**Low Freq limit of E fields** PhL 5.17.14

Early text on the low ω limits of $f_m$ etc and then of the E fields. This is where I had my slightly bogus cutoff frequency $\omega_c$ that was part of a loss model. I had to undo all that stuff and it was painful. I installed a better loss model somewhere else in lines doc that has no $\omega_c$ parameter. I then had to clean up Appendix M, it was all a big mess and took many hours. But basically what you see below is now in Section D.11. This loss model of course did not fix the $J_z$ problem. (10.8.14 review).


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## D.10 Low frequency limit of the round wire E fields

Recall from (D2.2.33) that,

**Summary of the E field solutions :** $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m$ $g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m$ $h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

Our goal here is to evaluate $f_m$, $g_m$ and $h_m$ in the low ω limit.

For low frequency we continue to use

$$\beta = e^{j3\pi/4}\,(\sqrt{2}/\delta) = e^{j3\pi/4}\sqrt{\sigma\omega\mu} \qquad (2.2.20) \qquad (2.2.21)$$

### (a) The effect of including loss on the parameter β'

If for the wave's wavenumber $\beta_d$ one uses the usual lossless $\beta_d = (\omega/v_d)$ one finds,

$$\frac{\beta_d}{\beta} = \frac{(\omega/v_d)}{e^{j3\pi/4}\sqrt{\sigma\omega\mu}} = \text{constant} * \sqrt{\omega} \qquad \text{// no loss} \qquad (D.10.1)$$

which results in $(\beta_d/\beta) \rightarrow 0$ for the low ω limit. Since $\beta'^2 = \beta^2 - \beta_d^2$ this again gives $\beta' = \beta$ as in the previous section. But now this β is very small at low ω, instead of very large as in the Section D.9. This is the correct limit, but we want to make sure it is still the correct limit if we include cable loss.

Including in $\beta_d$ a model for the unavoidable loss, we find at low ω (see section (b) below) that

$$\beta_d = (\omega - j\omega_c)/v_d \qquad \omega_c = (v_d^2/2)\,(R_{dc}C) \tag{D.10.2}$$

$$E(z) = E(0)\, e^{-j\beta_d z} = e^{-jz(\omega - j\omega_c)/v_d} = e^{-j(\omega/v_d)z}\, e^{-(\omega_c/v_d)z}$$

where $R_{dc}$ is the DC resistance per unit length of the transmission line (both conductors) and C is the capacitance per unit length. Reconsider now the low ω situation:

$$\frac{\beta_d}{\beta} = \frac{(\omega - j\omega_c)/v_d}{e^{j3\pi/4}\sqrt{\sigma\omega\mu}} \rightarrow \text{constant}/\sqrt{\omega} \tag{D.10.3}$$

Now $(\beta_d/\beta) \rightarrow \infty$ for the low ω limit, just the reverse of the lossless situation. Since $\beta'^2 = \beta^2 - \beta_d^2$ one finds from (D.10.2) that, for low ω,

$$\beta'^2 \approx -\beta_d^2 = -(j\omega_c)^2/v_d^2 = (\omega_c/v_d)^2$$

so

$$\beta' \approx (\omega_c/v_d)\,. \tag{D.10.4}$$

But since $\omega_c$ is usually fairly small and $v_d$ is fairly large, it turns out that $a\beta' << 1$ and we are then still interested in the small β' limit of our E field coefficients like $f_m$, as shown in an example below.

Alternatively one can write,

$$\beta_d = \beta_{d0} - j\alpha$$
$$\beta_{d0} = (\omega/v_d)$$
$$\alpha = (\omega_c/v_d) = (v_d/2)\,(R_{dc}C) \qquad \dim(\alpha) = \text{sec}/\text{m}^2 * (\text{m}/\text{sec}) = \text{m}^{-1} \tag{D.10.5}$$

where α is the amplitude loss factor for a propagating wave

$$e^{-j\beta_d z} = e^{-j\beta_{d0} z}\, e^{-\alpha z}$$

$$\text{Re}(e^{-j\beta_d z}) = e^{-\alpha z}\cos(\beta_{d0} z) = e^{-(\omega_c/v_d)z}\cos[(\omega/v_d)z]\,. \tag{D.10.6}$$

**(b) Derivation of the simple loss model** **[** here is that loss model! **]**

Start with this expression for power P in a wave,

$$P = IV = U v_d = [(1/2)L I^2 + (1/2)CV^2]\, v_d \tag{D.10.7}$$

Then

$$dP/dz = [L\, I\, dI/dz + C\, V\, dV/dz]\, v_d$$

$$= [\, L\, I\, (-\mathbf{y}\, V) + C\, V\, (-\mathbf{z}\, I)\,]\, v_d \qquad \text{// using (4.11.15)}$$

$$= -\, IV\, [\, L\mathbf{y} + C\mathbf{z}]\, v_d = -\, P[\, L(G + j\omega C) + C(R + j\omega L)]\, v_d$$

$$= -2\alpha\, P \qquad \text{(D.10.8)}$$

where

$$\alpha = [\, L(G + j\omega C) + C(R + j\omega L)]\, (v_d/2)\ . \qquad \text{(D.10.9)}$$

So then

$$P(z) = e^{-2\alpha z}\, P(0)$$
$$\mathbf{E}(z) = e^{-\alpha z}\, \mathbf{E}(0)\ . \quad \text{// electric field and all other non-energy non-power quantities} \qquad \text{(D.10.10)}$$

Ignoring G and assuming low ω, we obtain the result quoted above,

$$\alpha = [\, L(G + j\omega C) + C(R + j\omega L)]\, (v_d/2) \approx (LG+RC)\, (v_d/2) \approx (R_{dc}C)\, (v_d/2) \qquad \text{(D.10.11)}$$

The main point is that α and $\omega_c$ are constant, independent of ω.

As an example, consider Belden 8281 coaxial cable :

## Electrical Characteristics (Overall)

**Nom. Characteristic Impedance:**

| Impedance (Ohm) |
|---|
| 75 |

**Nom. Inductance:**

| Inductance (µH/m) |
|---|
| 0.429811 |

**Nom. Capacitance Conductor to Shield:**

| Capacitance (pF/m) |
|---|
| 68.901 |

**Nominal Velocity of Propagation:**

| VP (%) |
|---|
| 66 |

**Nominal Delay:**

| Delay (ns/m) |
|---|
| 5.05274 |

**Nom. Conductor DC Resistance:**

| DCR @ 20°C (Ohm/km) |
|---|
| 32.4819 |

**Nominal Outer Shield DC Resistance:**

| DCR @ 20°C (Ohm/km) |
|---|
| 3.6091 |

**Nom. Attenuation:**

| Freq. (MHz) | Attenuation (dB/100m) |
|---|---|
| 1 | 0.9843 |
| 3.6 | 1.6405 |
| 10.0 | 2.6248 |
| 71.5 | 6.8901 |
| 135 | 9.843 |
| 270 | 14.1083 |
| 360 | 16.7331 |
| 540 | 20.6703 |
| 720 | 24.2794 |
| 750 | 24.9356 |
| 1000 | 30.1852 |

For this standard 75Ω cable one finds that

$R_{dc}$ = (32.5 + 3.6) Ω/km = 36.1 x $10^{-3}$ ohms/m

C = 69 pF/m = 69 x $10^{-12}$ farad/m

$v_d$ = 0.66c = 0.66 x 3 x $10^8$ m/sec = 2 x $10^8$ m/sec

```
Rdc := 36.1e-3;
```

$$Rdc := .0361$$

```
C := 69e-12;
```

$$C := .69\ 10^{-10}$$

```
vd := 0.66 * 3e8;
```

$$vd := .198\ 10^{9}$$

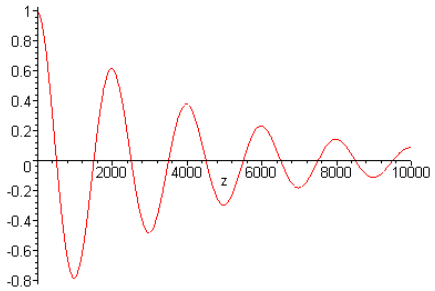
```
fc := Rdc*C*vd^2/(4*evalf(Pi));
```

$$fc := 7770.998213$$

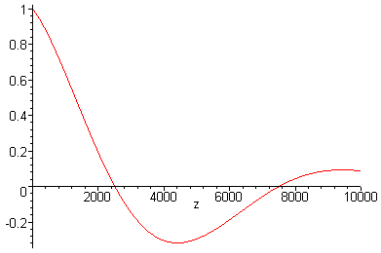
So $f_c$ = 7770 Hz and $\omega_c = 2\pi f_c$. So any line quantity like V(z) has this behavior,

$$V(z) = V(0)\, e^{-(\omega_c/v_d)z} \cos[(\omega/v_d)z] = V(0)\, e^{-(2\pi f_c/v_d)z} \cos[2\pi(f/v_d)z] \qquad \text{(D.10.12)}$$

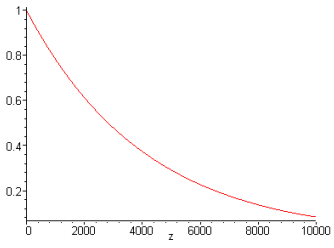
which we can plot for V(z) over 10 km of cable for several values of f:




f = 100 KHz




f = 10 KHz




f = 1 KHz

In all cases, some signal arrives at the end (ignoring noise), but in the right two graphs one would say the cable was quite "lossy".

For this example, we can compute β' for low ω:

$\beta' = (\omega_c/v_d) = 2\pi * 7770 / 2*10^8 = .00024 = 2.4 \times 10^{-4}\ m^{-1}$

$a = 394\mu = 394 \times 10^{-6}\ m = 0.394 \times 10^{-3}\ m$

$x_a = \beta' a = 2.4 \times 10^{-4} * 0.394 \times 10^{-3} = 0.95 \times 10^{-7}$ (D.10.13)

**(c) Low frequency evaluation of $f_m$, $g_m$ and $h_m$ and the E fields**

In the following we consider only $m \geq 0$ since we know from (D.9.2) that $f_{-m} = f_m$ , $g_{-m} = g_m$, $h_{-m} = h_m$ .

The small x limit for $J_m(x)$ is given by NIST 10.7.3,

$$J_n(x) = (x/2)^n / n! \quad . \qquad \text{for } n = 0,1,2,..... \qquad \text{(D.10.14)}$$

Since $J_{m-1}$ appears in our coefficient expressions and since m = 0 is encountered, we have to deal with m = 0 as a special case, since the above limit is not valid for n = -1. To this end we use NIST 10.2.2 which is valid for integer n,

$$J_{-n}(x) = (-1)^n J_n(x) \approx (-1)^n (x/2)^n / n! \qquad \text{(D.10.15)}$$

so that $J_{-1}(x) = - J_1(x) \approx - (x/2)$. Our limiting forms of interest are then

$$\begin{aligned} &J_n(x) = (x/2)^n / n! && \text{for } n = 0,1,2,..... \\ &J_{-1}(x) = - (x/2) && \text{for } n = -1 \end{aligned} \qquad \text{(D.10.16)}$$

We now examine the small x limits of $f_m$, $g_m$, and $h_m$ .

First $f_m$ for m > 0, and then for m = 0:

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m/m!}{(x_a/2)^{m+1}/(m+1)!} - \frac{(x/2)^m/m!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= [\,(m+1)\,(x/x_a)^m\,(2/x_a) - (1/m)\,(x/x_a)^m(x_a/2)\,]$$

$$= (x/x_a)^m\,[\,(m+1)\,(2/x_a) - (1/m)\,(x_a/2)\,]$$

$$\approx (x/x_a)^m\,(m+1)\,(2/x_a) \quad \text{as } x_a \rightarrow 0$$

$$f_0 = \left[\frac{J_0(x)}{J_1(x_a)} - \frac{J_0(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_0(x)}{J_1(x_a)} + \frac{J_0(x)}{J_1(x_a)}\right] = 2\,\frac{J_0(x)}{J_1(x_a)} = 2\,\frac{1}{(x_a/2)} = 4/x_a$$

First $g_m$ for m > 0, and then for m = 0:

$$g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^{m+1}/(m+1)!}{(x_a/2)^{m+1}/(m+1)!} - \frac{(x/2)^{m-1}/(m-1)!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= (x/x_a)^{m+1} - (x/x_a)^{m-1}$$

$$g_0 = \left[ \frac{J_1(x)}{J_1(x_a)} + \frac{J_{-1}(x)}{J_{-1}(x_a)} \right] = \left[ \frac{J_1(x)}{J_1(x_a)} + \frac{J_1(x)}{J_1(x_a)} \right] = 2 \frac{J_1(x)}{J_1(x_a)} = 2\,(x/x_a)$$

Results for $h_m$ are then obvious since there is only a sign change between the terms in $g_m$,

$$h_m = (x/x_a)^{m+1} + (x/x_a)^{m-1} \qquad h_0 = 0$$

**Small ω limit of the E field solutions :** $\qquad R_{dc} = \dfrac{1}{\sigma \pi a^2}$ $\qquad$ (D.10.17)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = (r/a)^{m+1} - (r/a)^{m-1} \qquad g_0 = 2\,(r/a)$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = (r/a)^{m+1} + (r/a)^{m-1} \qquad h_0 = 0$$

Recall now these field expansions for the case that n(θ) is an even function of θ,

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta')\, [\, f_0 + 2\, \Sigma_{m=1}^{\infty}\, f_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d)\, [\, g_0 + 2\, \Sigma_{m=1}^{\infty}\, g_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta_d)\, [\, h_0 + 2\, \Sigma_{m=1}^{\infty}\, h_m\, \eta_m\, \cos(m\theta)\,] \ . \qquad \textit{(D.9.4)}$$

For low ω we insert the expressions above to get

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, [\, 4 + 4\, \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d)\, [2\,(r/a) + 2\, \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} - (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta_d)\, [2\, \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} + (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,] \ . \qquad \text{(D.10.18)}$$

**Observations on the E fields for small ω**

1. The m = 0 term in $E_z$ is just $E_z = IR_{dc}$ which says $J_z = \sigma\, IR_{dc} = \sigma\, I(1/\sigma\pi a^2) = I/(\pi a^2)$. This is the current one would expect in a wire carrying DC current I.

2. Assuming a non-uniform n(θ) charge density on the wire surface, the $\eta_m$ moments are non-zero and one concludes that $J_z(r,\theta)$ is asymmetric even as $\omega \to 0$. This seems to conflict with the notion that the proximity effect should go away as $\omega \to 0$ as considered in Appendix P. One item to note is that

$$I = V/Z_0$$

and according to (4.11.16)

$$Z_0 \equiv V(z)/i(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (4.11.16)$$

and in the limit $\omega \to 0$ we get $Z_0 = \sqrt{R/G}$ = a very large number. As one lowers $\omega$, one must increase the size of the termination $Z_0$, assuming there is one, to maintain a properly terminated line. Thus, our situation here does not apply to taking the low frequency limit of a transmission line terminated with a 75Ω or 8Ω resistor.

3. The fields $E_r$ and $E_\theta$ are smaller than the field $E_z$ by factor $x_a = a\beta_d$ which we have assumed is small.