

$$E_{r,tot}(r,\theta,z,t) = -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_r(r,\theta;\beta_d)$$
$$E_{\theta,tot}(r,\theta,z,t) = -2j\sin(\beta_d[z-L])\, e^{j\omega t}\, E_\theta(r,\theta;\beta_d)$$
$$E_{z,tot}(r,\theta,z,t) = 2\cos(\beta_d[z-L])\, e^{j\omega t}\, E_z(r,\theta;\beta_d)$$

Assuming the functions on the right are non-singular in β_d we then have

$$E_{r,tot}(r,\theta,z,t) = 0$$
$$E_{\theta,tot}(r,\theta,z,t) = 0$$
$$E_{z,tot}(r,\theta,z,t) = 2\, e^{j\omega t}\, E_z(r,\theta;\beta_d = 0) \qquad \text{// this is the } \omega = 0 \text{ solution}$$

$$I_{tot}(z,t) = [\, 2\, e^{j\omega t}\,]\ I \qquad I \equiv 2\pi\omega\, (a/\beta_d)\, N_0$$

$$E_z(r,m;\beta_d) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m$$

So we end up somehow with NO E_r and NO E_θ, but somehow E_z is still asymmetric! Not clear yet what happens to I in this limit. If $N_0 = \langle n(\theta)\rangle$ as in (D.1.8), it would seem that

$$I = 2\pi\omega\,(a/\beta_d)\,N_0 = 2\pi\omega\,(a/k)\,N_0 \qquad \text{// in new notation}$$

But we have from Appendix Q:

Fact 4: In the low frequency limit with G = 0 , (Q.4)

$$\begin{aligned} \mathrm{Re}(k) &\approx \sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2} \\ \mathrm{Im}(k) &\approx -\sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2} \qquad\qquad \omega << R/L \end{aligned}$$

so it seems that we might end up with

$$I = 2\pi\omega\,(a/k)\,N_0 = 2\pi\omega\ \langle n(\theta)\rangle\ \frac{a}{\sqrt{RC/2}\ \omega^{1/2}\,(1-j)} = \text{stuff} * \omega^{1/2} \rightarrow 0$$

SO! We still end up with our I→0 situation in the DC limit for this reflection scenario! It does not yield a DC limit where there is a finite current, as we need to be doing Appendix P eddy currents. TBC.