High Freq Limits of f_m, g_m and h_m PhL 5.16.14

This was early work on doing these large ω limits. They are now derived inside lines doc D.10 in a derivation that was first done below relating to the Bessel function of large argument. I also confirmed these limits with at least one mws files in this folder. Unlike the low ω results, I think my high ω results are accurate. (10.8.14)

1. Large ω limit of f_m

Here is the f_m case taken from lines doc

Fact: In the extreme skin effect regime,

$$f_m(r) = -2j\,[\,(a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta}]$$

$$f_m(a) = -2j \quad . \qquad (6.5.22)$$

Proof: From NIST 10.7.8 we find that for large x argument,

$$J_m(x) \approx (2/\pi x)^{1/2} \cos(x-m\pi/2-\pi/4) \quad . \qquad \text{NIST 10.7.8}$$

Then for $x = \beta r = (j-1)(r/\delta)$ from (6.5.9) and with $\delta << r \le a$ we keep only the larger exponential term so that

$$J_m(x) \approx (2/\pi x)^{1/2} \cos(x-m\pi/2-\pi/4) \approx (2/\pi x)^{1/2}\; (1/2)\, e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]}$$

$$= (1/\sqrt{2}\,\pi\beta r)^{1/2}\; e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]} \quad . \qquad (6.5.23)$$

From (6.5.14),

$$f_m(r) \equiv \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \quad . \qquad \textit{(6.5.14)}$$

We then examine the two terms

$$\frac{J_m(x)}{J_{m+1}(x_a)} = \frac{(1/\sqrt{2}\,\pi\beta r)^{1/2}\; e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]}}{(1/\sqrt{2}\,\pi\beta a)^{1/2}\; e^{+a/\delta}\, e^{j[(a/\delta)+(m+1)\pi/2+\pi/4]}} = (a/r)^{1/2}\, e^{-(a-r)/\delta}\; e^{j[(r-a)/\delta-\pi/2]}$$

$$\frac{J_m(x)}{J_{m-1}(x_a)} = (a/r)^{1/2}\, e^{-(a-r)/\delta}\; e^{j[(r-a)/\delta+\pi/2]} \quad . \qquad (6.5.24)$$

Then

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = (a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta}\,[e^{-j\pi/2} - e^{+j\pi/2}]$$

$$= (a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta}\,[-j - j]$$

$$= -2j\,[\,(a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta}]\,.$$

which then verifies (6.5.22). QED

2. Large ω limit of g_m

Use this result from the previous section

$$J_m(x) \approx (1/\sqrt{2}\,\pi\beta r)^{1/2}\;\; e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]}$$

Now, from (***),

$$g_m \equiv \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right].$$

We then examine the three terms. We borrow this result from the previous section

$$\frac{J_m(x)}{J_{m-1}(x_a)} = (a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta+\pi/2]} = j\,(a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta]}$$

so then the first term becomes

$$\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} = \frac{2m}{\beta r}\, j\;\; (a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta]}$$

The second term is

$$\frac{J_{m+1}(x)}{J_{m+1}(x_a)} = \frac{(1/\sqrt{2}\,\pi\beta r)^{1/2}\;\; e^{+r/\delta}\, e^{j[(r/\delta)+(m+1)\pi/2+\pi/4]}}{(1/\sqrt{2}\,\pi\beta a)^{1/2}\;\; e^{+a/\delta}\, e^{j[(a/\delta)+(m+1)\pi/2+\pi/4]}}$$

$$= (a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta}$$

and the third term is

$$\frac{J_{m+1}(x)}{J_{m-1}(x_a)} = \frac{(1/\sqrt{2}\,\pi\beta r)^{1/2}\;\; e^{+r/\delta}\, e^{j[(r/\delta)+(m+1)\pi/2+\pi/4]}}{(1/\sqrt{2}\,\pi\beta a)^{1/2}\;\; e^{+a/\delta}\, e^{j[(a/\delta)+(m-1)\pi/2+\pi/4]}} = (a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta]}\;\; e^{j\pi}$$

$$= -\,(a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta]}$$

Combining the three terms we get

$$g_m \equiv \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right]$$

$$= \frac{2m}{\beta r}\, j\, (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta]}$$

$$+ (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta}$$

$$- [- (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta}$$

The last two terms are the same, so then

$$g_m = \{ \frac{2m}{\beta r}\, j + 2 \}\ (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta]}$$

But since we are doing $x = \beta r >> 1$ in this limit, this becomes just

$$g_m = 2\ (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta]}$$

where we ignore that fact that m can get large in distant partial waves.

3. Large ω limit of h_m

Same as previous section except

$$h_m \equiv \left[- \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right].$$

so now the last two terms will cancel leaving just the first term (assuming we can ignore correction terms in the asymptotic expansion), so

$$h_m = - \{ \frac{2m}{\beta r}\, j \}\ (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta]}$$

4. Implication for the Appendix D electric fields:

$$E_z(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\ (a\beta)\, f_m$$

$$E_r(r,m) = (j/4)\, \eta_m\, I\, R_{dc}\ (a\beta_d)\, g_m$$

$$E_\theta(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\ (a\beta_d)\, h_m$$

$$f_m = -2j\, [\ (a/r)^{1/2}\, e^{-(a-r)/\delta}\, e^{j[(r-a)/\delta]}$$

$$g_m = 2\ (a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta]}$$

$$h_m = -\{\frac{2m}{\beta r}\ j\ \}\ (a/r)^{1/2}\, e^{-(a-r)/\delta}\quad e^{j[(r-a)/\delta]}$$

Comments:

(a) Notice that for x>> 1 we get $h_m \approx 0$ and then $E_\theta \approx 0$ which I like in general terms.

(b) Both f_m and g_m are independent of m.

(c) Recall from lines doc that

$$J_z(r,\theta) = (-2j)\,[(a/r)^{1/2}\, e^{-(a-r)/\delta}\ e^{j[(r-a)/\delta}\,](I/\pi a^2)(\beta a/4)\ *$$
$$[\ 1 + 2\ \Sigma_{m=1}^{\infty}\ (-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\ ]\ .$$

(6.5.25)

and also

$$n_1(\xi_1,\theta) = (q/2\pi)[\ 1 + 2\sum_{m=1}^{\infty}\ (-1)^m\ e^{-m|\xi_1|}\cos(m\theta)\ ]\ .$$ *Bipolar (A.13)* *(6.5.5)*

allowing us to write

$$J_z(r,\theta) = (-2j)\,[(a/r)^{1/2}\, e^{-(a-r)/\delta}\ e^{j[(r-a)/\delta}\,](I/\pi a^2)(\beta a/4)\ (2\pi/q)\ n_1(\xi_1,\theta)$$

We can make then a similar conclusion about $J_r(r,\theta)$ by replacing -2j by 2, so

$$J_r(r,\theta) = 2\,[(a/r)^{1/2}\, e^{-(a-r)/\delta}\ e^{j[(r-a)/\delta}\,](I/\pi a^2)(\beta a/4)\ (2\pi/q)\ n_1(\xi_1,\theta)$$

This shows that *both* J_z and J_r "track n(θ)" in the extreme skin effect regime.

(d) From this last equation we can state that

$$E_r(r,\theta) \approx E_r(a,\theta)\sqrt{\frac{a}{r}}\ \ e^{(r-a)/\delta}\ e^{j(r-a)/\delta}$$ (6.5.18)

and this then justifies the assumption I made in (6.5.18) !! So I can improve that section with these extra facts.

Maple Computations

Start with NIST 10.17.2, keeping a few leading terms in each inverse power expansion :

$$J_m(x) \approx (2/\pi x)^{1/2}\,[\ \cos(w)*(1-a_2(m)/x^2)\ - \sin(w)*(a_1(m)/x)\ ]$$

where

$$w = x - m\pi/2 - \pi/4$$

$$e^{-w} = e^{-j(x-m\pi/2-\pi/4)} = e^{-j(z-\pi/4)}\, e^{jm\pi/2}$$

But

$$x = \beta r = e^{j3\pi/4}\,(\sqrt{2}/\delta)\, r$$

which has a large positive imaginary part for small δ, so w has a positive imaginary part. Then

$$\cos(w) = [\, e^{jw} + e^{-jw}]/2 \approx (1/2)\, e^{-jw}$$

$$\sin(w) = [\, e^{jw} - e^{-jw}]/2j \approx -(1/2j)\, e^{-jw} = (j/2)e^{-jw}$$

Then our expansion above becomes

$$J_m(x) \approx (1/2)\,(2/\pi x)^{1/2}\,[e^{-jw}\,(1-a_2/x^2) - j\, e^{-jw}\,(a_1/x)\,]$$

$$= (1/2\pi x)^{1/2}\, e^{-jw}\,[\, 1 - j\, a_1(m)\,(1/x) - a_2(m)\,(1/x^2) + \ldots\,]$$

$$= (1/2\pi x)^{1/2}\, e^{-jx}\, e^{j\pi/4} e^{jm\pi/2}\,[\, 1 - j\, a_1(m)\,(1/x) - a_2(m)\,(1/x^2) + \ldots\,]$$

Now set

$$x = \beta r = (j-1)(r/\delta)$$

so then

$$e^{-jx} = e^{-j(j-1)(r/\delta)} = e^{(1+j)(r/\delta)}$$

$$J_m(x) \approx (1/2\pi x)^{1/2}\, e^{(1+j)(r/\delta)}\, e^{j\pi/4} e^{jm\pi/2}\,[\, 1 - j\, a_1(m)\,(1/x) - a_2(m)\,(1/x^2) + \ldots\,]$$

Now look at

$$\frac{J_{m+1}(x)}{J_{m+1}(x_a)} = \sqrt{\frac{x_a}{x}}\;\frac{e^{(1+j)(r/\delta)}}{e^{(1+j)(a/\delta)}} = \sqrt{\frac{x_a}{x}}\; e^{(1+j)(r-a)/\delta}$$

and there are no more terms because m is the same top and bottom.

Next, look at

$$\frac{J_{m+1}(x)}{J_{m-1}(x_a)} = \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta}\, e^{j2\pi/2}\, \frac{1 - j\, a_1(m+1)\,(1/x) - a_2(m+1)\,(1/x^2) + \ldots}{1 - j\, a_1(m-1)\,(1/x_a) - a_2(m-1)\,(1/x_a^2) + \ldots}$$

$$= -\sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta}\, \frac{1 - j\, a_1(m+1)\,(1/x) - a_2(m+1)\,(1/x^2) + \ldots}{1 - j\, a_1(m-1)\,(1/x_a) - a_2(m-1)\,(1/x_a^2) + \ldots}$$

When we add, we get

$$\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} = \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} *$$

$$\left[\, 1 - \frac{1 - j\, a_1(m+1)\,(1/x) - a_2(m+1)\,(1/x^2) + \ldots}{1 - j\, a_1(m-1)\,(1/x_a) - a_2(m-1)\,(1/x_a^2) + \ldots} \,\right]$$

Now both x and x_a are large so approx the ratio as

$$\frac{1 - j\, a_1(m+1)\,(1/x) - a_2(m+1)\,(1/x^2) + \ldots}{1 - j\, a_1(m-1)\,(1/x_a) - a_2(m-1)\,(1/x_a^2) + \ldots} \approx 1 - j a_1(m+1)(1/x) + j a_1(m-1)(1/x_a)$$

Then we have

$$\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} = \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} * \left[\, j a_1(m+1)(1/x) - j a_1(m-1)(1/x_a) \,\right] \quad (*)$$

Now look at the other term

$$J_m(x) \approx (1/2\pi x)^{1/2}\, e^{(1+j)(r/\delta)}\, e^{j\pi/4} e^{jm\pi/2}\, \left[\, 1 - j\, a_1(m)\,(1/x) - a_2(m)\,(1/x^2) + \ldots \,\right]$$

$$\frac{2m x^{-1} J_m(x)}{J_{m-1}(x_a)} = (2m/x) \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta}\, e^{j\pi/2}\, \frac{1 - j\, a_1(m)\,(1/x) - a_2(m)\,(1/x^2) + \ldots}{1 - j\, a_1(m-1)\,(1/x_a) - a_2(m-1)\,(1/x_a^2) + \ldots}$$

and again we do our approximation on the ratio to get

$$= (2m/x) \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta}\, e^{j\pi/2}\, \left[\, 1 - j a_1(m)(1/x) + j a_1(m-1)(1/x_a) \,\right]$$

$$= j\,(2m/x) \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta}\, \left[\, 1 - j a_1(m)(1/x) + j a_1(m-1)(1/x_a) \,\right]$$

Now look at h_m :

$$h_m \equiv [- \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}].$$

$$= - j\,(2m/x) \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} [\, 1 - ja_1(m)(1/x) + ja_1(m-1)(1/x_a)]$$

$$+ \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} * [\, ja_1(m+1)(1/x) - ja_1(m-1)(1/x_a)\,]$$

$$= \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} *$$

$$\{ - j\,(2m/x) [\, 1 - ja_1(m)(1/x) + ja_1(m-1)(1/x_a)] + [ja_1(m+1)(1/x) - ja_1(m-1)(1/x_a)\,] \}$$

first term; last two terms

$$= \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} *$$

$$\{ - j\,(2m/x) + ja_1(m+1)(1/x) - ja_1(m-1)(1/x_a) \}$$

$$= \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta} \{ \{ j[- 2m + a_1(m+1)](1/x) - ja_1(m-1)(1/x_a) \}$$

$$= \sqrt{\frac{x_a}{x}}\, e^{(1+j)(r-a)/\delta}\, j\{ [- 2m + a_1(m+1)](1/x) - a_1(m-1)(1/x_a) \}$$

This whole thing is then order 1/x so we get

$$h_m \approx 0$$

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Compare my limit to the M notation:

$$J_m(x) = (1/2\pi x)^{1/2}\, e^{-jx}\, e^{j\pi m/2}\, e^{j\pi/4} [\, 1 - j\, c_m\,(1/x) - d_m\,(1/x^2) + \ldots\,]$$

Pause. Compare this to

$$J_m(e^{j3\pi/4}z) = M_m(z)\, e^{j\theta_m(z)}. \qquad (2.3.3)$$

so we have

$$x = e^{j3\pi/4}z = (j-1)z/\sqrt{2}$$

$e^{-jx} = e^{-j(j-1)z/\sqrt{2}} = e^{(1+j)z/\sqrt{2}}$

My limit expressed in terms of z is then

$J_m(x) \approx (1/2\pi\, e^{j3\pi/4} z)^{1/2}\, e^{(1+j)z/\sqrt{2}}\; e^{j\pi m/2}\, e^{j\pi/4} [\, 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z) - e^{-j3\pi/2} d_m\, (1/z^2) + \ldots\,]$

$|J_m(x)| \approx (1/2\pi\, z)^{1/2}\, e^{z/\sqrt{2}} \;\; |\; 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z) - e^{-j3\pi/2} d_m\, (1/z^2) + \ldots\; |$

$\approx (1/2\pi\, z)^{1/2}\, e^{z/\sqrt{2}} \;\; |\; 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z)\,|$

and the leading factors agree with (2.3.5). Now,

$|\; 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z)\,| = |\; 1 - j\, c_m(-1-j)\, (1/\sqrt{2}\, z)\,| = |\; 1 + j\, c_m(1+j)\, (1/\sqrt{2}\, z)\,|$

$= |\; 1 + c_m(j-1)\, (1/\sqrt{2}\, z)\,|$

So

$|\; 1 + c_m(j-1)\, (1/\sqrt{2}\, z)\,|^2 = (1 + c_m(j-1)\, (1/\sqrt{2}\, z))\, (1 + c_m(-j-1)\, (1/\sqrt{2}\, z))$

$= 1 + c_m(-j-1)\, (1/\sqrt{2}\, z) + c_m(j-1)\, (1/\sqrt{2}\, z))$

$= 1 - 2\, c_m\, (1/\sqrt{2}\, z)$

$\Rightarrow |\; 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z)\,| \approx 1 - c_m\, (1/\sqrt{2}\, z) = 1 - a_1(m)\, (1/\sqrt{2}\, z)$

Now NIST says,

10.17.1

$$a_k(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2k-1)^2)}{k!8^k}, \qquad k \geq 1,$$

$a_1(m)$: 2k-1 = 1 so numerator is $(4m^2 - 1)$, denom = 8, so

$a_1(m) = (4m^2 - 1)/8$

Then my limit becomes

$|J_m(x)| \approx (1/2\pi\, z)^{1/2}\, e^{z/\sqrt{2}} \;\; [\; 1 - a_1(m)\, (1/\sqrt{2}\, z)\,]$

$= (1/2\pi\, z)^{1/2}\, e^{z/\sqrt{2}} \;\; [\; 1 - \frac{4m^2-1}{8}\, (1/\sqrt{2}\, z)\,]$

$$= (1/2\pi\, z)^{1/2}\, e^{z/\sqrt{2}}\ \left[\ 1 - \frac{4m^2-1}{8\sqrt{2}}\ (1/z)\ \right]$$

This agrees with my (now corrected) (2.3.5)

How about the phase part?

$$J_m(x) \approx\ (1/2\pi\, e^{j3\pi/4} z)^{1/2}\, e^{(1+j)z/\sqrt{2}}\ \ e^{jm\pi/2}\, e^{j\pi/4}\, [\ 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z) - e^{-j3\pi/2} d_m\, (1/z^2)\ + \ldots\ ]$$

$$\approx\ e^{-j3\pi/8} (1/2\pi\, z)^{1/2}\, e^{(1+j)z/\sqrt{2}}\ \ e^{jm\pi/2}\, e^{j\pi/4}\, [\ 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z) - e^{-j3\pi/2} d_m\, (1/z^2)\ + \ldots\ ]$$

The leading factor has this phase (ignoring the [...] for the moment):

$$\theta = -\pi 3/8 + z/\sqrt{2} + m\pi/2 + \pi/4\ \ = -3\pi/8 + z/\sqrt{2} + m\pi/2 + 2\pi/8\ \ = -\pi/8 + z/\sqrt{2} + m\pi/2$$

This agrees with the leading terms of (2.3.5). Now for the bracket

$$[\ 1 - j\, c_m\, e^{-j3\pi/4}\, (1/z)]\ \ =\ \ 1 + c_m(j-1)\, (1/\sqrt{2}\, z) \qquad \text{// from a foot above}$$

$$= (1 - c_m\, (1/\sqrt{2}\, z)\ + j\, c_m (1/\sqrt{2}\, z)$$

$$\theta = \tan^{-1}\, (y/x)\ \approx y/x\ \approx\ c_m(1/\sqrt{2}\, z) / 1\ \approx c_m(1/\sqrt{2}\, z) \approx (4m^2 - 1)/8(1/\sqrt{2}\, z)$$

$$= \frac{4m^2-1}{8\sqrt{2}}\ (1/z)$$

and this agrees also with (2.3.5).

Fact: I have proven that my two limit formulas are the same and fixed some bugs!