Appendix D large ω limits of f_m etc PhL 5.18.14

Earliest work on this subject, when it was Section D.9 of lines doc. That is now Section D.10. This is version 2 of this document, so I guess that supersedes the earlier version. Some of the stuff below is verbatim now in Section D.10. (10.8.14)

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D.9 Symmetry and the high frequency limit of the round wire E fields

Recall from (D2.2.33) that,

Summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m$ $g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m$ $h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

(a) Symmetry of f_m, g_m and h_m and expansions for $E_i(r,\theta)$

From NIST (10.4.1) we know that for integer m,

$$J_{-m}(x) = (-1)^m J_m(x) \qquad (D.9.1)$$

For f_m we find that

$$f_{-m} = [\frac{J_{-m}(x)}{J_{-m+1}(x_a)} - \frac{J_{-m}(x)}{J_{-m-1}(x_a)}] = -[\frac{J_m(x)}{J_{m-1}(x_a)} - \frac{J_m(x)}{J_{m+1}(x_a)}] = f_m$$

Next,

$$g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$$

$$g_{-m} = \left[\frac{J_{-m+1}(x)}{J_{-m+1}(x_a)} + \frac{J_{-m-1}(x)}{J_{-m-1}(x_a)}\right] = \left[\frac{J_{m-1}(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)}\right] = g_m$$

$$h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$h_{-m} = \left[\frac{J_{-m+1}(x)}{J_{-m+1}(x_a)} - \frac{J_{-m-1}(x)}{J_{-m-1}(x_a)}\right] = \left[\frac{J_{m-1}(x)}{J_{m-1}(x_a)} - \frac{J_{m+1}(x)}{J_{m+1}(x_a)}\right] = -\left[\frac{J_{m+1}(x)}{J_{m-1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = h_m$$

Thus we have shown that

$$f_{-m} = f_m$$
$$g_{-m} = g_m$$
$$h_{-m} = h_m \ . \qquad \text{(D.9.2)}$$

If the surface charge n(θ) happens to be even in θ, we know from (D.1.7) that $\eta_m = N_m/N_0 = \eta_{-m}$. In this case, the E field components in (r,θ) can be written as in (D.1.7),

$$E_i(r,\theta) = E_i(r,m=0) + 2 \sum_{m=1}^{\infty} E_i(r,m)\cos(m\theta) \qquad \text{(D.9.3)}$$

Then for even n(θ) the E fields are

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta')\, [\, f_0 + 2\, \Sigma_{m=1}^{\infty}\, f_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d)\, [\, g_0 + 2\, \Sigma_{m=1}^{\infty}\, g_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta_d)\, [\, h_0 + 2\, \Sigma_{m=1}^{\infty}\, h_m\, \eta_m\, \cos(m\theta)\,] \ . \qquad \text{(D.9.4)}$$

(c) High frequency evaluation of f_m, g_m and h_m and the E fields

For large ω we assume β' = β as discussed below (D.2.2).

From NIST 10.17.2, keeping a few leading terms in each inverse power expansion, we have this large x behavior for $J_m(x)$,

$$J_m(x) = (2/\pi x)^{1/2}\, \{\, \cos(w)\, [a_0(m) - a_2(m)/x^2 + O(1/x^4)] - \sin(w)\, [a_1(m)/x + O(1/x^3)]\,]\,\}$$

$$w = x - m\pi/2 - \pi/4 \qquad \Rightarrow \qquad e^{-jw} = e^{-j(x - m\pi/2 - \pi/4)} = e^{-jx}\, e^{jm\pi/2}\, e^{j\pi/4}$$

$$a_0(m) = 1 \qquad a_1(m) = \frac{4m^2-1}{8} \equiv c_m \qquad a_2(m) = \frac{(4m^2-1)(4m^2-9)}{128} \equiv d_m \ . \qquad \text{(D.9.5)}$$

The expansion is in fact valid for all real and complex values of the parameter m, but we shall only use the expansion for m = -1,0,1,2....∞. Using abbreviations c_m and d_m one gets,

$$J_m(x) = (2/\pi x)^{1/2}[\cos(w)\,(1-d_m/x^2) - \sin(w)\,(c_m/x)] \,. \qquad \text{(D.9.6)}$$

Recall that inside the round wire,

$$\delta \equiv \sqrt{2/\omega\mu\sigma} \;= \text{skin depth} \qquad // \;\; \omega\mu\sigma = 2/\delta^2 \qquad \textit{(2.2.20)}$$

$$\beta = e^{j3\pi/4}\,(\sqrt{2}\,/\delta) = (j-1)/\delta \qquad \textit{(2.2.21)}$$

so

$$x \;= \beta r = e^{j3\pi/4}\,(\sqrt{2}\,/\delta)\, r \;= (j-1)\,(r/\delta)$$

$$x_a = \beta a = e^{j3\pi/4}\,(\sqrt{2}\,/\delta)\, a \;= (j-1)\,(a/\delta)\,. \qquad \text{(D.9.7)}$$

Since x has a large positive imaginary part for small δ, so does w. Then

$$\cos(w) \;= [\,e^{jw} + e^{-jw}]/2 \;\approx\; (1/2)\,e^{-jw}$$

$$\sin(w) \;= [\,e^{jw} - e^{-jw}]/2j \;\approx\; -(1/2j)\,e^{-jw} \;= (j/2)e^{-jw}\,. \qquad \text{(D.9.8)}$$

Then our large-x expansion above becomes

$$J_m(x) = (2/\pi x)^{1/2}\,(1/2)\,[e^{-jw}\,(1-\,d_m/x^2) - j\,e^{-jw}\,(c_m/x)]$$

$$= (1/2\pi x)^{1/2}\,e^{-jw}\,[\,1 - j\,c_m\,(1/x) - d_m\,(1/x^2) + \ldots\,]$$

$$= (1/2\pi x)^{1/2}\,e^{-jx}\,e^{jm\pi/2}\,e^{j\pi/4}\;[\,1 - j\,c_m\,(1/x) - d_m\,(1/x^2) + \ldots\,] \qquad \text{(D.9.9)}$$

It is not hard to show that this agrees with (2.3.5) through order 1/x. Notice that

$$e^{-jx} \;= e^{-j(j-1)(r/\delta)} \;= e^{(1+j)(r/\delta)}$$

giving a convenient hybrid form

$$J_m(x) = (1/2\pi x)^{1/2}\,e^{(1+j)(r/\delta)}\;(j)^m\,e^{j\pi/4}\;[\,1 - j\,c_m\,(1/x) - d_m\,(1/x^2) + \ldots\,]\,. \qquad \text{(D.9.10)}$$

From (D.9.10) we see by inspection that, through O(1/x),

$$\frac{J_m(x)}{J_n(x_a)} \;= (j)^{m-n}\sqrt{\frac{a}{r}}\;\; e^{(1+j)(r-a)/\delta}\frac{1 - j\,c_m\,(1/x) - d_m\,(1/x^2) + \ldots}{1 - j\,c_n\,(1/x_a) - d_n\,(1/x_a^2) + \ldots}$$

$$\approx \;\; (j)^{m-n}\sqrt{\frac{a}{r}}\;\; e^{(1+j)(r-a)/\delta}\;[\,1 - jc_m/x + jc_n/x_a\,] \qquad \text{(D.9.11)}$$

and

$$\frac{J_m(x)}{J_m(x_a)} = \sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta} \qquad \text{// independent of m} \qquad \text{(D.9.12)}$$

Therefore for large ω,

$$g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = 2\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta}$$

$$h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = 0$$

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = (j)^{-1}\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta}\left[1 - jc_m/x + jc_{m+1}/x_a\right]$$

$$- (j)^{+1}\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta}\left[1 - jc_m/x + jc_{m-1}/x_a\right]$$

$$= -j\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta}\left[2 - 2jc_m(1/x) + j(c_{m+1}+c_{m-1})(1/x_a)\right]$$

$$\approx -2j\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta}$$

The results are then

Large ω limits of the E field solutions : $\qquad R_{dc} = \dfrac{1}{\sigma\pi a^2} \qquad \text{(D.9.13)}$

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta)\, f_m \qquad f_m = -2j\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta} \qquad x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = 2\sqrt{\frac{a}{r}}\; e^{(1+j)(r-a)/\delta} \qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = 0$

As observed earlier, the longitudinal current J_z is much larger than the radial current J_r by factor (β/β_d).

Notice the standard skin effect behavior both in amplitude and phase for all field components. We saw this earlier in several places:

$E(x,\omega) = E(0,\omega)\, e^{-x/\delta}\, e^{-jx/\delta}$. $\qquad x \rightarrow (a-r) \qquad$ 1D example $\qquad$ *(2.1.8)*

$$\frac{|E_z(r)|}{|E_z(a)|} = \sqrt{\frac{a}{r}}\; e^{(r-a)/\delta} \qquad r/\delta > 3/\sqrt{2} = 2.1 \quad . \qquad \textit{(2.3.7)}$$

The configuration space fields from ** are then, for even n(θ),

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta')\, \{-2j \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\}\, [\, 1 + 2\, \Sigma_{m=1}^{\infty}\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d)\, \{2 \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\}\, [\, 1 + 2\, \Sigma_{m=1}^{\infty}\, 1\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = 0 \qquad\qquad \eta_m = N_m/N_0 \qquad\qquad \text{(D.9.14)}$$

But for even n(θ) the [...] expansions shown here are just $n(\theta)/N_0$ from (D.1.7) where recall that

$$N_0 = (\beta_d/2\pi\omega a)\, I\ . \qquad \textit{(D.2.31)}$$

Now since $R_{dc} = (\pi a^2)/\sigma$ we find,

$$2\, (1/4)\, I\, R_{dc}(a\beta_d)/N_0 = (1/2)\, I\, R_{dc}\, (a\beta_d)\, 2\pi\omega a/(\beta_d I) = \quad R_{dc}\, \pi\omega a^2 \ = (\omega/\sigma)$$

so then for even n(θ) and large ω:

$$E_z(r,\theta) = -\, (j\omega/\sigma) \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\ n(\theta)\, (\beta'/\beta_d)$$

$$E_r(r,\theta) = \quad (j\omega/\sigma) \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\ n(\theta)$$

$$E_\theta(r,\theta) = 0 \qquad\qquad \text{(D.9.15)}$$

Observations on the E fields for large ω

In the extreme skin effect (small δ, large ω) regime, and for n(θ) an even function of θ :

1. There is no azimuthal field E_θ inside or on the surface of the round wire.

2. Both E_z and E_r exhibit the standard skin effect form for amplitude and phase

3. The ratio $E_z(r,\theta)/E_r(r,\theta) = -\, (\beta'/\beta_d)$ is very large and is constant in r and θ

4. Both E_z and E_r track the surface charge density n(θ) for azimuthal dependence

5. If n(θ) ≠ constant, then $J_z = \sigma E_z$ ≠ constant in θ and the longitudinal current density is *asymmetric* across the round wire cross section, which is known as the proximity effect.