Altering the CPBC for radial Hall PhL 6.25.14

An earlier doc alludes to pondering the radial Hall as the fall guy for low ω problems. Here I check it out and conclude that it is not the fall guy. I quote from below (page 12 or 30):

Conclusion: If I have done the above correctly, then adding the radial Hall effect to the CPBC does not rescue that J_z asymmetry paradox! It was just another idea of many that did not pan out.

I then carry on with 20 pages of flailing away about G = 0 and G > 0 cases and wondering about the magnetostatic problem. I think those situations are well handled now in lines doc.

I think this is the cause of the asymmetry paradox. I think it very unlikely that this attempt will succeed [correct, it did not succeed] but it is worth a try because it is exactly at the right place in the machine where I am having trouble. The n(θ) shape is "getting into" J_z through this boundary condition. I know at at ω = 0 we have $J_r = 0$ and $E_r \neq 0$, so maybe near ω = 0 this idea will somehow help.

The main issue here is trying to use the tensor magnetic Ohm's law in place of $J_r = \sigma E_r$. The BC I think is OK as is in terms of J_r.

(a) obtain the appropriate magnetic Ohm's law for the round wire situation

The magnetic Ohm's law discussion starts in Section N.4. There I was using a static version, so perhaps the dynamic version is this

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + q\mathbf{v}x\mathbf{B} - (m/\tau)\,\mathbf{v} \quad . \qquad (N.1.5)$$

$$mj\omega\mathbf{v} = q\mathbf{E} + q\mathbf{v}x\mathbf{B} - (m/\tau)\,\mathbf{v} \quad .$$

or

$$m[j\omega + 1/\tau]\,\mathbf{v} = q\mathbf{E} + q\mathbf{v}x\mathbf{B}$$

I know that $1/\tau$ is about 10^{14} from (N.1.12). Even at 100 GHz we get

$$\omega = 2\pi \times 10^{11} \sim 10^{13}$$

and I am willing to ignore the jω term for my interests. Thus,

$$(m/\tau)\,\mathbf{v} \approx q\mathbf{E} + q\mathbf{v}x\mathbf{B}$$

This is the same as (N.4.1) and I have done all the solution work in Section N.7 with this result

$$J_z = \sigma\,(E_z + \omega_c\tau\,E_r)\,\frac{1}{1+(\omega_c\tau)^2} \qquad \omega_c \equiv (qB/m)$$

$$J_r = \sigma (E_r - \omega_c\tau\, E_z) \frac{1}{1+(\omega_c\tau)^2} \qquad \mathbf{B} = B\,\hat{\boldsymbol{\theta}}$$

$$J_\theta = \sigma E_\theta \ . \qquad \sigma = (nq^2\tau/m) \qquad \text{(N.7.8)}$$

where this makes use of $\mathbf{B}(r) = B(r)\,\hat{\boldsymbol{\theta}}$ which I will assume for the round wire!

So this is our magnetic Ohm's law of interest even at some frequency ω. I know $\omega_c\tau$ is small, but I want to keep it for a while. My next step in radial Hall effect is to say $J_r = 0$ for the static situation, but I certainly DON'T want to say that here. Notice that if I set $\omega_c\tau = 0$, then we just have the regular Ohm's law and they we are not fixing anything for Appendix D.

The above equation is local to any point **r** inside the round wire.

(b) What is the new boundary condition for Appendix D ?

$$J_r(r=a-\varepsilon,\theta) = j\omega\, n(\theta) \qquad \text{non-conducting dielectric} \qquad \text{(D.2.23)}$$

This then says

$$[E_r(a,\theta) - \omega_c(a)\,\tau\, E_z(a,\theta)] \frac{1}{1+(\omega_c\tau)^2} = (j\omega/\sigma)\, n(\theta)$$

and this IS a different boundary condition. I guess I really need to add this:

$$\omega_c(r) = (qB(r)/m)$$

I could at this point install B(r) for a uniform J_z round wire. But no matter how the current is distributed in the wire, B(a) will be the same! Thus without any approximation we can use

$$B(r) = \frac{\mu_0 I}{2\pi a}\,(r/a) \qquad \Rightarrow \qquad B(a) = \frac{\mu_0 I}{2\pi a}$$

$$\Rightarrow\ \omega_c(a) = (q/m)\frac{\mu_0 I}{2\pi a}$$

$$\omega_c(a)\tau = (q\tau/m)\frac{\mu_0 I}{2\pi a} = \frac{\mu\mu_0 I}{2\pi a} \qquad \mu = \text{mobility} \qquad \mu_0 = \text{the usual}$$

Now let's confirm that this is very small

$$\frac{\mu\mu_0 I}{2\pi a} \qquad \text{dim} = \text{tesla}^{-1}\ \text{henry/m amp / m}$$

$$= m^2\ \text{amp}^{-1}\text{henry}^{-1} * \text{henry/m amp / m} = \text{dimensionless as expected}$$

$$\frac{\mu\mu_0 I}{2\pi a}$$

$\mu = (q\tau/m) = 1.6 \times 10^{-19} * 10^{-14} / [9.1 \times 10^{-31}]$

$= 1.6/9.1 * 10^{-19-14+31} = 0.17 \times 10^{-2}$ tesla^{-1}

Then

$$\frac{\mu\mu_0 I}{2\pi a} = 0.17 \times 10^{-2} \times 4\pi \times 10^{-7} / 2\pi * (I/a) = 0.17 \times 10^{-2} \times 2 \times 10^{-7} * (I/a)$$

$= .34 \times 10^{-9}$ I/a

Example 1: I = 1000 amps and a = 10^{-2} m I/a = 10^{5} $\frac{\mu\mu_0 I}{2\pi a} = .34 \times 10^{-4}$

Example 2: I = 1 amp and a = 400μ = 400x10^{-6} = 4 x 10^{-4} I/a = 1/ [4 x 10^{-4}] ~ 10^{3}

So in either of these wildly different cases, we have $\frac{\mu\mu_0 I}{2\pi a} << 1$, so I will now simplify our new boundary condition and just say

$$[E_r(a,\theta) - \frac{\mu\mu_0 I}{2\pi a} E_z(a,\theta)] = (j\omega/\sigma)\, n(\theta)$$

Notice that the mixture on the left is the same at any ω of interest. I am not yet sure what affect this would have on the solution in Appendix D!

<u>Does this thing make any sense at DC?</u> Assume uniform E_z. Then it says

$$E_r(a,\theta) = \frac{\mu\mu_0 I}{2\pi a} E_z(a,\theta)$$

At DC this says

$$E_r(a,\theta) = \frac{\mu\mu_0 I}{2\pi a} (1/\sigma) J_z(a,\theta) = \frac{\mu\mu_0 I}{2\pi a} (1/\sigma) (I/\pi a^2)$$

Now we know that

$\mu = q\tau/m$ $\sigma = (nq^2\tau/m)$ $\mu/\sigma = q\tau/m * m/(nq^2\tau) = 1/nq$

so then

$$E_r(a,\theta) = \frac{\mu\mu_0 I}{2\pi a}\ (1/\sigma)\ (I/\pi a^2) = \frac{\mu_0 I}{2\pi a}\ 1/nq\ \ (I/\pi a^2)$$

$$= \frac{\mu_0 I^2}{2\pi^2 a^3 nq} \qquad = E_z \text{ of (N.7.12)}$$

So at DC the result of this BC agrees exactly with the radial Hall effect, I like it so far!

(c) What is the impact on Appendix D?

First write

$$E_r(a,m) - \frac{\mu\mu_0 I}{2\pi a}\ E_z(a,m) = (j\omega/\sigma)\ N_m$$

$$E_\theta(r=a,m) = 0$$

First summary of the E field solutions (D.2.21)

$$E_z(r,m) = -j\ (\beta'/\beta_d)\ \frac{K_m}{2}\ J_m(x) \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(r,m) = a_m\ x^{-1}\ J_m(x) + \frac{K_m}{2}\ J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad \textit{(D.2.11)}$$

$$jE_\theta(r,m) = -a_m\ x^{-1}\ J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad \textit{(D.2.15)}$$

Things are going to be different now! I don't know how much different. Define

$$\frac{\mu\mu_0 I}{2\pi a} = \omega_c(a)\tau = \kappa$$ just for short, a small dimensionless parameter!

Then

$$E_r(a,m) - \kappa\ E_z(a,m) = (j\omega/\sigma)\ N_m$$

$$E_\theta(r=a,m) = 0$$

and we get

$$a_m\ x_a^{-1}\ J_m(x_a) + \frac{K_m}{2}\ J_{m+1}(x_a)\ -\kappa\left[-j\ (\beta'/\beta_d)\ \frac{K_m}{2}\ J_m(x_a)\right] = (j\omega/\sigma)\ N_m$$

$$-a_m\ x_a^{-1}\ J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a)$$

where the new stuff is shown in red. Write again now

$$a_m\ x_a^{-1}\ J_m(x_a) + \frac{K_m}{2}\ J_{m+1}(x_a)\ + \kappa j\ (\beta'/\beta_d)\ \frac{K_m}{2}\ J_m(x_a)\ =\ (j\omega/\sigma)\ N_m$$

$$-a_m\ x_a^{-1}\ J_m(x_a) + (\ \frac{K_m}{2}\ + \frac{a_m}{m}\ )\ J_{m+1}(x_a)$$

$$[\ J_{m+1}(x_a) + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)\ ]\ \frac{K_m}{2}\ +\ [x_a^{-1}\ J_m(x_a)]\ a_m\ = (j\omega/\sigma)\ N_m$$

$$[J_{m+1}(x_a)]\frac{K_m}{2}\ +\ [-x_a^{-1}\ J_m(x_a) + J_{m+1}(x_a)/m]\ a_m\ =\ 0$$

or

$$[\ J_{m+1}(x_a) + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)\ ]\ \frac{K_m}{2}\ +\ [x_a^{-1}\ J_m(x_a)]\ a_m\ = (j\omega/\sigma)\ N_m$$

$$[m\ J_{m+1}(x_a)]\frac{K_m}{2}\ +\ [-m\ x_a^{-1}\ J_m(x_a) + J_{m+1}(x_a)]\ a_m\ =\ 0$$

But right here we can use

NIST (10.6.1) Bessel identity $\qquad (2m/x)J_m(x)\ = J_{m-1}(x) + J_{m+1}(x).$

$$(m/x)J_m(x)\ = (J_{m-1}(x) + J_{m+1}(x))/2$$

which says

$$-mx_a^{-1}\ J_m(x_a) + J_{m+1}(x_a) = -\ (J_{m-1}(x) + J_{m+1}(x))/2 + J_{m+1}(x_a)\ = -\ (J_{m-1} - J_{m+1})/2 = (J_{m+1} - J_{m-1})/2$$

so our two equations are then

$$[\ J_{m+1}(x_a) + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)\ ]\ \frac{K_m}{2}\ +\ [x_a^{-1}\ J_m(x_a)]\ a_m\ = (j\omega/\sigma)\ N_m$$

$$[m\ J_{m+1}(x_a)]\frac{K_m}{2}\ +\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ a_m\ =\ 0$$

$$A\frac{K_m}{2} + B\ a_m = C$$

$$D\frac{K_m}{2} + E\ a_m\ = 0$$

$$A = [J_{m+1}(x_a) + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)]$$
$$B = x_a^{-1}\ J_m(x_a)$$
$$C = (j\omega/\sigma)\ N_m$$
$$D = m\ J_{m+1}(x_a)$$

$$E = (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]$$

Now let Maple solve the two equations:

```
eq1 := A*Km2 + B*am = C;
```

$$eq1 := A\,Km2 + B\,am = C$$

```
eq2 := D*Km2+E*am = 0;
```

$$eq2 := D\,Km2 + E\,am = 0$$

```
solve({eq1,eq2},{Km2,am});
```

$$\{am = \frac{C\,D}{-E\,A + B\,D}, Km2 = -\frac{E\,C}{-E\,A + B\,D}\}$$

So our result is a bit complicated

$$a_m = CD/den = (j\omega/\sigma)\, N_m\, mJ_{m+1}(x_a)\ /den$$

$$\frac{K_m}{2} = -\,CE/den = -\,(j\omega/\sigma)\, N_m\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]/den$$

$$= -\,(j\omega/2\sigma)\, N_m\,[\ J_{m+1}(x_a) - J_{m-1}(x_a)]/den$$

$$den = BD-EA = x_a^{-1}\, J_m(x_a)\, m\, J_{m+1}(x_a)\ -\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]\,[J_{m+1}(x_a) + \kappa j\ (\beta'/\beta_d)\, J_m(x_a)]$$

Note that if $\kappa = 0$ we get

$$den = x_a^{-1}\, J_m(x_a)\, m\, J_{m+1}(x_a)\ -\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]\,[J_{m+1}(x_a)]$$

$$= J_{m+1}(x_a)[\ x_a^{-1}\, J_m(x_a)\, m\ -\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ ]$$

$$= (1/2)\, J_{m+1}(x_a)[\ 2mx_a^{-1}\, J_m(x_a) - J_{m+1}(x_a) + J_{m-1}(x_a)\ ]$$

$$= (1/2)\, J_{m+1}(x_a)[\ J_{m-1}(x) + J_{m+1}(x) - J_{m+1}(x_a) + J_{m-1}(x_a)\ ]$$

$$= (1/2)\, J_{m+1}(x_a)[\ 2J_{m-1}(x_a)\ ]$$

$$= J_{m+1}(x_a)J_{m-1}(x_a)$$

In this case we get

$$a_m = (j\omega/\sigma)\, N_m\, mJ_{m+1}(x_a)\ /den = (j\omega/\sigma)\, N_m\, mJ_{m+1}(x_a)/\ [J_{m+1}(x_a)J_{m-1}(x_a)]$$

$$= (j\omega/\sigma)\, N_m\, m/\ J_{m-1}(x_a)]$$

and this does agree. Next,

$$\frac{K_m}{2} = -(j\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]/ \left[J_{m+1}(x_a)J_{m-1}(x_a)\right]$$

$$= -(j\omega/2\sigma)\, N_m \left\{1/ J_{m-1}(x_a) - 1/ J_{m+1}(x_a)\right\}$$

and this also agrees.

Now without further ado, I can write

$$E_z(r,m) = -j\,(\beta'/\beta_d)\, J_m(x)\, \frac{K_m}{2}$$

$$= -j\,(\beta'/\beta_d)\, J_m(x)\left[-(j\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]/den\right]$$

$$= j\,(\beta'/\beta_d)\, J_m(x)\left[(j\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]/den\right]$$

$$= -(\beta'/\beta_d)\, J_m(x)\left[(\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]/den\right]$$

$$= -(\beta'/\beta_d)\, J_m(x)\left[(\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right.$$

$$\left./ \left\{ x_a^{-1} J_m(x_a)\, m\, J_{m+1}(x_a) - (1/2)\left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\left[J_{m+1}(x_a) + \kappa j\,(\beta'/\beta_d)\, J_m(x_a)\right]\right\}$$

Now for $\kappa = 0$ this reads

$$E_z(r,m) = -(\beta'/\beta_d)\, J_m(x)\left[(\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right.$$

$$\left./ \left\{ x_a^{-1} J_m(x_a)\, m\, J_{m+1}(x_a) - (1/2)\left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\left[J_{m+1}(x_a)\right]\right\}$$

$$= -(\beta'/\beta_d)\, J_m(x)\left[(\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right.$$

$$\left.\left(1/ J_{m+1}(x_a)\right)\right.$$

$$/ \left\{ x_a^{-1} J_m(x_a)\, m - (1/2)\left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right\}$$

$$= -(\beta'/\beta_d)\, J_m(x)\left[(2\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right.$$

$$\left.\left(1/ J_{m+1}(x_a)\right)\right.$$

$$/ \left\{ 2x_a^{-1} J_m(x_a)\, m - \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right\}$$

$$= -(\beta'/\beta_d)\, J_m(x)\left[(2\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right.$$

$$\left.\left(1/ J_{m+1}(x_a)\right)\right.$$

$$/ \left\{ 2x_a^{-1} J_m(x_a)\, m - J_{m+1}(x_a) + J_{m-1}(x_a)\right\}$$

$$= -(\beta'/\beta_d)\, J_m(x)\left[(2\omega/2\sigma)\, N_m \left[J_{m+1}(x_a) - J_{m-1}(x_a)\right]\right.$$

$$\left.\left(1/ J_{m+1}(x_a)\right)\right.$$

$$/ \left\{ J_{m-1}(x) + J_{m+1}(x) - J_{m+1}(x_a) + J_{m-1}(x_a)\right\}$$

$= -(\beta'/\beta_d)\, J_m(x)[\ (2\omega/2\sigma)\, N_m\, [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\, (1/\ J_{m+1}(x_a))$

$/\ \{2J_{m-1}(x)\ \}$

$= -(\beta'/\beta_d)\, J_m(x)[\ (\omega/2\sigma)\, N_m\, [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\, (1/\ J_{m+1}(x_a)\ J_{m-1}(x))$

$= -(\beta'/\beta_d)\, J_m(x)[\ (\omega/2\sigma)\, N_m\, [1/\ J_{m-1}(x_a)\ -\ 1/\ J_{m+1}(x_a)\ ]$

$= (\beta'/\beta_d)\, J_m(x)(\omega/2\sigma)\, N_m\, [1/\ J_{m+1}(x_a)\ -\ 1/\ J_{m-1}(x_a)\ ]$

$= (\beta'/\beta_d)\, (\omega/2\sigma)\, N_m\, f_m\ = (\beta'/\beta_d)\, (\omega/\sigma)\, \eta_m\, N_0\ f_m$

$= (\beta'/\beta_d)\, (\omega/2\sigma)\, \eta_m\ (\beta_d/2\pi\omega a)\, I\ f_m$

$= \beta'\, (\omega/2\sigma)\, \eta_m\ (1/2\pi\omega a)\, I\ f_m$ $\qquad\qquad R_{dc} = \dfrac{1}{\sigma\pi a^2}$

$= \beta'\, \omega\, \pi a^2 R_{dc}\, \eta_m\ (1/4\pi\omega a)\, I\ f_m$

$= (\beta' a)\ R_{dc}\, \eta_m\ (1/4)\, I\ f_m$

which is correct.

So having done all these tests, here is my new E_z when the radial Hall effect is included:

$$E_z(r,m) = (\beta'/\beta_d)\, J_m(x)(\omega/2\sigma)\, N_m\, [\ J_{m-1}(x_a) - J_{m+1}(x_a)]$$

$$/\ \{\ x_a^{-1}\, J_m(x_a)\, m\, J_{m+1}(x_a)\ -\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]\, [J_{m+1}(x_a) + \kappa j\, (\beta'/\beta_d)\, J_m(x_a)]\ \}$$

$$= (\beta'/\beta_d)\, (\omega/2\sigma)\, N_m\, J_m(x)\ * \frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ x_a^{-1}\, J_m(x_a)\, m\, J_{m+1}(x_a)\ -\ (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]\, [J_{m+1}(x_a) + \kappa j\, (\beta'/\beta_d)\, J_m(x_a)]\ \}}$$

Say it again Sam:

$$E_z(r,m) = (\beta'/\beta_d)\, (\omega/\sigma)\, N_m\, J_m(x)\ * \frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ 2m x_a^{-1}\, J_m(x_a)\ J_{m+1}(x_a)\ -\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\, [J_{m+1}(x_a) + \kappa j\, (\beta'/\beta_d)\, J_m(x_a)]\ \}}$$

Now suppose I use the known fact that κ is small to write

$$A = [J_{m+1}(x_a) + \kappa j\, (\beta'/\beta_d)\, J_m(x_a)]\ = J_{m+1}(x_a)\ [\ 1 + \kappa j\, (\beta'/\beta_d)\, J_m(x_a)/\ J_{m+1}(x_a)\ ]$$

Then the big ratio thing becomes,

$$\frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ 2mx_a^{-1}\ J_m(x_a)\ \ J_{m+1}(x_a)\ -\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ [J_{m+1}(x_a)\ [\ 1 + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)/\ J_{m+1}(x_a)\ ]\ \}}$$

=

$$\frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ 2mx_a^{-1}\ J_m(x_a)\ \ -\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ [[\ 1 + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)/\ J_{m+1}(x_a)\ ]\ \}} \ *\ (1/J_{m+1}(x_a))$$

The denominator of this ratio can be written

$$\{\ 2mx_a^{-1}\ J_m(x_a)\ \ -\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ [\ 1 + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)/\ J_{m+1}(x_a)\ ]$$

$$= \{\ 2mx_a^{-1}\ J_m(x_a)\ \ -\ \ J_{m+1}(x_a) + J_{m-1}(x_a)]\ + \kappa j\ (\beta'/\beta_d)\ [J_m(x_a)/\ J_{m+1}(x_a)]\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]$$

$$= \{\ J_{m-1}(x_a) + J_{m+1}(x_a)\ \ -\ \ J_{m+1}(x_a) + J_{m-1}(x_a)]\ + \kappa j\ (\beta'/\beta_d)\ [J_m(x_a)/\ J_{m+1}(x_a)]\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]$$

$$= \{\ 2J_{m-1}(x_a)\ +\ \kappa j\ (\beta'/\beta_d)\ [J_m(x_a)/\ J_{m+1}(x_a)]\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ \}$$

$$= 2J_{m-1}(x_a)\ \{\ 1 +\ \kappa j\ (\beta'/\beta_d)\ [J_m(x_a)/\ J_{m+1}(x_a)]\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ /\ [2J_{m-1}(x)\ \}$$

$$= 2J_{m-1}(x_a)\ \{\ 1 +\ \kappa j\ (\beta'/\beta_d)\ [J_m(x_a)/\ 2J_{m+1}(x_a)\ J_{m-1}(x_a)]\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ \}$$

Now write

$$[\ J_{m+1}(x_a) - J_{m-1}(x_a)]/\ J_{m+1}(x_a)\ J_{m-1}(x_a)$$

$$= 1/\ J_{m-1}(x_a)\ -\ 1/\ J_{m+1}(x_a)\ \ =\ -\ (1/J_m(x_a))\ [\frac{J_m(x)}{J_{m+1}(x_a)}\ -\ \frac{J_m(x)}{J_{m-1}(x_a)}\ ]$$

$$= -\ (1/J_m(x_a))\ f_m(x_a)\ \ = -\ f_m(x_a)/\ J_m(x_a)$$

and then

$$J_m(x_a)\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]/\ J_{m+1}(x_a)\ J_{m-1}(x_a)\ = f_m(x_a)$$

Then the denominator above can be written

$$= 2J_{m-1}(x_a)\ \{\ 1 +\ (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}$$

Then the Big Ratio becomes this

$$(1/J_{m+1}(x_a))\frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{2J_{m-1}(x_a)\ \{\ 1 +\ (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}}$$

$$= (1/2)[\ J_{m+1}(x_a) - J_{m-1}(x_a)]/\ J_{m+1}(x_a)\ J_{m-1}(x_a) * \frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ 1 + (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}}$$

$$= (1/2)\ \{ - f_m(x_a)/\ J_m(x_a)\} * \frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ 1 + (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}}$$

But now in the numerator

$$J_m(x_a)\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]/\ J_{m+1}(x_a)\ J_{m-1}(x_a) = f_m(x_a)$$

so

$$[\ J_{m+1}(x_a) - J_{m-1}(x_a)]/\ J_{m+1}(x_a)\ J_{m-1}(x_a) = f_m(x_a)/\ J_m(x_a)$$

$$[\ J_{m+1}(x_a) - J_{m-1}(x_a)] = [\ J_{m+1}(x_a)\ J_{m-1}(x_a) / J_m(x_a)\ ]\ f_m(x_a)$$

So now the Big Ratio becomes

$$= (1/2)\ \{\ f_m(x_a)/\ J_m(x_a)\} * \frac{[\ J_{m+1}(x_a)\ J_{m-1}(x_a) / J_m(x_a)\ ]\ f_m(x_a)}{\{\ 1 + (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}}$$

$$= (1/2)\ \{\ [f_m(x_a)/\ J_m(x_a)]^2\} * \frac{[\ J_{m+1}(x_a)\ J_{m-1}(x_a)]}{\{\ 1 + (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}}$$

I seem now to get

$$E_z(r,m) = (\beta'/\beta_d)\ (\omega/\sigma)\ N_m\ J_m(x)\ * (1/2)\ \{\ [f_m(x_a)/\ J_m(x_a)]^2\} * \frac{[\ J_{m+1}(x_a)\ J_{m-1}(x_a)]}{\{\ 1 + (1/2)\ \kappa j\ (\beta'/\beta_d) f_m(x_a)\ \}}$$

But this seems to have the wrong limit for $\kappa \rightarrow 0$, since the correct limit from above is this

$$= (\beta'/\beta_d)\ J_m(x)(\omega/2\sigma)\ N_m\ [1/\ J_{m+1}(x_a)\ -\ 1/\ J_{m-1}(x_a)\ ]$$

$$= (\beta'/\beta_d)\ J_m(x)(\omega/2\sigma)\ N_m\ (J_{m-1}(x_a) - J_{m+1}(x_a))/[\ J_{m+1}(x_a)\ J_{m-1}(x_a)]$$

OK, I have had enough. This is really a task for Maple.

The Big Question: Does having the small κ term present in the E_z solution somehow make us symmetric in the DC limit?

$$E_z(r,m) = (\beta'/\beta_d)\ (\omega/\sigma)\ N_m\ J_m(x)\ * \frac{[\ J_{m-1}(x_a) - J_{m+1}(x_a)]}{\{\ 2m x_a^{-1}\ J_m(x_a)\ J_{m+1}(x_a)\ -\ [\ J_{m+1}(x_a) - J_{m-1}(x_a)]\ [J_{m+1}(x_a) + \kappa j\ (\beta'/\beta_d)\ J_m(x_a)]\ \}}$$

The κ term can only be useful if (β'/β_d) is large, I would say.

The verdict is still out on whether this can somehow yield

$E_z(r,m) \sim \delta_{m,0}$ (stuff) as $\omega \to 0$

Today it was all math. Tomorrow maybe I can ponder what this means. It just seems unlikely that having that little κ there can make any difference.

June 26, 2014

I am going to switch to a Maple approach. First I set κ = 0 and have Maple solve the BC's for a_m and K_m. That was quite easy, file is "solve Km and am v2.mws".

I work first with κ = 0 hoping to reproduce old results. I find

$$am_ := m \to -\frac{j\,\omega\,N0\,\eta(m)\,xa\,m}{\sigma\,(-2\,J(m,xa)\,m + J(m+1,xa)\,xa)}$$

$$Km2_ := m \to \frac{j\,\omega\,N0\,\eta(m)\,(-J(m,xa)\,m + J(m+1,xa)\,xa)}{\sigma\,(-2\,J(m,xa)\,m + J(m+1,xa)\,xa)\,J(m+1,xa)}$$

```
rat10 := Ez(1)/Ez(0);
```

$$rat10 := \frac{\eta(1)\,(-J(1,xa) + J(2,xa)\,xa)\,J(1,x)\,J(1,xa)}{(-2\,J(1,xa) + J(2,xa)\,xa)\,J(2,xa)\,\eta(0)\,J(0,x)}$$

I used Q = (β'/β_d') in the E_z expression, but this cancelled out in the ratio.

Now if $\eta_1 \neq 0$, the ratio of the m=1 field to the m=0 field is as shown. If I make no assumption at all about β', then I cannot plot anything since x = β'r . But I can see that rat10 ≠ 0. Thus, if the m = 0 E_z component is present, then so is the m = 1 component, and this is my Paradox at ω = 0.

Now I will rerun the program adding in the κ term! I now avoid doing all the manual algebra above and I just let Maple do it. Here are some results where again Q = (β'/β_d') :

```
am_ := unapply(rhs(s[N]),m);
```

$$am_ := m \to -\frac{xa\,\eta(m)\,N0\,\omega\,J(m+1,xa)\,m}{\sigma\,(2j\,J(m,xa)\,J(m+1,xa)\,m - j\,J(m+1,xa)^2\,xa - J(m,xa)^2\,Q\,\kappa\,m + J(m,xa)\,Q\,\kappa\,J(m+1,xa)\,xa)}$$

```
Km2_ := unapply(rhs(s[M]),m);;
```

$$Km2_ := m \to \frac{\eta(m)\,N0\,\omega\,(-J(m,xa)\,m + J(m+1,xa)\,xa)}{\sigma\,(2j\,J(m,xa)\,J(m+1,xa)\,m - j\,J(m+1,xa)^2\,xa - J(m,xa)^2\,Q\,\kappa\,m + J(m,xa)\,Q\,\kappa\,J(m+1,xa)\,xa)}$$

.

Perhaps my hand computations agree with these results, they at least look vaguely similar.

I then enter E_z as usual in terms of these constants and I compute this ratio rat10 for m=1 over m=1 E_z:

$$\frac{\eta(1)\,(-J(1,xa)+J(2,xa)\,xa)\,J(1,x)\,(J(1,xa)+j\,\kappa\,Q\,J(0,xa))}{(-2\,J(1,xa)\,J(2,xa)+J(2,xa)^2\,xa-j\,\kappa\,Q\,J(1,xa)^2+j\,\kappa\,Q\,J(1,xa)\,J(2,xa)\,xa)\,\eta(0)\,J(0,x)}$$

Now here is the Question: is there some assumption I can make about Q at small ω which causes the above ratio to be zero? I know that at DC, the field E_z(r,m=1) has to be completely gone and I also know that the component E_z(r,m=0) has to be a constant, so the ratio has to go to 0.

If I assume that Q → 0, I get this result :

```
factor(rat10);
```

$$\frac{\eta(1)\,(-J(1,xa)+J(2,xa)\,xa)\,J(1,x)\,J(1,xa)}{J(2,xa)\,(-2\,J(1,xa)+J(2,xa)\,xa)\,\eta(0)\,J(0,x)}$$

This ratio is NOT zero because the numerator is not zero for all x certainly. So that assumption about Q does not solve the problem. In fact, it just replicates the κ = 0 result shown earlier.

If I assume that Q → ∞, I get this result :

```
limit(rat10,Q = infinity);
```

$$\frac{J(0,xa)\,\eta(1)\,J(1,x)}{J(0,x)\,J(1,xa)\,\eta(0)}$$

This result is independent of κ and is also NOT zero so this assumption does not resolve Paradox!

The only other choice is that Q→Q_0, some constant. They I get the full rat10 shown above with Q_0. I would then need the numerator to be zero. That would require the following expression be 0 :

$$-j\,J(1,xa)+J(0,xa)\,Q\,\kappa$$

since it appears in the numerator of the mess above, and no other numerator factors cancel.

This would require that, as ω→ 0,

$$Q\,\kappa = j\,J_1(x_a)/\,J_0(x_a)$$

which is to say

$$(\beta'/\beta_d')\,\frac{\mu\mu_0 I}{2\pi a} = j\,J_1(x_a)/\,J_0(x_a)$$

But this makes no sense at all. We might have(β'/β_d') $\rightarrow$ 0, constant, or ∞ based on our choice of loss model. Certainly we can rule out the 0 and ∞ cases. The constant case would require that the constant ratio be proportional to 1/I and I no model which that gives that.

Conclusion: If I have done the above correctly, then adding the radial Hall effect to the CPBC does not rescue that J_z asymmetry paradox! It was not another idea of many that did not pan out.

Let's now go back to the modified CPBC :

$$E_r(a,\theta) - \frac{\mu\mu_0 I}{2\pi a} E_z(a,\theta) = (j\omega/\sigma)\, n(\theta) \qquad \kappa = \frac{\mu\mu_0 I}{2\pi a}$$

Here are arguments why this does NOT fix the Paradox.

1. The problem originally with

$$E_r(a,\theta) = (j\omega/\sigma)\, n(\theta)$$

was that n(θ) "got into" E_r and then into E_z . The new BC does not really stop n(θ) from getting into E_r and E_z. You might imagine some delicate balance that caused a magic cancellation. but that would depend on I.

2. Also, since I $\rightarrow$ 0 in our limit $\omega \rightarrow 0$, the BC is not really changed at all.

3. At any finite I, this is basically true for any I since $\kappa << 1$.

One More Check. The modified BC is really this:

$$[E_r(a,\theta) - (\omega_c(a)\tau)\, E_z(a,\theta)] \frac{1}{1+(\omega_c\tau)^2} = (j\omega/\sigma)\, n(\theta)$$

where

$$(\omega_c(a)\tau) = \frac{\mu\mu_0 I}{2\pi a} = \kappa$$

so to be precise I should be writing the modified BC this way

$$[E_r(a,\theta) - \kappa\, E_z(a,\theta)] \frac{1}{1+\kappa^2} = (j\omega/\sigma)\, n(\theta)$$

$$[E_r(a,\theta) - \kappa\, E_z(a,\theta)] = (j\omega/\sigma)\, n(\theta)(1+\kappa^2)$$

$$[E_r(a,m) - \kappa\, E_z(a,m)] = (j\omega/\sigma) N_m (1+\kappa^2)$$

If I do this in my Maple program, what happens? It just replaces N_0 by $N_0(1+\kappa^2)$ and then it cancels out in the ratio rat10 and so it makes no difference whatsoever.

Another Mystery.

The CPBC says

$$E_r(r=a,m) = (j\omega/\sigma)\, N_m$$

Thus, it seems that you MUST get

$$E_r(r=a,m) = 0 \quad \text{when } \omega = 0$$

But the general solution at small ω is this

$$E_r(r,m) = (j/4)\, \eta_m\, I\, R_{dc}\, (a\beta_d')\, [(r/a)^{m+1} + (r/a)^{m-1}]$$

$$E_r(a,m) = (j/2)\, \eta_m\, I\, R_{dc}\, (a\beta_d')$$

so isn't this a conflict? Not if I = 0, the same escape hatch. But I also have

$$I = 2\pi\omega\, (a/\beta_d')\, N_0$$

and then the above ways

$$E_r(a,m) = (j/2)\, \eta_m\, 2\pi\omega\, (a/\beta_d')\, N_0\, R_{dc}\, (a\beta_d')$$

$$= (j/2)\, \eta_m\, 2\pi\omega\, a^2 N_0\, R_{dc} \qquad (1)$$

$$= (j/2)\, N_m\, 2\pi\omega\, a^2\, R_{dc}$$

and now our conflict goes away.

I suppose this is the same as doing this

$$I = V/Z_0 \qquad Z_0 = \sqrt{R/j\omega C}$$

Then you get

$$E_r(a,m) = (j/2)\, \eta_m\, V \sqrt{j\omega C/R}\, R_{dc}\, (a\beta_d')$$

But why does this give a different looking result? Suppose I now use this App Q fact

$$\beta'_d = \sqrt{\omega RC/2}\, (1-j)$$

The result is then

$$E_r(a,m) = (j/2)\,\eta_m\, V \sqrt{j\omega C/R}\; R_{dc}\; (a\sqrt{\omega RC/2}\;(1-j))$$

$$= (j/2)\,\eta_m\, V\, C\, \omega \sqrt{j}\,(1-j)/\sqrt{2}\;\; a\, R_{dc} = (j/2)\,\eta_m\, V\, C\, \omega \sqrt{j}\; e^{-j\pi/4} a\, R_{dc}$$

$$= (j/2)\,\eta_m\, V\, C\, \omega\, a\, R_{dc}\, e^{-j\pi/4} e^{-j\pi/4} = (j/2)\,\eta_m\, V\, C\, \omega\, a\, R_{dc}$$

$$= (j/2)\,\eta_m\, q\, \omega\, a\, R_{dc} \qquad\qquad q = CV \qquad\qquad (2)$$

Compare this to result (1) above

$$E_r(a,m) = (j/2)\,\eta_m\; 2\pi\omega\, a^2 N_0\, R_{dc}$$

and recall that

$$N_0 = (1/2\pi a)\, q(0) \qquad\qquad (D.1.8)$$

so this last result is then

$$E_r(a,m) = (j/2)\,\eta_m\; 2\pi\omega\, a^2 (1/2\pi a)\, q(0)\, R_{dc}$$

$$= (j/2)\,\eta_m\;\, \omega\, a\, q(0)\, R_{dc} \qquad\qquad (3)$$

But this is the same as result (2), thank goodness. This shows that App Q really is right in this limit, otherwise we would not get the exact result.

Maybe things are clearer if I get rid of I and make this replacement

$$I = 2\pi\omega\, (a/\beta_d')\, N_0 \;\; = \;\; 2\pi\omega\, (a/\beta_d')\, (1/2\pi a)\, q(0) = \; [\omega\, q\, (1/\beta_d')\,]$$

This replacement is valid at any ω. Then write

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\; (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\; (a\beta_d')\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\; (a\beta_d')\;\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_z(r,m) = (1/4)\,\eta_m\; [\omega\, (1/\beta_d')\, q]\, R_{dc}\; (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$E_r(r,m) = (j/4)\,\eta_m\,[\omega\,(1/\beta_d')\,q]\,R_{dc}\,(a\beta_d')\,g_m$ $\qquad g_m = [\,\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\,]$ $\qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\,[\omega\,(1/\beta_d')\,q]\,R_{dc}\,(a\beta_d')\,h_m$ $\qquad h_m = [\,\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\,]$

or

$E_z(r,m) = (1/4)\,\eta_m\,[\omega aq]\,(\beta'/\beta_d')\,R_{dc}\,f_m$ $\qquad f_m = [\,\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\,]$ $\qquad x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\,[\omega aq]\,R_{dc}\,g_m$ $\qquad g_m = [\,\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\,]$ $\qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\,[\omega aq]\,R_{dc}\,h_m$ $\qquad h_m = [\,\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\,]$ $\qquad (***)$

Where q is the charge/length at z = 0, a quantity we regard as fixed in our limit. But the above form is valid for any ω, just another way to write things that I have never used before.

Now bring in these small β' results which are apropo for small ω

$f_m = (r/a)^m\,(m+1)\,(2/\beta' a)$ $\qquad f_0 = 4/(a\beta')$

$g_m = (r/a)^{m+1} + (r/a)^{m-1}$ $\qquad g_0 = 2\,(r/a)$

$h_m = (r/a)^{m+1} - (r/a)^{m-1}$ $\qquad h_0 = 0$

to get

For m > 0:

$E_z(r,m) = (1/2)\,\eta_m\,[\omega\,q\,(1/\beta_d')\,]R_{dc}\,(r/a)^m\,(m+1)$

$E_r(r,m) = (j/4)\,\eta_m\,[\omega\,q\,a\,]R_{dc}\;\;[(r/a)^{m+1} + (r/a)^{m-1}]$

$E_\theta(r,m) = (1/4)\,\eta_m\,[\omega\,q\,a\,]R_{dc}\;\;[(r/a)^{m+1} - (r/a)^{m-1}]$

For m= 0:

$E_z(r,0) = [\omega\,q\,(1/\beta_d')\,]R_{dc}$

$E_r(r,0) = (j/2)\,[\omega\,q\,a\,]R_{dc}\;\;(r/a)$

$E_\theta(r,0) = 0$

Now install this result for small ω,

$\beta'_d = \sqrt{\omega RC/2}\;(1-j) = \sqrt{\omega RC}\;\;(1-j)/\sqrt{2} = \sqrt{\omega RC}\;\;e^{-j\pi/4}$

$(1/\beta_d') = e^{j\pi/4} / \sqrt{\omega RC}$

and we then get

For m > 0:

$E_z(r,m) = (1/2)\,\eta_m\,[\sqrt{\omega}\;\;q\,e^{j\pi/4} / \sqrt{RC}\;\;]R_{dc}\,(r/a)^m\,(m+1)$

$E_r(r,m) = (j/4)\,\eta_m\,[\omega\,q\,a\,]R_{dc}\;\;[(r/a)^{m+1} + (r/a)^{m-1}]$

$E_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a\,]R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$

For m= 0:

$E_z(r,0) = [\sqrt{\omega}\; q\, e^{j\pi/4} / \sqrt{RC}\,]R_{dc}$ G = 0

$E_r(r,0) = (j/2)\,[\omega\, q\, a\,]R_{dc}\;\; (r/a)$

$E_\theta(r,0) = 0$

Here we see ALL fields going to 0 in the limit $\omega \to 0$. As the limit is approached, there exists E_z asymmetry as shown.

Question: How does the above work out if G ≠ 0?

First, here are my results for repaired Appendix D:

Second summary of the E field solutions : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ $\beta''^2 = \beta^2 - \beta''_d{}^2$ (D.9.25)

(conducting dielectric) $I'' = V/Z_0$ $\xi_d = \varepsilon_d + \sigma_d/j\omega$

$\beta_d'' = \sqrt{\xi_d/\varepsilon_d}\;\beta_d$ where $\beta_d = \omega\sqrt{\mu_d\varepsilon_d}$ ($I'' = \sqrt{\xi_d/\varepsilon_d}\;I$ where I was for $\sigma_d = 0$)

$E''_z(r,m) = (1/4)\,\eta_m\, I''\, R_{dc}\; (a\beta'')\, f_m$ $f_m = \left[\dfrac{J_m(x)}{J_{m+1}(x_a)} - \dfrac{J_m(x)}{J_{m-1}(x_a)}\right]$ $x = \beta''r$

$E''_r(r,m) = (j/4)\,\eta_m\, I''\, R_{dc}\; (a\beta''_d)\, g_m$ $g_m = \left[\dfrac{J_{m+1}(x)}{J_{m+1}(x_a)} + \dfrac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$ $x_a = \beta''a$

$E''_\theta(r,m) = (1/4)\,\eta_m\, I''\, R_{dc}\; (a\beta''_d)\; h_m$ $h_m = \left[\dfrac{J_{m+1}(x)}{J_{m+1}(x_a)} - \dfrac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

Now write (see D.9.32 and below)

$I'' = \sqrt{\xi_d/\varepsilon_d}\;I = \sqrt{\xi_d/\varepsilon_d}\;2\pi\omega\,(a/\beta_d)\,N_0 = 2\pi\omega\,(a/\beta_d'')\,N_0$

But then we always have same as before

$N_0 = (1/2\pi a)\; q(0)$

so that

$I'' = 2\pi\omega\,(a/\beta_d'')\;(1/2\pi a)\; q(0) = \omega\,(1/\beta_d'')\; q(0) = [\omega\, q\, (1/\beta_d'')]$

Install to get something almost identical to (***) above,

$$E''_z(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a(\beta''/\beta_d'')]\,R_{dc} f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta'' r$$

$$E''_r(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta'' a$$

$$E''_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

For the r and θ fields things are exactly the same! This is all for general ω so far.

Now a big difference is going to be the Appendix Q limit which for $G \neq 0$ is for $\omega \rightarrow 0$

$$\beta''_d = (\omega/2)\frac{RC+GL}{\sqrt{RG}} - j\sqrt{RG} \approx - j\sqrt{RG}$$

For β'' we then get $\beta''^2 = \beta^2 - \beta''^2_d = - \beta''^2_d = - [- j\sqrt{RG}\,]^2 = - (-1)\, RG = RG$

$\beta'' \rightarrow \sqrt{RG}$ as $\omega \rightarrow 0$, something different!

Then

$$(\beta''/\beta_d'') = \sqrt{RG} / [\, - j\sqrt{RG}\,] = j$$

We then get for $\omega \rightarrow 0$,

$$E''_z(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta'' r$$

$$E''_r(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta'' a$$

$$E''_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Even though $\omega \rightarrow 0$, we can not in general use the small x arguments of Bessels. But if G is small, then β'' is very small and we CAN. Then we get

$f_m = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$

$g_m = (r/a)^{m+1} + (r/a)^{m-1} \qquad g_0 = 2\,(r/a)$

$h_m = (r/a)^{m+1} - (r/a)^{m-1} \qquad h_0 = 0$

So we get these final results

$$E''_z(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, f_m \qquad f_m = (r/a)^m\,(m+1)\,(2/\beta'' a) \qquad x = \beta'' r$$

$$E''_r(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, g_m \qquad g_m = [(r/a)^{m+1} + (r/a)^{m-1}\,] \qquad x_a = \beta'' a$$

$$E''_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a]\,R_{dc}\, h_m \qquad h_m = [(r/a)^{m+1} - (r/a)^{m-1}\,]$$

$E''_z(r,0) = (j/4)\,[\omega\, q\, a]\, R_{dc}\, f_0$ $\quad f_0 = 4/(a\beta')$ $\quad x = \beta''r$

$E''_r(r,0) = (j/4)\,[\omega\, q\, a]\, R_{dc}\, g_0$ $\quad g_0 = 2\,(r/a)$ $\quad x_a = \beta''a$

$E''_\theta(r,0) = (1/4)\,[\omega\, q\, a]\, R_{dc}\, h_0$ $\quad h_0 = 0$

Install to get

$E''_z(r,m) = (j/4)\,\eta_m\,[\omega\, q\,]\,(2/\beta'')\, R_{dc}\, (r/a)^m\,(m+1)$ $\quad x = \beta''r$

$E''_r(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\, R_{dc}\, [(r/a)^{m+1} + (r/a)^{m-1}]$ $\quad x_a = \beta''a$

$E''_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a]\, R_{dc}\, [(r/a)^{m+1} - (r/a)^{m-1}]$

$E''_z(r,0) = (j/4)\,[\omega\, q\,]\, R_{dc}\, 4/(\beta'')$ $\quad x = \beta''r$

$E''_r(r,0) = (j/4)\,[\omega\, q\, a]\, R_{dc}\, 2\,(r/a)$ $\quad x_a = \beta''a$

$E''_\theta(r,0) = 0$

Now use $\beta'' = \sqrt{RG}$ to get

$E''_z(r,m) = (j/4)\,\eta_m\,[\omega\, q\,]\,(2/\sqrt{RG}\,)\, R_{dc}\, (r/a)^m\,(m+1)$ $\quad x = \beta''r$ $\quad G \neq 0$

$E''_r(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a]\, R_{dc}\, [(r/a)^{m+1} + (r/a)^{m-1}]$ $\quad x_a = \beta''a$

$E''_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a]\, R_{dc}\, h_m = [(r/a)^{m+1} - (r/a)^{m-1}]$

$E''_z(r,0) = (j/4)\,[\omega\, q\,]\, R_{dc}\, 4/\sqrt{RG}$ $\quad x = \beta''r$

$E''_r(r,0) = (j/4)\,[\omega\, q\, a]\, R_{dc}\, 2\,(r/a)$ $\quad x_a = \beta''a$

$E''_\theta(r,0) = 0$

which we can compare with our earlier result for G = 0,

For m > 0:

$E_z(r,m) = (1/2)\,\eta_m\,[\sqrt{\omega}\; q\, e^{j\pi/4} / \sqrt{RC}\,]R_{dc}\,(r/a)^m\,(m+1)$

$E_r(r,m) = (j/4)\,\eta_m\,[\omega\, q\, a\,]R_{dc}\; [(r/a)^{m+1} + (r/a)^{m-1}]$

$E_\theta(r,m) = (1/4)\,\eta_m\,[\omega\, q\, a\,]R_{dc}\; [(r/a)^{m+1} - (r/a)^{m-1}]$

For m= 0:

$E_z(r,0) = [\sqrt{\omega}\; q\, e^{j\pi/4} / \sqrt{RC}\,]R_{dc}$ $\quad G = 0$

$E_r(r,0) = (j/2)\,[\omega\, q\, a\,]R_{dc}\;\; (r/a)$

$E_\theta(r,0) = 0$

We again find that all fields → 0, but the E_z fields now go with a full factor ω, whereas for G = 0 they only went to 0 as $\sqrt{\omega}$. In the limit in EITHER CASE, we find that

$E_z(r,m) / E_z(r,0) = (1/2)\,\eta_m\,(r/a)^m\,(m+1)$ $\quad G = 0$ or $G \neq 0$

so AS the fields vanish, the J_z asymmetry remains.

Question: What happens when you drive a line at very low ω?

For small ω: $\mathrm{Re}(\beta'_{\mathbf{d}}) = \sqrt{\omega RC/2}$

$\mathrm{Im}(\beta'_{\mathbf{d}}) = -\sqrt{\omega RC/2}$ $\qquad$ G = 0 and $\omega << (R/L)$

Thus,

$L_{\mathbf{decay}} = 1/\alpha = 2/\sqrt{\omega RC}$

As $\omega \to 0$, the wavelength of course increases since $\lambda = 2/\sqrt{\omega RC}$ as well. And the decay distance also increases, so near the driving end, the decay seems to go away. The decay is due to R, and as $R \to 0$ it is the same result as if $\omega \to 0$, there is no decay.

For $G \neq 0$, we instead have

For small ω: $\mathrm{Re}(\beta'_{\mathbf{d}}) = (\omega/2)\dfrac{RC+GL}{\sqrt{RG}}$

$\mathrm{Im}(\beta'_{\mathbf{d}}) = -\sqrt{RG}$ $\qquad$ $\omega << (R/L)$ and $\omega << (G/C)$

For this line, the decay distance is fixed by R and G as you see, so

$L_{\mathbf{decay}} = 1/\alpha = 1/\sqrt{RG}$

but the wave's wavelength still goes to ∞.

Question: Consider just the very first "section" of a transmission line which has some G > 0. Across this thing we have 1/G resistance and C capacitance in parallel. I talk about having a charge q at the start of the line, but if the time constant of ω is small enough, it seems that the capacitor has no effect and the charge q would just leak across the conductance and go away, leaving you with q= 0. So my assumption that there is some finite q at z = 0 is wrong in this case at low ω.

Is this right? Or since V is across that 1/G resistor, there still is V there, and C charges up to Q = CV and then there IS a q sitting there. Yes, the driver keeps supplying new q. So above is wrong.

Paradox B. Consider a line with G > 0. We know that as $\omega \to 0$, we will have $Z_0 = \sqrt{R/G}$ and therefore some current $I = V/Z_0 = V\sqrt{G/R}$ will flow. The decay distance is $L_{\mathbf{decay}} = 1/\alpha = 1/\sqrt{RG}$ which could be very long. The paradox B is that the field equations above say **E** = 0 at DC. Then Ohm's law says you cannot have a current flowing. Maybe this is a hint as to my general problem.

Resuming 6.27.14.

I think something is wrong with my section D 9 (d) on generalizing the CPBC. I get down to (D.9.23) and I think the idea that $N_{\mathbf{m}} \to (\xi_{\mathbf{d}}/\varepsilon_{\mathbf{d}})\, N_{\mathbf{m}}$ is the main change. Then looking at (D.2.28) agree that this is

equivalent to doing $K_m \rightarrow (\xi_d/\varepsilon_d) K_m \equiv K'_m$ and $a_m \rightarrow (\xi_d/\varepsilon_d) a_m \equiv a'_m$. I would argue right now that I should keep β_d as a general complex parameter which is unchanged. It is a parameter of my theory. I can later evaluate it for special cases if I want. Looking at its derivation, I think D.2.31 says

$$I = 2\pi\omega (a/\beta_d) N_0 \quad \rightarrow \quad 2\pi\omega (a/\beta_d) (\xi_d/\varepsilon_d) N_0$$

and we can interpret this in 2 says. First, it is just an application of $N_m \rightarrow (\xi_d/\varepsilon_d) N_m$. Second, it seems that I should increase due to $G \neq 0$. But we are not really done yet. Well OK, I guess (D.8.25) is OK as it is stated, but the text $\beta_d' = \sqrt{\xi_d/\varepsilon_d}\ \beta_d$ where $\beta_d = \omega\sqrt{\mu_d\varepsilon_d}$ only applies to the lossless case. I guess I would write that

$$I' = 2\pi\omega (a/\beta_d') (\xi_d/\varepsilon_d) N_0$$

What is the meaning of N_0 and $n(\theta)$? I think $n(\theta)$ is still the true surface charge as in (D.9.19). So then N_0 is exactly what it was before. I think it would be clearer NOT to have I or I' appearing in my results.

Let's back up and restate the un-modified equations first. My second summary was this

Second summary of the E field solutions : $\quad R_{dc} = \dfrac{1}{\sigma\pi a^2} \quad \beta'^2 = \beta^2 - \beta_d^2 \quad$ (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Dimension check: $\dim E_z$ = amp ohm/m = volt/m = correct

But suppose I replace at this early point $I = 2\pi\omega (a/\beta_d) N_0$, I then get an earlier form which I never wrote down,

$$\dim I = \sec^{-1} m^2\ \text{Cou/m}^2 = \text{Cou/sec} = \text{amp} = \text{correct}$$

Second summary of the E field solutions : $\quad R_{dc} = \dfrac{1}{\sigma\pi a^2} \quad \beta'^2 = \beta^2 - \beta_d^2 \quad$ (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, 2\pi\omega (a/\beta_d) N_0\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, 2\pi\omega (a/\beta_d) N_0\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$E_\theta(r,m) = (1/4)\,\eta_m\ 2\pi\omega\,(a/\beta_d)\,N_0\,R_{dc}\ (a\beta_d)\ h_m \qquad h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

dim E_z = sec^{-1} m m Cou/m^2 ohm/m = sec^{-1} Cou ohm/m = amp ohm /m = volt/m = correct

which I cosmetically rewrite as

Second summary of the E field solutions : $\qquad R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad$ (D.2.33)

$E_z(r,m) = (1/4)\,\eta_m\ 2\pi a\ \omega\,(1/\beta_d)\,N_0\,R_{dc}\ (a\beta')\,f_m \qquad f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}] \qquad x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\ 2\pi a^2\ \omega\ N_0\,R_{dc}\ g_m \qquad g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}] \qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\ 2\pi a^2\ \omega\,N_0\,R_{dc}\ h_m \qquad h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

DimE_z = m sec^{-1} m Cou/m^2 ohm/m = sec^{-1} Cou ohm/m = amp ohm/m = volt/m = correct
DimE_r = m^2 $sec^{-1}$$Cou/m^2$ohm/m = sec^{-1}Cou ohm/m = correct

Now suppose at THIS point I <u>turn on the conducting dielectric.</u> I argued above that $N_m \rightarrow (\xi_d/\varepsilon_d)\,N_m$ is the fundamental change to make. So then the results are

Second summary of the E field solutions : $\qquad R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad$ (D.2.33)

dielectric conducts

$E_z(r,m) = (1/4)\,\eta_m\ 2\pi a\ \omega\,(1/\beta_d)\,(\xi_d/\varepsilon_d)\,N_0\,R_{dc}\ (a\beta')\,f_m \qquad f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$

$E_r(r,m) = (j/4)\,\eta_m\ 2\pi a^2\ \omega\ (\xi_d/\varepsilon_d)\,N_0\,R_{dc}\ g_m \qquad g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

$E_\theta(r,m) = (1/4)\,\eta_m\ 2\pi a^2\ \omega\,(\xi_d/\varepsilon_d)\,N_0\,R_{dc}\ h_m \qquad h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

where the following is still true

$N_0 = (1/2\pi a)\,q(0)$ $\qquad$ so rewrite as $\qquad$ (D.1.8)

Second summary of the E field solutions : $\qquad R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad$ (D.2.33)

dielectric conducts

$$E_z(r,m) = (1/4)\,\eta_m\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ (a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,m) = (j/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where q = q(0) is the charge per length at the start of the transmission line, and which I think is a good thing to think of as not changing when I turn on the conductivity of the dielectric. But now let's install the low $\omega\, x_m$ factors

$$f_m = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$$
$$g_m = (r/a)^{m+1} + (r/a)^{m-1} \qquad g_0 = 2\,(r/a)$$
$$h_m = (r/a)^{m+1} - (r/a)^{m-1} \qquad h_0 = 0$$

For example, this results in:

$$E_z(r,m) = (1/4)\,\eta_m\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ (a\beta')\,[(r/a)^m\,(m+1)\,(2/\beta' a)]$$
$$E_z(r,0) = (1/4)\ \ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ (a\beta')\,[4/(a\beta')]$$

or

$$E_z(r,m) = (1/4)\,\eta_m\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\ \ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ [4]$$

Dimension check:

$$\dim E_z = \sec^{-1}\ \text{Cou/m}\ \ \text{m}\ \ \text{ohm/m} = \sec^{-1}\ \text{Cou}\ \ \text{ohm/m} = \text{amp ohm/m} = \text{volt/m} = \text{correct}$$

Appendix Q says

For small ω:

$$\mathrm{Re}(\beta_d) = (\omega/2)\frac{RC+GL}{\sqrt{RG}}$$

$$\mathrm{Im}(\beta_d) = -\sqrt{RG}\left[1 + (\omega^2/8)\left(\frac{RC-GL}{RG}\right)^2\right] \qquad \omega << (R/L) \text{ and } \omega << (G/C)$$

which fully incorporates dielectric conductivity. We then have close to ω = 0,

$$\beta_d = (\omega/2)\frac{RC+GL}{\sqrt{RG}} - j\sqrt{RG}$$
$$= -j\sqrt{RG}\ \left\{\ 1 + (\omega/2)\frac{RC+GL}{\sqrt{RG}}\ /\ [-j\sqrt{RG}\,]\ \right\}$$
$$= -j\sqrt{RG}\ \left\{\ 1 + j\,(\omega/2)\frac{RC+GL}{RG}\ \right\}$$

I can then invert this since ω is small to get

$$(1/\beta_d) = [-j\sqrt{RG}\,]^{-1}\ \{\ 1 - j(\omega/2)\frac{RC+GL}{RG}\ \}$$

$$= (j/\sqrt{RG}\,)\ \{\ 1 - j(\omega/2)\frac{RC+GL}{RG}\ \}$$

I then get

$$E_z(r,m) = (1/4)\,\eta_m\,\omega\,q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad \omega\,q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ [4]$$

or

$$E_z(r,m) = (1/4)\,\eta_m\,\omega\,q\,(j/\sqrt{RG}\,)\ \{\ 1 - j(\omega/2)\frac{RC+GL}{RG}\ \}\,(\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad \omega\,q(j/\sqrt{RG}\,)\ \{\ 1 - j(\omega/2)\frac{RC+GL}{RG}\ \}\,(\xi_d/\varepsilon_d)R_{dc}\ [4]$$

or

$$E_z(r,m) = (1/4)\,\eta_m\,\omega\,q\,(j/\sqrt{RG}\,)\,(\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad \omega\,q\,(j/\sqrt{RG}\,)\,(\xi_d/\varepsilon_d)R_{dc}\ [4]$$

or

$$E_z(r,m) = (1/4)\,\eta_m\,\omega\,q\,(j/\sqrt{RG}\,)\,(1 - j\sigma_d/\varepsilon_d\omega)\,R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad \omega\,q\,(j/\sqrt{RG}\,)\,(1 - j\sigma_d/\varepsilon_d\omega)\,R_{dc}\ [4]$$

Now take the limit $\omega\rightarrow 0$ and we get

$$E_z(r,m) = (1/4)\,\eta_m\ q\,(j/\sqrt{RG}\,)\,(-j\sigma_d/\varepsilon_d)\,R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad q\,(j/\sqrt{RG}\,)\,(-j\sigma_d/\varepsilon_d)\,R_{dc}\ [4]$$

or

$$E_z(r,m) = (1/4)\,\eta_m\ q\,(1/\sqrt{RG}\,)\,(\sigma_d/\varepsilon_d)\,R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad q\,(1/\sqrt{RG}\,)\,(\sigma_d/\varepsilon_d)\,R_{dc}\ [4]$$

Dim E_z = Cou/m m ohm^{-1}/m m/far ohm/m = Cou/m 1/far = volt/m = correct

Now we can write for a medium using (4.11.34),

$$G = C\,(\sigma_d/\varepsilon_d) \quad => \quad (\sigma_d/\varepsilon_d) = G/C$$

Then the results above become

$$E_z(r,m) = (1/4)\,\eta_m\ q\,(1/\sqrt{RG}\,)\,G/C\,R_{dc}\ [(r/a)^m\,(m+1)\,2]$$
$$E_z(r,0) = (1/4)\quad q\,(1/\sqrt{RG}\,)\,G/C\,R_{dc}\ [4]$$

or

$E_z(r,m) = (1/4)\,\eta_m\ (q/C)\sqrt{G/R}\ R_{dc}\ [(r/a)^m\,(m+1)\,2]$

$E_z(r,0) = (1/4)\quad (q/C)\sqrt{G/R}\ R_{dc}\ [4]$

or

$E_z(r,m) = (1/4)\,\eta_m\ V\sqrt{G/R}\ R_{dc}\ [(r/a)^m\,(m+1)\,2]$

$E_z(r,0) = (1/4)\quad V\sqrt{G/R}\ R_{dc}\ [4]$

or

$E_z(r,m) = (1/2)\,\eta_m\ V\sqrt{G/R}\ R_{dc}\ [(r/a)^m\,(m+1)]$

$E_z(r,0) = \qquad V\sqrt{G/R}\ R_{dc}$

where remember R is for the total transmission line.

Dimension check: volt mho ohm/m = volt/m= correct

Now let's consider this a different way. We know that as $\omega \to 0$ we get

$$Z_0 = \sqrt{R/G}$$

$$I = V/Z_0 = V\sqrt{G/R}$$

$$J_z = I/(\pi a^2) = V\sqrt{G/R}\,/(\pi a^2)$$

$$E_z = J_z/\sigma = V\sqrt{G/R}\,/(\sigma\pi a^2) \quad = V\sqrt{G/R}\ R_{dc}$$

This agrees exactly with the $E_z(r,0)$ term. But, we still have the $E_z(r,m)$ term sitting there causing asymmetry.

Comment: In my original work, I always had G = 0 and I got my asymmetric result for J_z and I argued that $I \to 0$ so it did not matter for the infinite transmission line, you could sort of "sweep the asymmetry paradox under the rug".

But now I have redone the analysis for G > 0, and this time you don't get I = 0, so this sweeping under the rug does not work. The uniform J_z school says one thing for E_z, while my theory says another thing for E_z, and they really do seem to disagree!! I cannot argue that it doesn't matter since I = 0.

Note that in both cases G = 0 and G > 0, my $E_z(r,m=0)$ term comes out exactly right!!!

So the mystery continues with this better example. I now have a simple DC situation and both theories cannot be correct!

Let's look at the other field components instead of just E_z . Go back to

Second summary of the E field solutions : $\qquad R_{dc} = \dfrac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad$ (D.2.33)

dielectric conducts

$$E_z(r,m) = (1/4)\,\eta_m\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ (a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,m) = (j/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

and next we install

$$f_m = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$$
$$g_m = (r/a)^{m+1} + (r/a)^{m-1} \qquad g_0 = 2\,(r/a)$$
$$h_m = (r/a)^{m+1} - (r/a)^{m-1} \qquad h_0 = 0$$

to get

$$E_z(r,m) = (1/4)\,\eta_m\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ (a\beta')\,(r/a)^m\,(m+1)\,(2/\beta' a)$$
$$E_r(r,m) = (j/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = (1/4)\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}\ (a\beta')\,4/(a\beta')$$
$$E_r(r,0) = (j/4)\ a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ 2\,(r/a)$$
$$E_\theta(r,0) = (1/4)\ a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ 0$$

or

$$E_z(r,m) = (1/2)\,\eta_m\ \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}\ [(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = \omega\, q\,(1/\beta_d)\,(\xi_d/\varepsilon_d)R_{dc}$$
$$E_r(r,0) = (j/2)\ a\omega\, q\ (\xi_d/\varepsilon_d)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

Now for $\omega \approx 0$ we make the replacement,

$$(\xi_d/\varepsilon_d) \approx (-\,j\sigma_d/\varepsilon_d\omega)$$

to get

$$E_z(r,m) = (1/2)\,\eta_m\ \ q\,(1/\beta_d)\,(-\,j\sigma_d/\varepsilon_d)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, a\, q\ (-\,j\sigma_d/\varepsilon_d)R_{dc}\ [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, a\, q\ (-\,j\sigma_d/\varepsilon_d)R_{dc}\ [(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = q\,(1/\beta_d)\,(-j\sigma_d/\varepsilon_d)R_{dc}$$
$$E_r(r,0) = (j/2)\;a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

or

$$E_z(r,m) = (1/2)\,\eta_m\;\; q\,(1/\beta_d)\,(-j\sigma_d/\varepsilon_d)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\,a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}\;[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\,a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = q\,(1/\beta_d)\,(-j\sigma_d/\varepsilon_d)R_{dc}$$
$$E_r(r,0) = (j/2)\;a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

Next as above we do this

$$(1/\beta_d) = (j/\sqrt{RG})$$

which gives

$$E_z(r,m) = (1/2)\,\eta_m\;\; q\;(j/\sqrt{RG})\,(-j\sigma_d/\varepsilon_d)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\,a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}\;[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\,a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = q\,(j/\sqrt{RG})\,(-j\sigma_d/\varepsilon_d)R_{dc}$$
$$E_r(r,0) = (j/2)\;a\,q\;(-j\sigma_d/\varepsilon_d)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

The next step is to use

$$\sigma_d/\varepsilon_d = G/C$$

to get

$$E_z(r,m) = (1/2)\,\eta_m\;\; q\;(j/\sqrt{RG})\,(-j\,G/C)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\,a\,q\;(-jG/C)R_{dc}\;[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\,a\,q\;(-j\,G/C)R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = q\,(j/\sqrt{RG})\,(-j\,G/C)R_{dc}$$
$$E_r(r,0) = (j/2)\;a\,q\;(-j\,G/C)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

or

$$E_z(r,m) = (1/2)\,\eta_m\; q\; (1/\sqrt{RG})\,(G/C)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (1/4)\,\eta_m\, a\, q\; (G/C)R_{dc}\;[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (-j/4)\,\eta_m\, a\, q\; (G/C)R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = q\,(1/\sqrt{RG})\,(G/C)R_{dc}$$
$$E_r(r,0) = (1/2)\; a\, q\; (G/C)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

or

$$E_z(r,m) = (1/2)\,\eta_m\; (q/C)\; (1/\sqrt{RG})\,(G)R_{dc}(r/a)^m\,(m+1)$$
$$E_r(r,m) = (1/4)\,\eta_m\, a\, (q/C)\; (G)R_{dc}\;[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (-j/4)\,\eta_m\, a\, (q/C)\; (G)R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = (q/C)\,(1/\sqrt{RG})\,(G)R_{dc}$$
$$E_r(r,0) = (1/2)\; a\; (q/C)\; (G)R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

or

$$E_z(r,m) = (1/2)\,\eta_m\; V\sqrt{G/R}\,R_{dc}(\,r/a)^m\,(m+1)$$
$$E_r(r,m) = (1/4)\,\eta_m\, a\, V\; (G)\; R_{dc}\;[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (-j/4)\,\eta_m\, a\, (V\; (G)\; R_{dc}\;[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = V\sqrt{G/R}\,R_{dc}$$
$$E_r(r,0) = (1/2)\; a\; V\,(G)\; R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

Comment on the Radial Current Density. Somehow I expect that E_r might be a function of θ just looking offhand at the problem. Outside the wires, you would expect E to be strongest between the conductors, so you would perhaps expect a larger current there! That then *requires* that some $E_r(r,m)$ terms be present!

Question: Is there a simple way to solve the *magnetostatics* problem of the two cylinders?

I can model the *exterior* problem using the ladder network and I know *that* for that problem,

$$Z_0 = \sqrt{R/G} \qquad \Rightarrow \qquad I = V/Z_0 = V\sqrt{G/R}$$

This is the total current going into the line at z = 0. I would expect this to decay since

$$\mathrm{Im}(\beta'_d) = -\sqrt{RG} = \alpha$$

so I then expect to have

$$I(z) = V\sqrt{G/R}\ \exp(-\sqrt{RG}\ z\ )$$

But the exterior transmission line model tells me nothing about solution inside the wires.

But I *do* know how to compute the exterior E fields for the two wires and from that I could compute E_r just outside the wire surface and this will be basically $E_r(a,\theta) = n(\theta)/\varepsilon_d$ (confirmed). So the current density $J_r(a,\theta)$ just *outside* the wire is going to be asymmetric.

If we apply this rule at the surface, as near D.2.23

$$\text{div } \mathbf{J} = -j\omega\rho \qquad \Leftrightarrow \qquad -j\omega[\int_V \rho\, dV] = \int_S \mathbf{J} \bullet d\mathbf{S}\ . \qquad \text{(D.2.22)}$$

we get J_r being continuous through the surface! I already wrote this in D.9 (d) as

$$J_r(a-\alpha,\theta) - J_r(a+\alpha,\theta) = 0 \qquad \text{for } \omega = 0$$

Thus, just BELOW the surface, I know that

$$J_r(a,\theta) = \sigma_d\, E_r(a,\theta) = \sigma_d\, n(\theta)/\varepsilon_d = (\sigma_d/\varepsilon_d)\, n(\theta) \qquad (*)$$

and there is your strong asymmetry.

<u>Does this agree with my theory ???</u> My theory above says:

$$E_z(r,m) = (1/2)\,\eta_m\ V\sqrt{G/R}\ R_{dc}(\,r/a)^m\,(m+1)$$
$$E_r(r,m) = (1/4)\,\eta_m\, a\, V\ (G)\ R_{dc}\ [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (-j/4)\,\eta_m\, a\,(V\ (G)\ R_{dc}\ [(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = V\sqrt{G/R}\ R_{dc}$$
$$E_r(r,0) = (1/2)\ a\ V\,(G)\ R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

which I can write at r = a as

$$E_z(a,m) = (1/2)\,\eta_m\ V\sqrt{G/R}\ R_{dc}\,(m+1)$$
$$E_r(a,m) = (1/4)\,\eta_m\, a\, V\ (G)\ R_{dc}\ 2$$
$$E_\theta(a,m) = (-j/4)\,\eta_m\, a\,(V\ (G)\ R_{dc}\ 0$$

$$E_z(a,0) = V\sqrt{G/R}\ R_{dc}$$
$$E_r(a,0) = (1/2)\ a\ V\,(G)\ R_{dc}$$
$$E_\theta(a,0) = 0$$

So the facts here are then

$$E_r(a,m) = \eta_m (1/2)\, a\, V\, G\, R_{dc}$$
$$E_r(a,0) = (1/2)\, a\, V\, G\, R_{dc}$$

Using these two facts, I find that

$$E_r(a,\theta) = \sum_{m=-\infty}^{\infty} E_r(r,m)\, e^{jm\theta}$$

$$= (1/2)\, a\, V\, G\, R_{dc} \sum_{m=-\infty}^{\infty} \eta_m\, e^{jm\theta} = (1/2)\, a\, V\, G\, R_{dc}/N_0 \sum_{m=-\infty}^{\infty} N_m\, e^{jm\theta}$$

$$= (1/2)\, a\, V\, G\, R_{dc}/N_0\, n(\theta)$$

But

$$N_0 = (1/2\pi a)\, q(0)$$

so then

$$E_r(a,\theta) = (1/2)\, a\, V\, G\, R_{dc}/N_0\, n(\theta) = (1/2)\, a\, V\, G\, R_{dc}\, n(\theta)\, 2\pi a/q(0)$$

$$= (1/C)\, G\, R_{dc}\, n(\theta)\, \pi a^2 = (G/C)\,(1/\sigma)\, n(\theta)$$

$$= (\sigma_d/\varepsilon_d)\,(1/\sigma)\, n(\theta)$$

and then

$$E_r(a,\theta) = (\sigma_d/\varepsilon_d)\, n(\theta)$$

and this is the same as (*) above! So my quick little analysis of the J_r situation shows that my general theory model is correct in its statements of $E_r(a,m)$ and $E_r(a,0)$. This then lends credence to my full solution which is this

$$E_z(r,m) = (1/2)\, \eta_m\, V \sqrt{G/R}\, R_{dc} (r/a)^m\, (m+1)$$
$$E_r(r,m) = (1/4)\, \eta_m\, a\, V\, (G)\, R_{dc}\, [(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (-j/4)\, \eta_m\, a\, (V\, (G)\, R_{dc}\, [(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = V \sqrt{G/R}\, R_{dc}$$
$$E_r(r,0) = (1/2)\, a\, V\, (G)\, R_{dc}(r/a)$$
$$E_\theta(r,0) = 0$$

Remember that this is the exact solution of the Helmholtz equation and Maxwell equations with the limits of Appendix Q, so this HAS TO BE RIGHT! We really do end up with asym E_z in this case. In this limit, the CPBC just says that J_r is continuous at r = a, no big question about that.