Why Superposition does not Work (Reflection Scenario) PhL 8.8.14

[I set out to show why superposition does not work, and ended up showing it does work. This then led to a later doc called "Plan C Revised". This is just another in my series of reflection scenario docs as a way to understand the asym J_z mystery at DC.]

Motivation: My idea was to superpose two infinite transmission line waves going in opposite directions in order to represent the physical situation of a finite line with a shorting bar at z = L.

Discussion.

One can think about this superposition for lots of different quantities, such as J_z, i, V E_r , B_θ and so on. Perhaps the most "physical" quantity is V(z), the voltage on the transmission line at z. So let's start off by thinking about V(z).

The superposition for V(z) then has this most general form [it is really V(z,t)] :

$$V(z) = A\, e^{j(\omega t - k[z-L])} + B\, e^{j(\omega t + k[z-L])}$$

We am certainly allowed to offset the z axis by an amount L. The first wave goes to the right, the second wave goes to the left. Since each wave separately is a solution of the transmission line wave equation with parameter $k = -j\sqrt{zy}$, the sum is *also* a solution -- that is the superposition idea. We sweep all other issues under the rug such as: (1) What do the other quantities look like with this superposition? (2) Does the superposition of those other quantities satisfy Maxwell's equations and the Helmholtz equations, etc? (3) Does the superposed solution satisfy the boundary conditions of Appendix D? Here we are looking only at V(z), and only if this works out will we worry about these other matters.

The main goal is to cause V(L) = 0 at all times, since there is a shorting bar at z = L. Thus, the superposed solution must satisfy this equation:

$$V(L) = A\, e^{j(\omega t - k[L-L])} + B\, e^{j(\omega t + k[L-L])} = (A + B)\, e^{j\omega t} = 0 .$$

This then forces B = - A, and then our superposed wave solution is

$$V(z) = A\, e^{j(\omega t - k[z-L])} - A\, e^{j(\omega t + k[z-L])}$$

$$= A\, e^{j\omega t} \{ e^{-jk[z-L]} - e^{+jk[z-L]} \} = -A\, e^{j\omega t} \{ e^{+jk[z-L]} - e^{-jk[z-L]} \}$$

$$= -A\, e^{j\omega t} \{ 2j \sin(k[z-L]) \}$$

$$= -2jA\, e^{j\omega t} \sin(k[z-L])$$

This then is the only possible V(z) you can construct from oppositely going transmission line waves such that V(L) = 0 at all times.

Note that $k = \beta - j\alpha$ and k "knows about" both phase factor β and attenuation factor α. The notion of "loss" is built into this theory in the form of parameter α .

When G = 0, we know that as $\omega \to 0$, we have $\mathbf{y} = G + j\omega C \to 0$ and therefore $k = -j\sqrt{\mathbf{zy}} \to 0$. It also happens that $\mathbf{z} \to 2R$ which is finite (R for each conductor).

Therefore, as $\omega \to 0$, our solution becomes

$$V(z) = 2jA\, e^{j\omega t} \sin(0[z-L]) = 0$$

so V(z) vanishes everywhere on the line from z = 0 to z = L.

But this is NOT the physical situation we are trying to represent at DC! In that situation, we have some total resistance 2RL and we expect to have I = V(0)/(2RL) = constant, and V(z) = V(0) [1 - z/L] = linear. Notice that these forms for I and V satisfy the first order transmission line equations:

$$\frac{di(z)}{dz} = -\mathbf{y}\, V(z) = 0 \qquad \Rightarrow \qquad i(z) = I = \text{constant}$$

$$\frac{dV(z)}{dz} = -\mathbf{z}\, i(z) = -2RI \qquad \Rightarrow \qquad V(z) = -2RIz + V(0)$$

We know that V(L) = 0 so this last right side equation says 0 = -2RIL + V(0) so V(0) = 2RIL and therefore that I = V(0)/(2RL). Finally, V(z) = -2Rz [V(0)/(2RL)] + V(0) = V(0)[1 - z/L] as claimed.

Therefore, the superposition model proposed above does not have as its DC limit the physical situation we want it to describe. In other words, the superposition model does not work.

There is then no need to delve into what the "other quantities" look like under this superposition.

How about a triple superposition

Suppose we start with this general form for all ω

$$V(z) = A\, e^{j(\omega t - k[z-L])} + B\, e^{j(\omega t + k[z-L])} + V(0)\,[\,1 - z/L]$$

The problem now is that this last term does NOT satisfy the 2nd order transmission line equations for general ω. For example, this last term has $\partial^2_z V(z) = 0$ but we need it to satisfy $[\partial^2_z + k^2]V(z) = 0$.

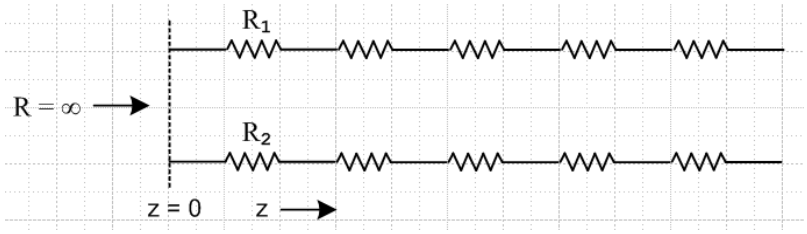
More Discussion

I have tried to model a certain physical situation (shorted finite line) by superposing two infinite transmission line wave solutions. Each of those infinite transmission line solutions involves

$$k \equiv -j\sqrt{(R+j\omega L)(G+j\omega C)}$$

where R,L,G,C are parameters of the infinite transmission line and also of the finite line.

In each of the two superposed infinite transmission line solutions (with G = 0), we know that k→0 as ω→0. This means that for each solution separately, we have V(z) = V(0) = constant when ω = 0. This makes perfect sense for an infinite transmission line according to this picture of such a line at ω = 0 :



The line has infinite resistance, so I = 0, so there is no voltage drop going down the line, so V(z) = V(0).

In the superposition of these two solutions, as ω→0 each solution becomes a constant. In order to make the sum solution vanish at z = L, it must then vanish at all z.

The shorted finite line is a different physical system than the infinite line. You therefore should not expect that a solution to the finite line situation would be a superposition of solutions to the infinite line situation. This is our old scenario of Situations 1,2 and 3. The shorted line has a boundary condition which is not present in the infinite line.

OK Fine. But then just how DO you analyze the shorted bar finite line at general ω ?

[here I give the reflection scenario another go, but this is only for V(z) and results are reasonable.]

Maybe you would start with an Appendix K "network approach" and try to find the general nature of the solution prior to worrying about what happens inside a conductor. We know that the solution must solve the TL equations, so the only possible form is

$$V(z) = A(k)\, e^{j(\omega t - k[z-L])} + B(k)\, e^{j(\omega t + k[z-L])}$$

where now I have something new that I omitted above: A and B are allowed to be functions of k. Following the development above, we then end up with

$$V(z) = -2jA(k)\, e^{j\omega t} \sin(k[z-L])$$

Any function A(k) is allowed here as far as the TL wave equation is concerned.

What conditions exist on A(k) ? We assume that V(0) = some $V_0 e^{j\omega t}$ we apply to the system which I imagine is independent of anything, so we then have

$$V_0 = -2jA(k) \sin(k[-L])$$

which tells us that

$$A(k) = V_0 / [\, -2j \sin(-kL) \,] \;\; = V_0 / [\, 2j \sin(kL) \,]$$

Our solution is then

$$V(z) = -2jA(k)\, e^{j\omega t} \sin(k[z-L]) \;\; = \;\; -2j \{\, V_0 / [\, 2j \sin(kL) \,] \} \, e^{j\omega t} \sin(k[z-L])$$

$$= - \{\, V_0 / [\, \sin(kL) \,] \} \, e^{j\omega t} \sin(k[z-L]) \;\; = - V_0 \, e^{j\omega t} \frac{\sin(k[z-L])}{\sin(kL)}$$

Now we take $\omega \rightarrow 0$ and have $k \rightarrow 0$ and this gives

$$V(z) = - V_0 \, e^{j\omega t} \frac{(z-L)}{L} \;\; = - V_0 \, e^{j\omega t} \, (z/L - 1) \;\; = V_0 \, e^{j\omega t} \, (1 - z/L)$$

and we have (finally!) arrived at the correct DC solution.