Math Problem with R,L,C,G PhL 6.22.14

Early version of what was developed in Appendix Q. But *that* Appendix Q assumed constant parameters, which I had to replace by a new Appendix Q allowing for ω variation of the parameters R,L,G,C. But that only affected large and small ω limits. The result just *below* still appear as is in current Appendix Q, getting the real and imaginary part of k(ω) in essence. (10.8.14)

What is shown here:

$$s \equiv \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{\mathbf{zy}}$$

$$\mathrm{Re}(s) = (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4}\;\frac{1}{\sqrt{2}}\sqrt{1+\frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}}}$$

$$\mathrm{Im}(s) = (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4}\;\frac{1}{\sqrt{2}}\sqrt{1-\frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}}}$$

In simpler form, first define

$a \equiv (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4}$ dim(a) = R/m $a^2 \to \omega^2LC$ as $\omega \to \infty$

$c \equiv RG - \omega^2LC$ $\dim(c) = 1/m^2$

$$b \equiv \frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}} = c/a^2$$ dim(b) = 1 $b \to -1$ as $\omega \to \infty$; $1-b \to 2$

so that

$$\mathrm{Re}(s) = \frac{1}{\sqrt{2}}\,a\sqrt{1+b}$$

$$\mathrm{Im}(s) = \frac{1}{\sqrt{2}}\,a\sqrt{1-b}$$

$$s = \frac{1}{\sqrt{2}}\,a\,[\sqrt{1+b} + j\sqrt{1-b}\,]$$

<u>Verification</u> of this solution

$$s = \frac{1}{\sqrt{2}}\,a\,[\sqrt{1+b} + j\sqrt{1-b}\,]$$

$$= \frac{1}{\sqrt{2}}\, a\, [\sqrt{1+c/a^2} + j\sqrt{1-c/a^2}\,]$$

$$= \frac{1}{\sqrt{2}}\, [\sqrt{a^2+c} + j\sqrt{a^2-c}\,]$$

If I draw a triangle, the radius is a which is in fact correct. But how do I show the angle? From the above I can see that

$$\sin\varphi = \frac{\sqrt{a^2-c}}{a}/\sqrt{2} = [\ \frac{1-c/a^2}{2}\ ]^{1/2} =$$
$$\cos\varphi = \frac{\sqrt{a^2+c}}{a}/\sqrt{2} = [\ \frac{1+c/a^2}{2}\ ]^{1/2}$$

Now ansatz that $c/a^2 = \cos\theta$, then consistent with $\varphi = \theta/2$, but is the ansatz right?

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Consider

$$q \equiv (R+j\omega L)(G+j\omega C) = (RG-\omega^2 LC) + j\omega(LG+RC)$$

This q has positive imaginary part and unknown sign real part, so this argument is a vector in the upper half plane. We can then write

$$q = |q|\, e^{j\theta}$$

$$|q|^2 = |\,(R+j\omega L)(G+j\omega C)\,|^2 = |\,(R+j\omega L)|^2 |(G+j\omega C)\,|^2 = (R^2+\omega^2L^2)\,(G^2+\omega^2C^2)$$

$$\tan\theta = \omega(LG+RC) / (RG-\omega^2 LC)$$

$$|q| = \sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}$$

$$\theta = \tan^{-1}[\omega(LG+RC)] / (RG-\omega^2 LC)]$$

Note that the tangent jumps from $+\infty$ to $-\infty$ but the angle θ is continuous through $\pi/2$. Note that:

$$\cos\theta = (RG-\omega^2 LC) / |q| = (RG-\omega^2 LC)/\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)} = c/a^2$$

$$\sin\theta = \omega(LG+RC) / |q| = \omega(LG+RC)/\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}$$

So now define

$$s \equiv \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{q}$$

We can then write

$$s = |s|\, e^{i\varphi} \qquad |s| = \sqrt{|q|} \qquad \varphi = \theta/2$$

so then

$$|s| = (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4} = a$$

$$\varphi = (1/2)\tan^{-1}[\omega(LG+RC)] / (RG-\omega^2LC)]$$

Now try to write out the real and imaginary parts of this:

$$\mathrm{Re}(s) = |s|\cos\varphi = |s|\cos(\theta/2)$$
$$\mathrm{Im}(s) = |s|\sin\varphi = |s|\sin(\theta/2)$$

Now use

$$\cos\varphi = \cos(\theta/2) = \sqrt{(1+\cos\theta)/2} = \sqrt{(1 + (RG-\omega^2LC)/\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}\,)/2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+\cos\theta} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}}} = \frac{1}{\sqrt{2}}\sqrt{1 + c/a^2}$$

Note that the magnitude of the big fraction is always less than 1, so everything is real. Similarly

$$\sin\theta/2 = \sqrt{(1-\cos\theta)/2} = \sqrt{1-\frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}}}\,/\sqrt{2} = \frac{1}{\sqrt{2}}\sqrt{1 - c/a^2}$$

Then here is my conclusion:

$$s \equiv \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{\mathbf{zy}}$$

$$\mathrm{Re}(s) = (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4}\;\frac{1}{\sqrt{2}}\sqrt{1+\frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}}}$$

$$\mathrm{Im}(s) = (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4}\;\frac{1}{\sqrt{2}}\sqrt{1-\frac{(RG-\omega^2LC)}{\sqrt{(R^2+\omega^2L^2)\,(G^2+\omega^2C^2)}}}$$

If I ask Maple to do this, I get

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> restart; alias(I=I,j=sqrt(-1));
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$$j$$

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> assume(omega>0,R>0,G>0,L>0,C>0);
> z := R+j*omega*L;
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$$z := R + j\,\omega\,L$$

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> y := G + j*omega*C;
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$$y := G + j\,\omega\,C$$

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> s := sqrt(z*y);
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$$s := \sqrt{(R + j\,\omega\,L)\,(G + j\,\omega\,C)}$$

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> s := evalc(s);
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$$s := \frac{1}{2}\sqrt{2\sqrt{R^2 G^2 + \omega^4 L^2 C^2 + R^2 \omega^2 C^2 + \omega^2 L^2 G^2} + 2RG - 2\omega^2 LC} + \frac{1}{2} j \sqrt{2\sqrt{R^2 G^2 + \omega^4 L^2 C^2 + R^2 \omega^2 C^2 + \omega^2 L^2 G^2} - 2RG + 2\omega^2 LC}$$

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factor(s);
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$$\frac{1}{2}\sqrt{2\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + 2RG - 2\omega^2 LC} + \frac{1}{2} j \sqrt{2\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} - 2RG + 2\omega^2 LC}$$

but then Maple does not realize that, if you take the positive root of the inner root, then the argument of each outer root is positive and you get my answer. Maple is worried you might take the negative root for the inner root. That is why we get

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> Re(s);
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$$\frac{1}{2}\Re\left(\sqrt{2\sqrt{R^2 G^2 + \omega^4 L^2 C^2 + R^2 \omega^2 C^2 + \omega^2 L^2 G^2} + 2RG - 2\omega^2 LC} + j\sqrt{2\sqrt{R^2 G^2 + \omega^4 L^2 C^2 + R^2 \omega^2 C^2 + \omega^2 L^2 G^2} - 2RG + 2\omega^2 LC}\right)$$

and it refuses to take the real part.