Very Low Frequency Limit of the Appendix D fields PhL 3.9.14

[On 5.15.14 I carefully reviewed this document. Interestingly, whether or not you include loss in the model, you find that $\beta' \to 0$ as $\omega \to 0$ and this leads to the small argument Bessel limits as noted below and this leads to the fact that J_z is asymmetric is maintained at arbitrarily low ω and in fact approaches the 3D plot shown below. There just seems no way out of this fact beyond that escape hatch fact that $I \to 0$ in this limit as well. I added various red comments in the text below as I went. At the very end I show the most convincing fact:

As $\beta' \to 0$ as $\omega \to 0$, Helmholtz for $E_z \to$ Laplace for E_z which => $A_m r^m \cos(m\theta)$ solution

and unless A_m coefficient is 0 ($I \to 0$), you must have asymmetry in E_z . The only way to make A_m vanish is to have $I \to 0$ or to have $\eta_m \to 0$ .This all assumes the two BC's of Appendix D!]

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1. First shot at computing the low ω limit of the E fields.

I used to have a section in Appendix D on this low frequency limit, but I removed it. Now, I am having confusions in this limit, and that is why I am starting this new doc you are reading.

When the boundary conditions are applied, we find that

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad a = \text{radius} \qquad \eta_m \equiv \frac{N_m}{N_0}$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

One thing we must do carefully is obtain the correct expressions for β and β_d and make sure they are correct for very small values of ω. In general we know that

$$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu(\varepsilon - j\sigma/\omega) = \omega^2\mu\,\xi \qquad \xi \equiv \varepsilon - j\sigma/\omega \quad . \qquad (1.5.1)$$

Case 1: Inside the conductor, where β is called β

$$\beta^2 = \omega\mu(\omega\varepsilon - j\sigma)$$

At some finite frequency ω, we usually argue that we can say $\beta^2 = -j\sigma\omega\mu$ since σ is so large. You can see that for very small ω, this is even more valid, since then $\omega\varepsilon <<< -j\sigma$ so to speak. So this seems a viable expression for β^2. We choose our usual root of -j and then get

$$\beta = [\,(j-1)/\sqrt{2}\,]\,\sqrt{\sigma\omega\mu} = e^{-3\pi j/4}\,\sqrt{\sigma\omega\mu} \qquad \text{ok}$$

Case 2: Outside the conductor, where β is called β_d

$$\beta_d^2 = \omega\mu_d(\omega\varepsilon_d - j\sigma_d)$$

I am willing to restrict everything to a vacuum dielectric which really has $\sigma_d \equiv 0$. Then

$$\beta_d^2 = \omega\mu_d(\omega\varepsilon_d) = \omega^2(\mu_d\varepsilon_d) = \omega^2/v_d^2 = (1/v_d)^2\,\omega^2$$

Summary

$$\beta = e^{3\pi j/4}\,\sqrt{\sigma\mu}\;\;\omega^{1/2} \qquad // \qquad \beta = e^{j3\pi/4}\,(\sqrt{2}/\delta)$$

$$\beta_d = (1/v_d)\,\omega \qquad // \text{ later this becomes } \beta_d = (1/v_d)(\omega - j\omega_c)$$

[The red correction includes the effect of loss as discussed in "DC limit...". For 8281 f_c = 7.7 KHz.]

You see that these β's differ in their power of ω.

We are often interested in the ratio:

$$|\beta_d/\beta| = (1/v_d)\,(1/\sqrt{\sigma\mu}\,)\,\omega^{1/2} \qquad // = (1/v_d)\,(1/\sqrt{\sigma\mu}\,)\,\frac{\omega - j\omega_c}{\omega^{1/2}}$$

We often argue that this ratio is very small since σ is so large for a conductor. Due to the factor $\omega^{1/2}$, this smallness is even more dramatic for very small values of ω. [But with ω_c included, we get the reverse result that $|\beta_d/\beta|$ becomes very LARGE as $\omega \rightarrow 0$]

Since $\beta'^2 = \beta^2 - \beta_d^2$, if the ratio $|\beta_d/\beta|$ is very small, then we expect $\beta' \approx \beta$ for all practical purposes. Especially for very small ω we expect this to be true, as just noted. so we will assume

$\beta' = \beta$

Just to make sure of this, here is more detail:

$$\beta'^2 = \beta^2 - \beta_d^2 = -j\sigma\mu\omega - \omega^2/v_d^2 = \omega\,[\,-j\sigma\mu - \omega/v_d^2\,]$$

so for small ω we get

$$\beta' \approx \sqrt{-j\sigma\mu\omega} = e^{3\pi j/4}\sqrt{\sigma\mu}\;\omega^{1/2} = \beta \qquad [\text{ valid for } \omega >> \omega_c\,]$$

[But with our red correction, as $\omega \to 0$ we have $|\beta/\beta_d| \to 0$ so this means $\beta'^2 = -\beta_d^2$. Then

$$\beta_d \approx (1/v_d)(-j\omega_c) \text{ low } \omega \quad \Rightarrow \quad \beta'^2 = -\beta_d^2 = -(-1)(1/v_d)^2\,\omega_c^2 \Rightarrow \quad \beta' = \omega_c/v_d$$

For 8281 coax what is β'a? Recall that a = 394μ and $v_d = 2 \times 10^8$ m/sec and f_c = 7770 Hz, so then

```
a := 394e-6;

fc := 7770;

wc := evalf(2*Pi*fc);

vd := 0.66 * 3e8;

betap := wc/vd;

a*betap;
```

$$a := .000394$$

$$fc := 7770$$

$$wc := 48820.34984$$

$$vd := .198\ 10^{9}$$

$$betap := .0002465674234$$

$$.9714756482\ 10^{-7}$$

This shows then that as $\omega \to 0$, if we include the effect of loss, we have $x_a = \beta'a \approx 10^{-7}$ = very small. But when we ignored loss, we *also* found that $\beta' \to 0$ as shown above, so with or without loss, you want to know about Bessel functions of small argument!]

Examine a Typical Bessel function for low ω

1. We are really interested in $J_m(x)$ for $|x| << 1$.

Consider

$$J_m(x) = J_m(\beta' r) = J_m(\beta r) = J_m\,[e^{3\pi j/4}\sqrt{\sigma\mu}\;\omega^{1/2}\,r\;]$$

Let's work at the maximum value of r which is a, then we have

$$\beta\, a = e^{-3\pi j/4}\ \sqrt{\sigma\mu}\ \ \omega^{1/2}\, a$$

For a normal transmission line, we have

$$\beta a = \sqrt{2}\ e^{j3\pi/4}\ (a/\delta)$$

so if we are in the skin depth regime, we think of $\beta a > 1$. But in the small region, we will instead have the fact that $\beta a < 1$ and in fact βa can be very small as we lower ω, which is our interest today.

So we are then interested in $J_m(x)$ where $|x| << 1$ in our limit of small ω. correct

Digression on the Symmetries of the Square Bracket Objects

If we define

$$f_m \equiv \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

then consider

$$f_{-m} = \left[\frac{J_{-m}(x)}{J_{-m+1}(x_a)} - \frac{J_{-m}(x)}{J_{-m-1}(x_a)}\right] = -\left[\frac{J_m(x)}{J_{m-1}(x_a)} - \frac{J_m(x)}{J_{m+1}(x_a)}\right] = f_m$$

On the other hand, if we define

$$g_m \equiv \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

then we find

$$g_{-m} = \left[-\frac{2mx^{-1}J_{-m}(x)}{J_{-m-1}(x_a)} + \frac{J_{-m+1}(x)}{J_{-m+1}(x_a)} - \frac{J_{-m+1}(x)}{J_{-m-1}(x_a)}\right]$$

$$= \left[+\frac{2mx^{-1}J_m(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)} - \frac{J_{m-1}(x)}{J_{m+1}(x_a)}\right] \qquad \text{// negative all orders}$$

But there is a Bessel function identity of interest here

$$\mathscr{C}_{\nu-1}(z) + \mathscr{C}_{\nu+1}(z) = (2\nu/z)\,\mathscr{C}_\nu(z),$$

or

$$J_{m-1}(x) + J_{m+1}(x) = (2m/x)J_m(x)$$

We let Maple internally use this identify and we end up with $g_{-m} = g_m$ after all:

```
restart: alias(I=I,j=sqrt(-1),J = BesselJ);
```

```
gm := 2*m*(1/x)*J(m,x)/J(m-1,xa) +  J(m+1,x)/J(m+1,xa) - J(m+1,x)/J(m-1,xa);
```

$$gm := 2\frac{m\,J(m,x)}{x\,J(m-1,xa)} + \frac{J(m+1,x)}{J(m+1,xa)} - \frac{J(m+1,x)}{J(m-1,xa)}$$

```
gmm := 2*m*(1/x)*J(m,x)/J(m+1,xa) +  J(m-1,x)/J(m-1,xa) - J(m-1,x)/J(m+1,xa);
```

$$gmm := 2\frac{m\,J(m,x)}{x\,J(m+1,xa)} + \frac{J(m-1,x)}{J(m-1,xa)} - \frac{J(m-1,x)}{J(m+1,xa)}$$

```
gm - gmm;
```

$$2\frac{m\,J(m,x)}{x\,J(m-1,xa)} + \frac{J(m+1,x)}{J(m+1,xa)} - \frac{J(m+1,x)}{J(m-1,xa)} - 2\frac{m\,J(m,x)}{x\,J(m+1,xa)} - \frac{J(m-1,x)}{J(m-1,xa)} + \frac{J(m-1,x)}{J(m+1,xa)}$$

```
simplify(%);
```

$$0$$

Finally, let's define the third square bracket, where notice that two signs are different(red)

$$h_m \equiv [\, -\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\,]$$

so following the above, where we again negate all orders, but now we have two new signs,

$$h_{-m} \equiv [\, -\frac{2mx^{-1}J_m(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)} + \frac{J_{m-1}(x)}{J_{m+1}(x_a)}\,]$$

In this case we find that

$$h_{-m} = -\,h_m$$

as Maple shows:

```
hm := -2*m*(1/x)*J(m,x)/J(m-1,xa) +  J(m+1,x)/J(m+1,xa) + J(m+1,x)/J(m-1,xa);
```

$$hm := -2\frac{m\,J(m,x)}{x\,J(m-1,xa)} + \frac{J(m+1,x)}{J(m+1,xa)} + \frac{J(m+1,x)}{J(m-1,xa)}$$

```
hmm := -2*m*(1/x)*J(m,x)/J(m+1,xa) +  J(m-1,x)/J(m-1,xa) + J(m-1,x)/J(m+1,xa);
```

$$hmm := -2\frac{m\,J(m,x)}{x\,J(m+1,xa)} + \frac{J(m-1,x)}{J(m-1,xa)} + \frac{J(m-1,x)}{J(m+1,xa)}$$

```
hm + hmm;
```

$$-2\frac{m\,J(m,x)}{x\,J(m-1,xa)} + \frac{J(m+1,x)}{J(m+1,xa)} + \frac{J(m+1,x)}{J(m-1,xa)} - 2\frac{m\,J(m,x)}{x\,J(m+1,xa)} + \frac{J(m-1,x)}{J(m-1,xa)} + \frac{J(m-1,x)}{J(m+1,xa)}$$

```
simplify(%);
```

$$0$$

Summary of results: I think these symmetries are all correct

$$f_m \equiv \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad f_{-m} = f_m$$

$$g_m \equiv \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad g_{-m} = g_m$$

$$h_m \equiv \left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad h_{-m} = -h_m$$

We will find below that only f_m blows up for small ω because only f_m has a term where the power of the numerator is less than the power of the denominator.

What is this small x limit of $J_m(x)$?

If we keep *only the leading term* in the $J_m(x)$ expansion, we get this

10.7.3 $J_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$

therefore

$J_m(x) = (x/2)^m / m!$ for m = 0,1,2,.....

But the rule

$J_{-m}(x) = (-1)^m J_m(x)$

says then that

$J_{-m}(x) = (-1)^m J_m(x) = (-x/2)^m / m!$ for m = 0,1,2....

However, due to our symmetry rules found above, we can restrict our interest to $m \geq 0$! ok

What is the small x limit of f_m ? **$m \geq 0$**

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m / m!}{(x_a/2)^{m+1} / (m+1)!} - \frac{(x/2)^m / m!}{(x_a/2)^{m-1} / (m-1)!}\right]$$

$$= [\,(m+1)\,(x/2)^m\,(x_a/2)^{-m-1} - (1/m)\,(x/2)^m (x_a/2)^{-m+1}]$$

$$= (x/2)^m\,(x_a/2)^{-m}\,[\,(m+1)\,(x_a/2)^{-1} - (1/m)\,(x_a/2)\,]$$

$$= (r/a)^m\,[\,(m+1)\,(2/\beta' a)\,] \qquad \text{since } x_a \to 0 \quad m > 0$$

For m = 0 we start over and get twice the first term so

$$f_0 = 2\ (x/x_a)^0\ [\ (0+1)\ (x_a/2)^{-1}\ ]\ = 2\ (2/x_a) = 4/x_a \qquad\qquad m = 0$$

But then I can use the symmetry of f_m to extend this to say

$$f_m = (x/x_a)^{|m|}\ [\ (|m|+1)\ (2/x_a)\ ] \qquad\qquad m \neq 0$$

I am concerned that we might need to keep two leading terms due to this subtraction in f_m ?

10.2.2 $$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$J_m(x) = (x/2)^m\ [\ 1/m!\ \ -\ (x^2/4)\ /\ (m+1)!\ +\ O(x^4)\ ]$$

Well no, this would only matter if the leading terms cancelled, which they do not, so forget that.

[In "low w limit of fm.mws" I allow the use of any number of leading terms in the $J_m(x)$ limit, but the results are always the same and are as given above.]

What is the small ω limit of E_z ? $\qquad m \geq 0$

$$E_z(r,m) = (1/4)\ \eta_m\ I\ R_{dc}\ (a\beta')\ f_m \qquad\qquad R_{dc} = \frac{1}{\sigma\pi a^2}$$

$$= (1/4)\ \eta_m\ I\ R_{dc}\ (a\beta')\ (x/x_a)^m\ [\ (m+1)\ (x_a/2)^{-1}\ ]$$

$$= (1/4)\ \eta_m\ I\ R_{dc}\ (a\beta')\ (r/a)^m\ [\ (m+1)\ (2/\beta' a)\ ]$$

$$= (1/2)\ (|m|+1)\ \eta_m\ I\ R_{dc}\ (r/a)^{|m|} \qquad [\text{ no change here despite } \beta_d \text{ change due to loss }]$$

I am rather astounded to find that for m > 0, $E_z(r,m) \neq 0$ as we go to very low ω. I always thought the z current was uniform at super low frequency (I will avoid ω = 0 exactly).

[So there it is! For either method of handling β' (loss or no loss), conclusion is the same: J_z is asym at ω=0. This says that even at ω = 0, the moments η_m "get into" E_z as shown above.]

Calculate and Plot $E_z(r,\theta)$ at very low frequency.

$$E_z(r,\theta) = \sum_{m=-\infty}^{\infty} E_z(r,m)\ e^{jm\theta} \qquad \text{// expansion} \qquad (A.1)$$

$$E_z(r,m) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\ E_z(r,\theta)\ e^{-jm\theta}\ . \qquad \text{// projection} \qquad (A.2)$$

Method A:

$$E_z(r,\theta) = \sum_{m=-\infty}^{\infty} E_z(r,m)\ e^{jm\theta} = \sum_{m=-\infty}^{\infty} (1/4)\ \eta_m\ I\ R_{dc}\ (a\beta')\ f_m\ e^{jm\theta}$$

But I know that both η_m and f_m are symmetric in m, so we get

$$= (1/4)\ I\ R_{dc}\ (a\beta')\ f_0 + (1/2)\ I(a\beta')R_{dc} \sum_{m=1}^{\infty} \eta_m\ f_m \cos(m\theta)$$

$$= (1/4)\ I\ R_{dc}\ (a\beta')\ (4/x_a) + (1/2)\ I(a\beta')R_{dc} \sum_{m=1}^{\infty} \eta_m\ f_m \cos(m\theta)$$

$$= (1/4)\ I\ R_{dc}\ (a\beta')\ (4/a\beta') + (1/2)\ I(a\beta')R_{dc} \sum_{m=1}^{\infty} \eta_m\ f_m \cos(m\theta)$$

$$= I\ R_{dc} + (1/2)\ I(a\beta')R_{dc} \sum_{m=1}^{\infty} \eta_m\ f_m \cos(m\theta)$$

$$= I\ R_{dc}\ [\ 1 + (1/2)\ (a\beta') \sum_{m=1}^{\infty} \eta_m\ f_m \cos(m\theta)\ ] \qquad // \eta_m \equiv N_m/N_0 = (-1)^m\ e^{-|m\xi_1|}$$

$$= I\ R_{dc}\ [\ 1 + (1/2)\ (a\beta') \sum_{m=1}^{\infty} \eta_m\ (x/x_a)^{|m|}\ [\ (|m|+1)\ (2/x_a)\ ] \cos(m\theta)\ ]$$

$$= I\ R_{dc}\ [\ 1 + \sum_{m=1}^{\infty} \eta_m\ (x/x_a)^{|m|}\ [\ (|m|+1)] \cos(m\theta)\ ]$$

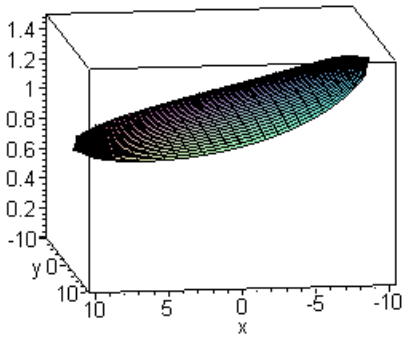
$$= I\ R_{dc}\ [\ 1 + \sum_{m=1}^{\infty} (-1)^m\ e^{-|m\xi_1|}\ (r/a)^{|m|}\ [\ (|m|+1)] \cos(m\theta)\ ]$$

Therefore I find that as $\omega \rightarrow 0$, $E_z(r,\theta)$ approaches this asymmetric shape which is indep of ω :

$$E_z(r,\theta) = I\ R_{dc}\ [\ 1 + \sum_{m=1}^{\infty} (-1)^m\ e^{-m|\xi_1|}\ (r/a)^m\ [\ (m+1)] \cos(m\theta)\ ]$$

and here is a plot where I set $\xi_1 = -2$, which is supposedly the wide-spaced twin lead

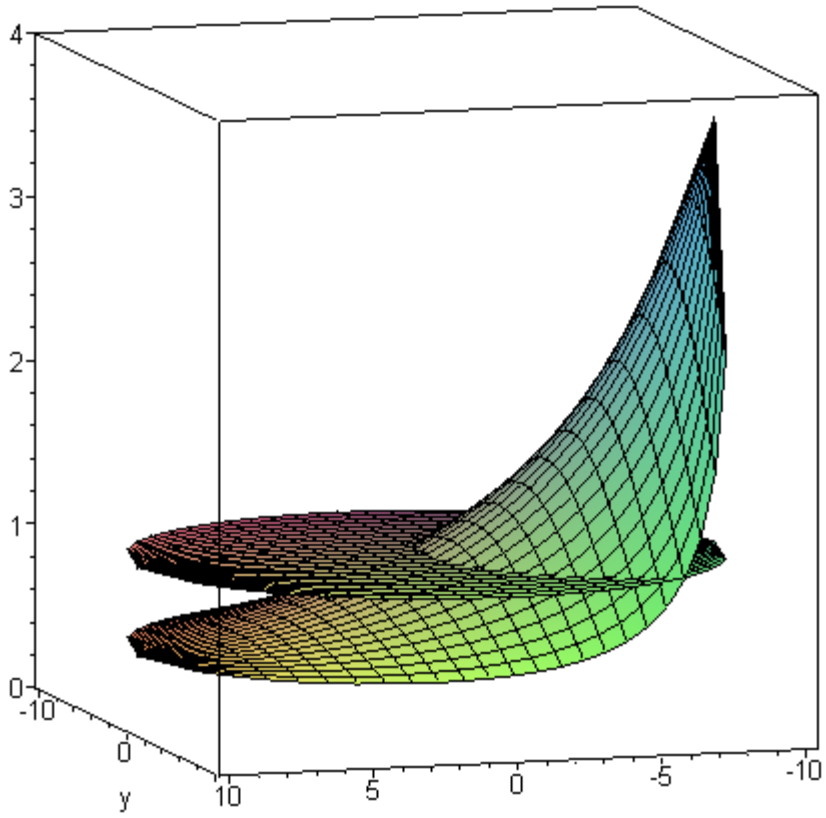
```
restart;
theta := arctan(y,x):
r := sqrt(x^2+y^2):
a := 10:
Ez := 1 + sum((-1)^m*exp(-m*2)*(r/a)^m*(m+1)*cos(m*theta),m=1..20):
plot3d(Ez,x=-a..a,y=-sqrt(a^2-x^2)..sqrt(a^2-x^2),axes=boxed, view =0..1.5 );
```



I don't believe this large amount of asymmetry. I need an external source on this subject!!!

This is very mysterious, I just don't believe it. Everything has now come to a complete halt! I have to decide whether or not I believe my Appendix D fields. If I do, then I need to understand why DC current would be asymmetric as the above plot shows. The web is turning up nothing on this subject , I don't have a search handle that works. still mysterious on 5.14.14

[Here is a later plot for $\xi = 0.75$ which shows an even stronger effect, from "very low freq 1.mws"



This then is the constant shape indep of ω which you get for low ω. Notice there is no "skin effect" visible and in fact current seems larger toward the conductor center.]

What are the other currents doing for very small ω?

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d) g_m$$

$$g_m \equiv \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right]$$

$$= \left[\frac{2mx^{-1}\,(x/2)^m / m!}{(x_a/2)^{m-1} / (m-1)!} + \frac{(x/2)^{m+1} / (m+1)!}{(x_a/2)^{m+1} / (m+1)!} - \frac{(x/2)^{m+1} / (m+1)!}{(x_a/2)^{m-1} / (m-1)!} \right]$$

Now

$$\frac{2mx^{-1}\,(x/2)^m / m!}{(x_a/2)^{m-1} / (m-1)!} = \frac{2mx^{-1}(x/2)^m}{(x_a/2)^{m-1}\, m} = \frac{(x/2)^{m-1}}{(x_a/2)^{m-1}} = (x/x_a)^{m-1} = (r/a)^{m-1}$$

$$\frac{(x/2)^{m+1} / (m+1)!}{(x_a/2)^{m+1} / (m+1)!} = (x/x_a)^{m+1}$$

$$\frac{(x/2)^{m+1}/(m+1)!}{(x_a/2)^{m-1}/(m-1)!} = \frac{(x/2)^{m+1}}{(m+1)m(x_a/2)^{m-1}} = (x/x_a)^m\,(x/2)(x_a/2)/[m(m+1)]$$

Therefore

$$g_m = (r/a)^{m-1} + (r/a)^{m+1} + (r/a)^m\,(\beta^2 ra/4)/[m(m+1)]$$

It would seem that for small ω, we can neglect the β^2 term and we then have

$$g_m = (r/a)^{m-1} + (r/a)^{m+1} \qquad \text{// small } \omega$$

[the g_m mws gives this same result for m > 0. We need a different expression for g_0 since our J expansion is not valid for J_{-1}.]

Then

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\,(a\beta_d)g_m = (j/4)\,\eta_m\, I\, R_{dc}\,(a\beta_d)\,[(r/a)^{m-1} + (r/a)^{m+1}]$$

and thanks to β_d this term → 0 as ω → 0 for all m. Thus $E_r(r,\theta) \to 0$ as ω→0.

[Change: As ω → 0, we get $\beta_d \to -j\omega_c/v_d$ and then we have

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\,(a[-j\omega_c/v_d])\,[(r/a)^{m-1} + (r/a)^{m+1}]$$

$$= (1/4)\,\eta_m\, I\, R_{dc}\,(a\omega_c/v_d)\,[(r/a)^{m-1} + (r/a)^{m+1}]$$

$$\Rightarrow E_r(a,m) = (1/2)\,\eta_m\, I\, R_{dc}\,(a\omega_c/v_d) \neq 0 \qquad \omega_c = v_d\alpha$$

$$\Rightarrow E_r(a,m) = (1/2)\,\eta_m\, I\, R_{dc}\,(a\alpha) \neq 0 \qquad \alpha = \text{loss coefficient}$$

$$= (1/2)\,\eta_m\, I\, R_{dc}\,(a\, R_{dc}C\,(v_d/2)) = (1/2)\,\eta_m\,(I\, R_{dc}^2)\, a\, C\,(v_d/2)$$

and this does NOT vanish as ω → 0. This needs an interpretation!]

Next

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\,(a\beta_d)\, h_m$$

Now h_m differs by two signs, so I expect to get

$$h_m = -(r/a)^{m-1} + (r/a)^{m+1} \qquad \text{very small } \omega$$

and then

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\,[-\ (r/a)^{m-1}\ +\ (r/a)^{m+1}]$$

and this too → 0. So E_z is the only Problem Child. The reason is that for the other fields, the g_m and h_m don't blow up as ω→0, but f_m does blow up!

Conclusion: My equations are saying that the J_z current is asymmetric even at ω= 0, and I cannot find anything wrong. agreed 5.14.14

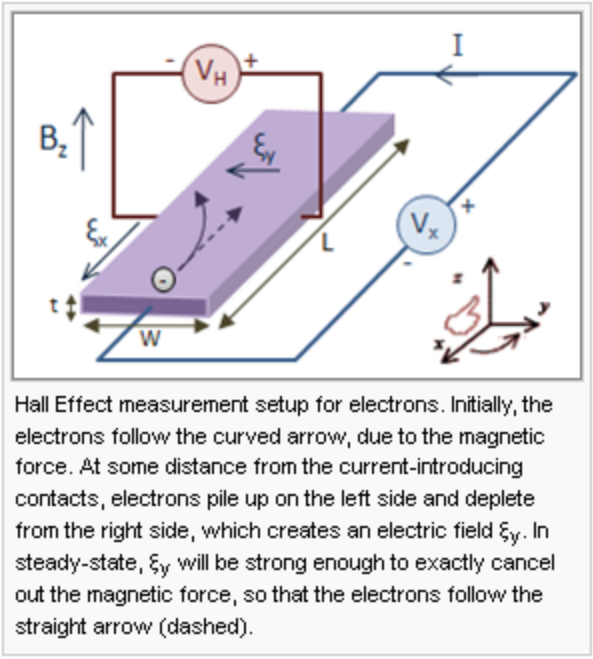
If this is correct, then the asymmetry is being caused by one of two things:

(1) the magnetic field due to the other wire

(2) the magnetic field gradient due to the other wire. [both in fact for ω>0, see App P]

[so this I guess is how I got all involved with the Hall effect at this point.]

If (1) were true, then you would expect to see this asymmetry effect in a conductor sitting in a uniform external magnetic field! That might be a problem I can find somewhere. The Lorentz force on electrons might create a transverse voltage and charge density on the conductor surfaces, maybe this is related to the Hall effect. Well, that is precisely the Hall effect, here from wiki



Hall Effect measurement setup for electrons. Initially, the electrons follow the curved arrow, due to the magnetic force. At some distance from the current-introducing contacts, electrons pile up on the left side and deplete from the right side, which creates an electric field ξ_y. In steady-state, ξ_y will be strong enough to exactly cancel out the magnetic force, so that the electrons follow the straight arrow (dashed).

Web searching, hard to find stuff.

" The current distribution on the conductor cross-section is not uniform as long as the current varies with time"

What about the B fields from Appendix D?

[Well, I don't care so much about B on 5.15.14, but this did lead me to write the various limit mws programs so I could verify all those limit results used above.]

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2}+\frac{a_m}{m}\right)J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\{ + \left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{\beta_d}{\beta'}\left(\frac{K_m}{2}+\frac{a_m}{m}\right) J_{m+1}(x) \}$$

$$B_\theta(r,m) = (\beta'/\omega)\{ - \left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{K_m}{2}\left(\frac{\beta_d}{\beta'}+\frac{\beta'}{\beta_d}\right) J_{m+1}(x) \}\ , \qquad (D.4.8)$$

$$a_m = (j/4)(a\beta_d)\,\eta_m\, I\, R_{dc} \ * \ 2m\ \frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = (j/4)(a\beta_d)\,\eta_m\, I\, R_{dc} \ * \ \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\left(\frac{K_m}{2}+\frac{a_m}{m}\right) = (j/4)(a\beta_d)\,\eta_m\, I\, R_{dc} \ * \ \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right].$$

First, look at B_z . There we see this combination of coefficients:

$$\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)\,(\beta'/\omega)\,(a\beta_d)$$

Now $\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$ is the same as in E_z with sign change second term which did not contribute anything, so this term should be the exact same as f_m in our limit:

$$f_m = (r/a)^m\,[\ (m+1)\,(2/\beta a)\ ] \qquad m \neq 0 \qquad \text{and} \qquad f_0 = 4/\beta a$$

Thus, the combined coefficient on B_z for all values of m looks like

$$(r/a)^m\,[\ (m+1)\,(2/\beta a)\ ]\ (\beta/\omega)\,(a\beta_d) = (r/a)^m\,[\ (m+1)\ 2\ ]\ (1/\omega)\,(\beta_d)$$

$$= (r/a)^m\,[\ (m+1)\ 2\ ]\ (1/\omega)\,(1/v_d)\,\omega = (r/a)^m\,[\ (m+1)\ 2\ ]\,(1/v_d)$$

And thus we get

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2}+\frac{a_m}{m}\right)J_m(x)$$

$$= (\beta'/\omega)\,J_m(x)\,(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc} \ * \ \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right]$$

$$= \{\,(\beta'/\omega)\,J_m(x)\,(a\beta_d)\left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right]\}\ (j/4)\,\eta_m\, I\, R_{dc}$$

$= (j/4)\,\eta_m\, I\, R_{dc}\;\; (r/a)^m\,[\;(m+1)\;2\;]\,(1/v_d)$ $\qquad m > 0$

$= (j/4)\, I\, R_{dc}\, 4\, (1/v_d)$ $\qquad m = 0$

Now this is a VERY strange result and I am sure must be wrong.

I think I need a more automated way to compute all these limits using Maple, there is too much room for algebraic error. I learn just now that the series thing returns a special non-usable type=series data structure and only works for specific m. Suppose I roll my own using the following which I regard OK m = 0,1,2

10.2.2 $$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

This works very well. For example,

```
restart; alias(I=I,j=sqrt(-1));
J := (m,z) -> (z/2)^m * sum((-1)^k*(z/2)^(2*k)/(k!*(m+k)!),k=0..3);
```

$$J := (m,z) \to \left(\frac{1}{2}z\right)^m \left(\sum_{k=0}^{3} \frac{(-1)^k \left(\frac{1}{2}z\right)^{(2k)}}{k!\,(m+k)!} \right)$$

```
J(3,z);
```

$$\frac{1}{8}z^3\left(\frac{1}{6} - \frac{1}{96}z^2 + \frac{1}{3840}z^4 - \frac{1}{276480}z^6\right)$$

so as a fringe benefit I can keep any number of series terms I want. However, this expansion should not be used for m = -1,-2 and so on because our factorial (m+k)! is then ill-defined. Just don't use it!

Now look at

$$f_m \equiv \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

For $m \geq 1$ we can use the above expansion, while m = 0 needs to be a special case where we know we can drop the second term and double the first term.

OK, I have now written "low w limit of fm.mws" and the results it gives are these:

$\beta\, f_m \;\to\; (r/a)^m\;(m+1)\;(2/a)$	compare	$f_m = (x/x_a)^{\lvert m\rvert}\;[\;(\lvert m\rvert+1)\;(2/x_a)\;]$
$\beta\, f_0 \;\to\; (4/a)$	compare	$f_0 = 4/x_a$

You have to rerun the program until it does a certain power simplification of $(\beta r)^m\,(\beta a)^{-m} = (r/a)^m$. When it does not do this, you get a divide by 0 error.

Now let's make a version of this program that does my B_z calculation

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x)$$

$$= (\beta'/\omega)\,(j/4)\,\eta_m\, I\, R_{dc}\;(a\beta_d) * \left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$= (\beta'/\omega)\,(j/4)\,\eta_m\, I\, R_{dc}\;(a\beta_d) * s_m$$

Notice that $s_0 \equiv 0$ so right off we know there is no m = 0 component!
Program is "low w limit of sm.mws" . The result is

$$\beta\, s_m \;\rightarrow\; (r/a)^m\;(m+1)\,(2/a)$$

which is of course the same result as for E_z. We then get

$$B_z(r,m) = (\beta/\omega)\,(j/4)\,\eta_m\, I\, R_{dc}\;(a\beta_d) * s_m =$$

$$= (1/\omega)\,(j/4)\,\eta_m\, I\, R_{dc}\;a\beta_d * (\beta s_m) =$$

$$= (1/\omega)\,(j/4)\,\eta_m\, I\, R_{dc}\;a[(1/v_d)\,\omega] * (r/a)^m\;(m+1)\,(2/a)$$

$$= (j/4)\,\eta_m\, I\, R_{dc}\;a[(1/v_d)\,] * (r/a)^m\;(m+1)\,(2/a)$$

and this replicates our strange result of B_z activity in the m≠0 partial waves, though $B_z(r,0) = 0$.

Let's now try B_θ :

$$B_\theta(r,m) = (\beta'/\omega)\left\{ -\left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{K_m}{2}\left(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}\right) J_{m+1}(x) \right\}$$

I am going to get more Maple oriented now. Write

$$a_m = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc} * 2m\,\frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc} * \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

Let's define

$$Q = (j/4)\,\eta_m\, I\, R_{dc}$$

so then

$$a_m = Q(a\beta_d) * 2m \frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = Q(a\beta_d) * \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

Next write

$$\beta = e^{3\pi j/4}\sqrt{\sigma\mu}\ \ \omega^{1/2} = c\,\omega^{1/2} = cu \qquad c = e^{3\pi j/4}\sqrt{\sigma\mu}$$

$$\beta_d = (1/v_d)\,\omega = (w/v) = u^2/v$$

I now have a more sophisticated Maple program which I can now describe.
Program is "**low w limit of Btheta.mws**".

1. First, get the quantities of interest entered, where I replace ω everywhere by u^2 in order to avoid having the factor $\sqrt{\omega}$ lying around which complicates life for Maple. Also b = β and bd = β_d . We can easily check the entries against above.

```
restart; alias(I=I,j=sqrt(-1)):
assume(m,integer);
#### I am going to replace omega = w  with u^2 everywhere to avoid square roots
Bth1 := (b/u^2)*(-m*(Km/2)*(b/bd) + (bd/b)*am)*J(m,x)/x;
```

$$Bth1 := \frac{b\left(-\frac{1}{2}\frac{m\,Km\,b}{bd} + \frac{bd\,am}{b}\right)J(m,x)}{u^2\,x}$$

```
Bth2 := (b/u^2)* (Km/2)*((bd/b) + (b/bd))*J(m+1,x);
```

$$Bth2 := \frac{1}{2}\frac{b\,Km\left(\frac{bd}{b} + \frac{b}{bd}\right)J(m+1,x)}{u^2}$$

```
Bth := Bth1+Bth2;
```

$$Bth := \frac{b\left(-\frac{1}{2}\frac{m\,Km\,b}{bd} + \frac{bd\,am}{b}\right)J(m,x)}{u^2\,x} + \frac{1}{2}\frac{b\,Km\left(\frac{bd}{b} + \frac{b}{bd}\right)J(m+1,x)}{u^2}$$

```
Km := 2*Q*a*bd*(1/J(m+1,xa)-1/J(m-1,xa));
```

$$Km := 2\,Q\,a\,bd\left(\frac{1}{J(m+1,xa)} - \frac{1}{J(m-1,xa)}\right)$$

```
am := Q*a*bd*2*m /J(m-1,xa);
```

$$am := 2\frac{Q\,a\,bd\,m}{J(m-1,xa)}$$

```
Bth :
```

Now set in the constants and series for the J function

```
J := (m,z) -> (z/2)^m * sum((-1)^k*(z/2)^(2*k)/(k!*(m+k)!),k=0..1):
x := b*r:
xa := b*a:
bd := u^2/v:
b := c*u:
```

At this point, Bth is a big mess, but if we expand and then simplify,

```
Btha := simplify(expand(Bth,u));
```

we end up with

Btha = -(Q/4)(poly(u) / [u^2 poly(u)]

We can then have it compute

```
Bthau2 := Btha*u^2;
```

and then just set u = 0 to get a result. The result is this:

```
u := 0: factor(Bthau2); unassign('u'):
```

$$-2\frac{Q\left(\frac{r}{a}\right)^{m} m\,(m+1)}{r}$$

.

which is valid only for m = 1,2,3... Notice that constant c does not appear. There are lots of c's sitting in Btha above, but they are always multiplied by powers of u and these go away.

Now in the above I kept the first two terms in the $J_m(x)$ expansion. If I increase to three terms, program is a little slower, but get exact same answer!

The result here is this:

$\omega B_\theta(r,m) = -2Qm(m+1)(r/a)^m (1/r)$

$= -2 (j/4) \eta_m I R_{dc} m(m+1)(r/a)^m (1/r)$

Dimensions seem odd.

LHS = sec^{-1} amp henry/m^2 = sec^{-1} amp ohm-sec /m^2 = amp ohm/m^2 = volt/m^2

RHS = amp (ohm/m) (1/m) = volt/m^2 so OK

So fine, I now have a reliable method of computing things like this in Maple.

I can do the m = 0 result by hand:

$$B_\theta(r,m) = (\beta'/\omega)\{ - (m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'} a_m) x^{-1} J_m(x) + \frac{K_m}{2}(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}) J_{m+1}(x) \}$$

$$B_\theta(r,0) = (\beta'/\omega)\{ - (0\frac{K_0}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'} a_0) x^{-1} J_0(x) + \frac{K_0}{2}(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}) J_1(x) \}$$

$$= (\beta'/\omega)\{ \frac{K_0}{2}(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}) J_1(x) \}$$

$$a_0 = Q(a\beta_d) * 20 \frac{1}{J_{m-1}(x_a)} = 0$$

$$\frac{K_0}{2} = 2\ Q(a\beta_d) * \frac{1}{J_1(x_a)}$$

I added code at the end of the above program and it shows that

$$B_\theta(r,0) = 2c^2 Q r = 2Qr(e^{3\pi j/4}\sqrt{\sigma\mu})^2 = -j2Qr\sigma\mu$$

$$= -j2(j/4)\eta_0 I R_{dc} r\sigma\mu$$

$$= (1/2) I R_{dc} r\sigma\mu = (1/2) I(1/\sigma\pi a^2) r\sigma\mu$$

$$= (\mu I/2\pi)(r/a^2) \qquad \text{// dim(RHS) = henry/m * amp * 1/m = correct!}$$

which says

$$H_\theta = (I/2\pi)(r/a^2)$$

But my usual method says $2\pi r H_\theta = I(r^2/a^2) \Rightarrow H_\theta = (I/2\pi)(r/a^2)$ which is what I got. So that is good news. At least the m = 0 component here does what I expect!

But again, how should I interpret this low frequency result:

$$\omega B_\theta(r,m) = -2(j/4)\eta_m I R_{dc} m(m+1)(r/a)^m (1/r) \qquad m = 1,2,3....$$

It does not even make sense as $\omega \rightarrow 0$!! Something is very wrong as usual.

Dimensions: since curl E = -jωB, I know that dim(ωB) = dim(E)/m = volts/m^2 = dim(LHS)

dim(RHS) = volts/m * 1/m = volts/m^2 so OK

So my question above is really a question about curl E = -jωB . If you have some fixed curl E due to some fixed E field, then you must have $B \rightarrow \infty$ as $\omega \rightarrow 0$. In my case I have

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}}[r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}[r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r]$$

But as $\omega \to 0$, the E_r and E_θ $m > 0$ moments all vanish, but E_z does not, so I guess I have

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}} \left[r^{-1} \partial_\theta E_z \right] + \hat{\boldsymbol{\theta}} \left[- \partial_r E_z \right] = \hat{\mathbf{r}} \left[r^{-1} jm E_z \right] + \hat{\boldsymbol{\theta}} \left[- \partial_r E_z \right]$$

and then I get

$$-j\omega B_\theta = [\text{curl } \mathbf{E}]_\theta = - \partial_r E_z = -\partial_r \{ (1/2) (|m|+1) \eta_m I R_{dc} (r/a)^{|m|} \}$$

$$= -(1/2) (m+1) \eta_m I R_{dc} m (r/a)^{m-1} (1/a)$$

$$= -(1/2) (m+1) \eta_m I R_{dc} m (r/a)^{m} (1/r)$$

$$\Rightarrow \quad \omega B_\theta = - j(1/2) (m+1) \eta_m I R_{dc} m (r/a)^{m} (1/r)$$

which is exactly my result from above

$$\omega B_\theta(r,m) = -2 (j/4) \eta_m I R_{dc} m(m+1)(r/a)^{m} (1/r)$$

so I guess this result is correct. Once again

$$\omega B_\theta(r,m) = - (j/2a) \eta_m I R_{dc} m(m+1)(r/a)^{m-1}$$ dim checks!

But still, if I am trying to take the limit $\omega \to 0$, what does this say? I am totally confused by the interpretation of this result! It really does say that $B_\theta(r,m) \to \infty$. Maybe this is a clue as to my overall problem. What is says is that I need to have $E_z(r,m) \to 0$ or else I have this disaster.

Question: Why is $B_\theta(r,m) = \infty$ as we approach DC ?

I just add this to the list for a while. // Well, in the lines edit log, I think I have an explanation of this fact.

[The Z_0 Escape Hatch says $I \to 0$ and then this problem goes away!]

2. Let's have a look at the z equation itself.

$$[r^2 \partial_r^2 + r \partial_r + (r^2 \beta'^2 - m^2)] E_z(r,m) = 0 \qquad (D.2.1)$$

This nice thing is that this stands entirely alone, no entanglement with E_r and E_φ. Just suppose (not justified) that for low ω we could neglect the β'^2 term to get

$$[r^2 \partial_r^2 + r \partial_r - m^2] E_z(r,m) = 0 \qquad (D.2.1)$$

Maple says

```
eq := r^2*diff(E(r),r,r) + r*diff(E(r),r) - m^2*E(r);
```

$$eq := r^2\left(\frac{\partial^2}{\partial r^2}\mathrm{E}(r)\right) + r\left(\frac{\partial}{\partial r}\mathrm{E}(r)\right) - m^2\,\mathrm{E}(r)$$

```
dsolve(eq);
```

$$\mathrm{E}(r) = _C1\,\sinh(m\,\ln(r)) + _C2\,\cosh(m\,\ln(r))$$

For m ≠ 0, the term cosh(mln(r)) is like $e^{mlnr} + e^{-mlnr} = r^m + r^{-m}$. These are the real solutions and we have to throw out r^{-m} for m>0 so we basically have

$E_z(r,m) = K\ r^m$

and this in fact agrees with the result I could above which was

$E_z(r,m) = (1/2)\,(|m|+1)\,\eta_m\,I\,R_{dc}\,(r/a)^{|m|}$ [no change here despite β_d change due to loss]

[This last method is quite convincing! If you go right to the bare bones Helmholtz equation for $E_z(r,m)$, you find that the m > 0 solutions do not vanish as ω→0 and you have the asymmetry, apart from the escape hatch fact that I → 0.]

[Basically as β'→0, the Helm $(\nabla^2+\beta'^2)E_z = 0$ equation becomes Laplace $(\nabla^2)E_z(r,\theta) = 0$ and we know the atom forms for this equation to be [$r^{\pm m}$, $e^{\pm im\theta}$] and this is what we are seeing here!]