Appendix D new piece PhL 3.26.05

I presume this is superseded by version 2.

D.9 Symmetry and high frequency limit of the round wire E fields

Recall from earlier in this appendix that,

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ *(D.2.33)*

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \quad \text{where } f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \quad x = \beta' r \quad x_a = \beta' a$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \quad \text{where } g_m = \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \quad \text{where } h_m = \left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

(a) Symmetry of f_m, g_m and h_m.

From NIST (10.4.1) we know that for integer m,

$$J_{-m}(x) = (-1)^m J_m(x)$$

For f_m we find that

$$f_{-m} = \left[\frac{J_{-m}(x)}{J_{-m+1}(x_a)} - \frac{J_{-m}(x)}{J_{-m-1}(x_a)}\right] = -\left[\frac{J_m(x)}{J_{m-1}(x_a)} - \frac{J_m(x)}{J_{m+1}(x_a)}\right] = f_m$$

Next,

$$g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$g_{-m} = \left[\frac{J_{-m+1}(x)}{J_{-m+1}(x_a)} + \frac{J_{-m-1}(x)}{J_{-m-1}(x_a)}\right] = \left[\frac{J_{m-1}(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)}\right] = g_m$$

$$h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$h_{-m} = \left[\frac{J_{-m+1}(x)}{J_{-m+1}(x_a)} - \frac{J_{-m-1}(x)}{J_{-m-1}(x_a)}\right] = \left[\frac{J_{m-1}(x)}{J_{m-1}(x_a)} - \frac{J_{m+1}(x)}{J_{m+1}(x_a)}\right] = -\left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = h_m$$

For large ω we assume β' = β as discussed below (D.2.2).

From NIST 10.17.2, keeping a few leading terms in each inverse power expansion, we have this large x behavior for $J_m(x)$,

$$J_m(x) = (2/\pi x)^{1/2} \{ \cos(w) [a_0(m) - a_2(m)/x^2 + O(1/x^4)] - \sin(w) [a_1(m)/x + O(1/x^3)]] \}$$

$$w = x - m\pi/2 - \pi/4 \qquad \Rightarrow \qquad e^{-jw} = e^{-j(x-m\pi/2-\pi/4)} = e^{-jx}\, e^{jm\pi/2}\, e^{j\pi/4}$$

$$a_0(m) = 1 \qquad a_1(m) = \frac{4m^2-1}{8} \equiv c_m \qquad a_2(m) = \frac{(4m^2-1)(4m^2-9)}{128} \equiv d_m \ . \qquad \text{(D.9.1)}$$

The expansion is in fact valid for all real and complex values of the parameter m, but we shall only use the expansion for m = -1,0,1,2....∞. Using abbreviations c_m and d_m one gets,

$$J_m(x) = (2/\pi x)^{1/2}[\cos(w) (1-d_m/x^2) - \sin(w) (c_m/x)] . \qquad \text{(D.9.2)}$$

Recall that inside the round wire,

$$\delta \equiv \sqrt{2/\omega\mu\sigma} \ = \text{skin depth} \qquad // \ \omega\mu\sigma = 2/\delta^2 \qquad \textit{(2.2.20)}$$

$$\beta = e^{j3\pi/4} (\sqrt{2}\ /\delta) = (j-1)/\delta \qquad \textit{(2.2.21)}$$

so

$$x = \beta r = e^{j3\pi/4} (\sqrt{2}\ /\delta)\ r = (j-1)\ (r/\delta)$$

$$x_a = \beta a = e^{j3\pi/4} (\sqrt{2}\ /\delta)\ a = (j-1)\ (a/\delta) . \qquad \text{(D.9.3)}$$

Since x has a large positive imaginary part for small δ, so does w. Then

$$\cos(w) = [e^{jw} + e^{-jw}]/2 \ \approx \ (1/2)\ e^{-jw}$$

$$\sin(w) = [e^{jw} - e^{-jw}]/2j \ \approx \ -(1/2j)\ e^{-jw} = (j/2)e^{-jw} . \qquad \text{(D.9.4)}$$

Then our large-x expansion above becomes

$$J_m(x) = (2/\pi x)^{1/2}\ (1/2)\ [e^{-jw}\ (1-\ d_m\ /x^2) \ - j\ e^{-jw}\ (c_m\ /x)\ ]$$

$$= (1/2\pi x)^{1/2}\ e^{-jw}\ [\ 1\ -j\ c_m\ (1/x) - d_m\ (1/x^2) \ + ...\ ]$$

$$= (1/2\pi x)^{1/2}\ e^{-jx}\ e^{jm\pi/2}\ e^{j\pi/4} \ [\ 1\ -j\ c_m\ (1/x) - d_m\ (1/x^2) \ + ...\ ] \qquad \text{(D.9.5)}$$

It is not hard to show that this agrees with (2.3.5) through order 1/x. Notice that

$$e^{-jx} = e^{-j(j-1)(r/\delta)} = e^{(1+j)(r/\delta)}$$

giving a convenient hybrid form

$$J_m(x) = (1/2\pi x)^{1/2}\, e^{(1+j)(r/\delta)}\, e^{j\pi m/2}\, e^{j\pi/4}\, [\, 1 - j\, c_m\,(1/x) - d_m\,(1/x^2) + \ldots\,] \ . \qquad \text{(D.9.6)}$$

Using this asymptotic formula, we start examining the ratios that appear in our E field coefficients.

From (D.9.6) we see by inspection that

$$\frac{J_{m+1}(x)}{J_{m+1}(x_a)} = \sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta} \ . \qquad \text{(D.9.7)}$$

Next , keeping only leading terms in 1/x and $1/x_a$,

$$\frac{J_{m+1}(x)}{J_{m-1}(x_a)} = \sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, e^{j2\pi/2}\, \frac{1 - j\, c_{m+1}\,(1/x) - d_{m+1}\,(1/x^2) + \ldots}{1 - j\, c_{m-1}\,(1/x_a) - d_{m-1}\,(1/x_a^2) + \ldots}$$

$$\approx -\sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, [\ 1 - jc_{m+1}(1/x) + jc_{m-1}(1/x_a)\,] \quad . \qquad \text{(D.9.8)}$$

Our third object of interest for g_m and h_m is

$$\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} = (2m/x)\sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, e^{j\pi/2}\, \frac{1 - j\, c_m(1/x) - d_m\,(1/x^2) + \ldots}{1 - j\, c_{m-1}\,(1/x_a) - d_{m-1}(1/x_a^2) + \ldots}$$

$$+ j\ (2m/x)\sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, [\, 1 - jc_m\,(1/x) + j\, c_{m-1}(1/x_a)] \ . \qquad \text{(D.9.9)}$$

For f_m we have

$$\frac{J_m(x)}{J_{m+1}(x_a)} = \sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, e^{j(-1)\pi/2}\, \frac{1 - j\, c_m\,(1/x) - d_m\,(1/x^2) + \ldots}{1 - j\, c_{m-1}\,(1/x_a) - d_{m-1}\,(1/x_a^2) + \ldots}$$

$$\approx -j\sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, [\, 1 - jc_m\,(1/x) + j\, c_{m+1}(1/x_a)] \ . \qquad \text{(D.9.10)}$$

$$\frac{J_m(x)}{J_{m-1}(x_a)} = \sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, e^{j(+1)\pi/2}\, \frac{1 - j\, c_m\,(1/x) - d_m\,(1/x^2) + \ldots}{1 - j\, c_{m-1}\,(1/x_a) - d_{m-1}\,(1/x_a^2) + \ldots}$$

$$\approx +j\sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}\, [\, 1 - jc_m\,(1/x) + j\, c_{m-1}(1/x_a)] \quad . \qquad \text{(D.9.11)}$$

Keeping only the leading terms, we find then that

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = -2j\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} \quad . \qquad \text{(D.9.12)}$$

Next, again keep terms through O(1/x),

$$g_m = \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$= +j\ (2m/x)\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} + \sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta} + \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}$$

$$= \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\ [\,2 + 2jm/x\,] \approx 2\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} \quad . \qquad \text{(D.9.13)}$$

Finally,

$$h_m = \left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$= -j\ (2m/x)\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} + \sqrt{\frac{a}{r}}\ e^{(1+j)(r-a)/\delta}$$

$$- \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\,[\,1 - jc_{m+1}\,(1/x) + j\,c_{m-1}(1/x_a)]$$

$$= \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\ \{\,-2mj/x + 1 - 1 + jc_{m+1}\,(1/x) - j\,c_{m-1}(1/x_a)\,\}$$

$$= j\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\ \{\ (c_{m+1} - 2m)\,(1/x) - c_{m-1}(1/x_a)\,\}$$

$$\approx 0\,. \qquad \text{(D.9.14)}$$

The results are then

Large ω limits of the E field solutions : $\qquad R_{dc} = \dfrac{1}{\sigma\pi a^2} \qquad \text{(D.9.15)}$

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta)\, f_m$ $\quad$ where $f_m = -2j\sqrt{\dfrac{a}{r}}\, e^{(1+j)(r-a)/\delta}$ $\quad x = \beta' r$ $\quad x_a = \beta' a$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\, g_m$ $\quad$ where $g_m = 2\sqrt{\dfrac{a}{r}}\, e^{(1+j)(r-a)/\delta}$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\ h_m$ $\quad$ where $h_m = 0$

As observed earlier, the longitudinal current J_z is much larger than the radial current J_r by factor (β/β_d).

Notice the standard skin effect behavior both in amplitude and phase for all field components. We saw this earlier in several places:

$$E(x,\omega) = E(0,\omega)\, e^{-x/\delta}\, e^{-jx/\delta} . \qquad x \rightarrow (a-r) \qquad \text{1D example} \qquad \textit{(2.1.8)}$$

$$\frac{|E_z(r)|}{|E_z(a)|} = \sqrt{\frac{a}{r}}\; e^{(r-a)/\delta} \qquad r/\delta > 3/\sqrt{2} = 2.1 \quad . \qquad \textit{(2.3.7)}$$