Various Appendix A work PhL 9.6.13

Here I ponder various issues relating to Appendix A, such as Poisson with boundary metal present.

0. Lemma: The Poisson equation $-\nabla^2\varphi = \rho$ has a unique solution.

Imagine some charge distribution $\rho(\mathbf{x})$ that is constrained to a localized region within infinite space. Perhaps the charge is "glued" to some non-conducting material to keep it from moving around. There are no conductors anywhere in this space. In this idealized scenario, the solution to the Poisson equation $-\nabla^2\varphi(\mathbf{x}) = \rho(\mathbf{x})$, subject to the added condition that $\varphi(|\mathbf{x}|=\infty) = 0$, is unique and is given by the following expression:

$$\varphi(\mathbf{x}) = \frac{1}{4\pi}\int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \qquad (A.0.1)$$

where the integral is over all space. Since the charge is localized to some region, in practice the integral is over that finite region of space. We will first show that this $\varphi(\mathbf{x})$ is in fact a solution of the Poisson equation, and secondly, we will show that it is a unique solution. In an operational sense, we know the solution is unique since we put the charge there and it creates some potential and we know intuitively that there will be only one potential field possible.

To see that the above $\varphi(\mathbf{x})$ is a solution to the Poisson equation $-\nabla^2\varphi(\mathbf{x}) = \rho(\mathbf{x})$, we first consider a simpler equation $-\nabla^2 g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$. Here $g(\mathbf{x},\mathbf{x}')$ is the potential φ at an arbitrary observation point $\mathbf{x}$ which is created by a unit positive delta-function point charge located at position $\mathbf{x}'$. We know from elementary electrostatics that the potential of a point charge (in our units) is $\varphi = 1/(4\pi r)$ where r is the distance between $\mathbf{x}$ and $\mathbf{x}'$. Formally then our solution is $g(\mathbf{x},\mathbf{x}') = 1/(4\pi|\mathbf{x} - \mathbf{x}'|)$. Thus,

$$-\nabla^2 g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \quad \Rightarrow \quad g(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

so

$$-\nabla^2\{\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = \delta(\mathbf{x}-\mathbf{x}') \quad \text{or} \quad -\nabla^2\{\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = 4\pi\,\delta(\mathbf{x}-\mathbf{x}') \qquad (A.0.2)$$

Now apply $-\nabla^2$ to the right side of (A.0.1). Since the $-\nabla^2$ variable is $\mathbf{x}$, it passes through the (convergent) integral and hits $1/(|\mathbf{x} - \mathbf{x}'|)$ yielding $4\pi\,\delta(\mathbf{x}-\mathbf{x}')$, according to (A.0.2) so we find

$$-\nabla^2\varphi(\mathbf{x}) = \frac{1}{4\pi}\int d^3x'\,\rho(x')\,\{-\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = \frac{1}{4\pi}\int d^3x'\,\rho(x')\,4\pi\,\delta(\mathbf{x}-\mathbf{x}') = \rho(\mathbf{x}) \quad \text{QED.}$$

Thus the $\varphi(x)$ given in (A.0.1) is a solution to Poisson's equation. The assisting function $g(\mathbf{x},\mathbf{x}')$ is called a Green's Function or fundamental solution or free-space propagator; there are many names for such an object.

Suppose there were two different Poisson solutions φ and φ_1 which both vanish on the boundary at $|x| = \infty$. Let the difference be $u = \varphi - \varphi_1$. Then we must have $-\nabla^2 u(\mathbf{x}) = 0$ with $u = 0$ at $|x| = \infty$. If we can

show that the only solution of this last problem is u(x) ≡ 0, then any alternate solution like φ_1 must be the same as φ, so the solution φ is unique. The equation $-\nabla^2 u(\mathbf{x}) = 0$ is known as the Laplace Equation, and its solutions are called harmonic functions. It is not hard to show that any harmonic function must take both is max and min values on the boundary, which here is the great sphere. Thus $u_{max} = 0$ and $u_{min} = 0$, so the only possibility is that u(x) ≡ 0 everywhere. Thus, the φ(x) of (A.0.1) is unique.

Comment: In this Appendix, we shall continue to assume everything happens in free space without the presence of conductors or other "boundaries" other than the great sphere at infinity. Our analysis is complex enough without the added complication of dealing with such conductors which form "boundary conditions". Rest assured that these boundary conditions can be properly handled.

As an example, perhaps there is a piece of metal nearby our point charge which has a certain constant potential caused by a 9 volt battery connecting it to the great sphere at ∞. We can then regard our point charge as being enclosed in a region consisting of the surface of this piece of metal and the great sphere. The potential is specified everywhere on this boundary (it is 0 on the great sphere and perhaps +9 on the metal). A famous result of electrostatics (the Dirichlet Problem) is that if a point charge or any charge distribution is surrounded by a closed boundary on which the potential is prescribed, then if one first solves the problem for a Green's Function $g(\mathbf{x},\mathbf{x}')$ which vanishes everywhere on the boundary, one obtains an explicit and unique solution of the problem which consists of the original solution of (A.0.1) plus other terms which are solutions of $-\nabla^2 u(\mathbf{x}) = 0$ and which assure that the boundary conditions are met. The new Green's Function $g(\mathbf{x},\mathbf{x}')$ in this context is $1/(4\pi|\mathbf{x} - \mathbf{x}'|)$ plus a second term which solves $\nabla^2 u(\mathbf{x}) = 0$ and makes g = 0 on the entire boundary.

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I am wondering about this entire Appendix and what its main point is. I want to show that the **E** and **B** fields can be determined by the potentials φ and **A**, and that a gauge transformation on φ and **A** does not affect **E** and **B**, and that in fact by doing a suitable transformation, you can arrange to have div **A** = any f(x) you want including 0. I think these are the main points. It seems to me that by using integral forms, I have complicated things a lot and that the whole thing can be done with differential forms only.

Question: How is this appendix used in the main doc?

- used for (1.3.4)
- below 1.5.9:

See Appendix 1.1 for an explanation of Green's Functions and why the above integrals solve (1) and (2).

Answer is that I hardly use it all except for the first item. So I am free I think to redo it.

Question 1: how do you show that E and B can be obtained from the potentials?

Here are the important equations.

B = **curl A**
E = - **grad** φ - ∂**A** /∂t

This does not give a definition of A and φ however. So how do you show that this gives the most general form possible for B and E? I guess the first step is to show that the E and B obtained satisfy the four Maxwell equations, I think that would be fairly easy. But why does this include all possible solutions? Here is my quote of the Maxwell equations for a conducting medium.

$$\text{curl } \mathbf{H} = \partial\mathbf{D}/\partial t + \mathbf{J_T} = \partial\mathbf{D}/\partial t \; + \; [\, \sigma\mathbf{E} + \mathbf{J_a} \,] \tag{1.1.1}$$

$$\text{curl } \mathbf{E} = -\, \partial\mathbf{B}/\partial t \tag{1.1.2}$$

$$\text{div } \mathbf{D} = \rho_{\mathbf{T}} \; = \; [\, \rho + \rho_{\mathbf{a}} \,] \tag{1.1.3}$$

$$\text{div } \mathbf{B} = 0 \tag{1.1.4}$$

$$\mathbf{B} = \mu\mu_0\mathbf{H} \tag{1.1.5}$$

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} \tag{1.1.6}$$

Maybe ask a simpler question: electrostatics in vacuum. Why can you say $\mathbf{E} = -\nabla\varphi$? The Maxwell equations in this case away from charges are div$\mathbf{E} = 0$ and curl$\mathbf{E} = 0$. If you seem the most general possible solution to these equations, why do you get $\mathbf{E} = -\nabla\varphi$? This method allows you to construct a set of solutions, that is true. But why does this include all possible solutions? Jackson uses the integral form on page 7 as I do. This integral form is Coulomb's law applied to ρ. So how do "boundaries" affect this? If there are pieces of metal lying around, they might have charge ρ and that charge would then be included. If there is a Dirichlet specification say on a sphere. again that might result in some charges on the sphere, and then those are included. So that is a good way to handle BC's.

OK, I am going to attempt a section 0 rewrite from this point of view.

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Here I make a longer version of Section 0 including those boundaries in effect.

0. Fact 0: The Poisson equation $-\nabla^2\varphi = \rho/\varepsilon_0$ has a unique solution as stated below.

(a) Imagine some static charge distribution $\rho(\mathbf{x})$ that is constrained to a perhaps large but localized region within infinite space. The distribution ρ(x) includes *all* charges in this region. Here are some types of charges which would be included in ρ(x):

- point charges which are "glued down" to certain points in space.
- linear continuous charge densities that are glued down along curved filaments in space.
- surface charge densities that are either glued to certain surfaces, or which are stable because they lie on the surfaces of pieces of conductor (like metal). These two charge types account for "Dirichlet boundary conditions", "Neumann boundary conditions", and "mixed boundary conditions", but these phrases need not concern us here. In the Neumann case, a piece of surface has two charge density layers of opposite sign very closely spaced (but not superposed), which is sometimes called a dipole layer.

- 3D continuous charge densities that are "glued down" somehow in 3D space so they cannot move.

By including all these types of charge in ρ, we are able to avoid the complicating issue of "boundary conditions" in our discussion below, and our only boundary of interest is The Great Sphere which is a sphere of infinite radius surrounding our localized region of interest.

From Coulomb's Law (in SI units) we know that the electric potential φ of a point charge q located at point $\mathbf{x}'$ is $\varphi(\mathbf{x}) = (1/4\pi\varepsilon_0)q/r$ where ε_0 is a certain constant appropriate to SI units and $r = |\mathbf{x}-\mathbf{x}'|$ is the distance between charge q at $\mathbf{x}'$ and an observation point $\mathbf{x}$. Such a point charge is described by $\rho(\mathbf{x}) = q\delta(\mathbf{x}-\mathbf{x}')$. The general equation which relates $\varphi(\mathbf{x})$ to $\rho(\mathbf{x})$ is the Poisson Equation,

$$-\nabla^2\varphi(\mathbf{x}) = \rho(\mathbf{x})/\varepsilon_0 \ . \qquad \text{(A.0.1)}$$

Since this is a linear equation (the operator ∇^2 is linear), we may superpose the potentials of multiple charges to get the potential resulting from a distribution of charges. Thus, we at once obtain this superposed version of Coulomb's law

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \quad . \qquad \text{(A.0.2)}$$

where $d^3\mathbf{x}'\ \rho(\mathbf{x}') = dq(\mathbf{x}')$ = a differential chunk of charge located at $\mathbf{x}'$ contained in tiny volume $d^3\mathbf{x}'$. Thus, (A.0.2) must be a solution of (A.0.1). If we allow the observation point $\mathbf{x}$ to move right on top of some point charge in the distribution ρ, we will get $\varphi = \infty$, so we generally avoid such points.

(b) We would like to explicitly show that (A.0.2) is a solution of (A.0.1) for a general distribution ρ. To this end, we digress to consider the following equation and its solution,

$$-\nabla^2 g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')/\varepsilon_0 \qquad \Rightarrow \qquad g(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x}-\mathbf{x}'|} \qquad \text{(A.0.3)}$$

The equation on the left is Poisson's Equation where ρ consists of a positive point charge of q = 1 unit sitting at position $\mathbf{x}'$. Recall from above that $\rho(\mathbf{x}) = q\delta(\mathbf{x}-\mathbf{x}')$ for a point charge. Coulomb's law gives the solution shown on the right. Therefore it must be true that

$$-\nabla^2\{\frac{1}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = \delta(\mathbf{x}-\mathbf{x}')/\varepsilon_0 \qquad \text{or} \qquad -\nabla^2\{\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = 4\pi\,\delta(\mathbf{x}-\mathbf{x}') \ . \qquad \text{(A.0.4)}$$

This fact can be verified directly using the theory of distributions which says $-\nabla^2(1/r) = 4\pi\delta(\mathbf{r})$, but we have already shown it is true, given Coulomb's Law and superposition. We can now show that (A.0.2) is a solution of (A.0.1) for an arbitrary distribution ρ as follows:

$$-\nabla^2\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\,\rho(\mathbf{x}')\,\{-\nabla^2\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\,\rho(\mathbf{x}')\,4\pi\,\delta(\mathbf{x}-\mathbf{x}') = \rho(\mathbf{x})/\varepsilon_0 \qquad \text{QED.}$$

The assisting function g(**x**,**x'**) has various names with respect to (A.0.3): the Green's Function, the fundamental solution, the free-space propagator. It is nothing more than the potential created by a point charge of 1 unit located at **x**' and viewed from **x**.

(c) We have found the particular solution of (A.0.1) given by (A.0.2). There are many *other* solutions which can be obtained by adding to the solution (A.0.2) a solution of $-\nabla^2 u = 0$. This last equation, usually written $\nabla^2 u = 0$, is called the Laplace Equation, and it is the "homogeneous" form of the Poisson Equation, that is, the right side of the Poisson Equation is set to 0. Solutions u are called homogeneous solutions. One obvious solution is u = 2, so we could then add 2 to (A.0.2) and get a new solution to (A.0.1). Since we have specified that our charge distribution ρ(x) is localized to some region of space, we expect that as $x \to \infty$, we must have $\varphi \to 0$. The solution (A.0.2) meets this requirement, but if we add 2, then our physical requirement is not met, so we must rule out adding a 2. We would also rule out 2x + 3, for example, or 7xy. Recall that $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$.

It turns out that the only solution of $\nabla^2 u = 0$ which meets the requirement $u \to 0$ as $\mathbf{x} \to \infty$ in all directions is the trivial function u(**x**) = 0. In 2D one intuitively sees this because a massless thin rubber sheet tied down to height u = 0 around a large circular perimeter is going to be a *flat* rubber sheet with u = 0 everywhere. The solutions to the 3D equation $\nabla^2 u = 0$ are called harmonic functions, and it is not hard to show that any harmonic function must take both is max and min values on the boundary, which here is a 3D great sphere. Thus $u_{max} = 0$ and $u_{min} = 0$, so the only possibility is that $u(x) \equiv 0$ everywhere.

The implication of the previous paragraph is that (A.0.2) is the only possible solution of (A.0.1) because the only homogeneous solution one is allowed to add to (A.0.2) is u = 0. One can suppose there are two different solutions of $-\nabla^2\varphi = \rho/\varepsilon_0$ called φ and φ' both of which go to 0 on the great sphere. Then $-\nabla^2(\varphi-\varphi') = 0$ with $(\varphi-\varphi') \to 0$ on the great sphere. But then $(\varphi-\varphi') = 0$ so $\varphi' = \varphi$ and there cannot then exist two different physical solutions of (A.0.1).

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So the above version "deals with" the boundaries at least in some manner. How then are later appendix sections affected?

Fact 1 section: I did in-place editing here, adding ε_0. I claim that curl B "drops off" so there is no parts contribution. At least I mention the parts, but don't show precisely what the required drop off condition is on various operators acting on B. Presumably as with ρ, curl B includes contributions to curl B from any "boundary situations", though I don't explicitly say that.

Now let's try Fact 2: it looks good too!

Fact 3 is also OK, small edits.

Fact 4 is OK as well.

Fact 5 and most of Fact 6 are also OK.

Everything I have done up to this point is in SI units, but then in 7 I seem to screw up. So let's try an SI version of section 7. But I have to first pick a metric tensor so I pick BD which is diag(1,-1,-1,-1)

I had issues with the proper scaling of the 4 vector components, maybe this is being treated below.

7. The Lorentz Gauge and QED: This section is certainly off the transmission-lines beaten path, but the author thought the reader might find it interesting. It is true that the nature of a transmission line results from photons "jumping back and forth" between the conductors. Unlike elsewhere in this document, everything is not "explained" in the following quick outline.

In relativistic notation one uses 4-vectors which have one time component and three spatial components such as $x^\mu = (t,x,y,z)$. This is a contravariant 4-vector and the corresponding covariant 4-vector is $x_\mu = (t,-x,-y,-z)$. Thus, one has $x^0 = x_0$ but $x^i = -x_i$. We are assuming here the "Bjorken-Drell metric" $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$. The gradient operator ∂_i "transforms as" the covariant part of a 4-vector, and one can write $\partial^i = -\partial_i$ just as $x^i = -x_i$ for i = 1,2,3. A four-vector gradient operator can be written $\partial_\mu = (\partial_0, \partial_i)$ and $\partial^\mu = (\partial^0, \partial^i) = (\partial_0, -\partial_i)$ where $\partial^0 = \partial_0 = \partial_t = \partial/\partial t$. The Laplacian is $\nabla^2 = \partial_i\partial_i$ (implied sum on i) while the corresponding object $\Box \equiv \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$ is the D'Alembertian which appears in wave equations. As is normal in discussions of this type, the speed of light is taken to be c = 1.

Consider the gauge transformation (A.5.1) which in relativistic notation is

$$A'^i = A^i + \partial_i\Lambda = A^i - \partial^i\Lambda \qquad i = 1,2,3$$
$$\varphi' = \varphi - \partial^0\Lambda \qquad \text{(A.7.1)}$$

The components of a classical vector like **A**, normally written as A_i, are in fact the contravariant components A^i in relativistic notation. If we now identify $\varphi \equiv A^0$ we can combine the two gauge transformation equations into a single equation involving three 4-vectors (one of which is $\partial^\mu\Lambda$),

$$A'^\mu = A^\mu - \partial^\mu\Lambda \qquad \mu = 0,1,2,3. \qquad \text{(A.7.2)}$$

Suppose we want $\partial_\mu A'^\mu = 0$ (implicit sum on $\mu = 0,1,2,3$). If we could find a potential A'^μ with this property, that would be very convenient for the following reason: In general $a_\mu b^\mu$ ($= a^\mu b_\mu = a \bullet b$) is the same in all frames of reference related by Lorentz Transformations. If $\partial_\mu A'^\mu = 0$ in one frame, it is 0 in all frames, and that makes computational life simple.

So, is it possible to have $\partial_\mu A'^\mu = 0$? Writing this out we get

$$\partial_t\varphi + \partial_i A'^i = 0$$

or

$$\partial_t\varphi + \text{div } \mathbf{A}' = 0$$

or

$$\text{div } \mathbf{A}' = -\partial_t\varphi. \qquad \text{(A.7.3)}$$

But we showed in Fact 2 that given any **A**, we can find an equivalent **A'** which has div **A'** = any f(**x**) we want, so we just select our arbitrary function f(**x**) to be $-\partial\varphi/\partial t$. By selecting this f(x), we are **selecting the Lorentz Gauge**. In this gauge (dropping the prime on A), we have $\partial_\mu A^\mu = 0$. As just noted, this gauge is so named because the resulting equation $\partial_\mu A^\mu = 0$ is "covariant" under all Lorentz transformations. That means the equation has the same form in all frames of references which are related by Lorentz transformations (which include rotations, velocity transformations, and combinations of same). One can

interpret $\partial_\mu A^\mu = 0$ as $\partial \bullet A = 0$ where ∂ is a 4-divergence operator. Thus, in the Lorentz gauge, the 4-divergence of A^μ is always exactly 0 at every point in spacetime.

In relativistic quantum field theory (aka quantum electrodynamics, or QED), the potential A^μ is interpreted as the quantum field of a vector particle called the photon. In the Lagrangian density which describes the interaction between the photon and electron, the term $J_\mu A^\mu$ appears,

$$\mathcal{L} = \ldots - J_\mu A^\mu \qquad J_\mu = e_0 \bar{\psi} \textbf{ Error! Bookmark not defined.}\gamma_\mu \psi \qquad \text{(A.7.4)}$$

where J_μ is the electric current, an operator built from the quantum field ψ of the electron. Because of gauge invariance (A.7.2), a gauge transformation creates a new term $J_\mu \partial^\mu \Lambda$ in the Lagrangian density. The physics of QED is determined by $S = \int d^4x \mathcal{L} = \int d^3x \int dt \mathcal{L}$ which is called the action. If we insert the gauge term $J_\mu \partial^\mu \Lambda$ into the action and do part integration to move ∂^μ from Λ to J_μ we end up with a gauge term of the form $\Delta S = \int d^4x (\partial^\mu J_\mu)\Lambda$. But at every point in spacetime, we know that $\partial^\mu J_\mu = 0$ (shown in a moment) so we find that $\Delta S = 0$ which means the action S is invariant under a gauge transformation. The reason $\partial^\mu J_\mu = \partial_\mu J^\mu = 0$ is because $J^\mu = (\rho, J^i)$ where ρ is charge density and J^i is electric current, and then the statement $\partial_\mu J^\mu = 0$ says that $\partial_t \rho + \partial_i J^i = 0$ or div $\mathbf{J} = -\partial\rho/dt$. This is the equation of continuity which says that if there is a current flowing out of a tiny volume of space, the charge density in that volume must be correspondingly decreasing. In other words, charge is "conserved". We can reverse our logic to conclude that the reason electric charge is conserved and cannot leak away "into the vacuum" is due to the invariance of the QED action under gauge transformations (A.7.2). More generally, symmetries (invariances) of the action always result in conserved quantities. Since 1949, unusual names have been given to similar conversed quantities: isospin, strangeness, color, charm, etc. The association of a conserved quantity with a differential symmetry of the action is known as Noether's Theorem, in honor of Emmy Noether who first showed this connection in 1915.

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I am removing this text that no longer seems useful and has units problems,

If writes down the Lagrangian density $\mathcal{L}$ for the electromagnetic field and its interaction with charges and currents, one finds that (Goldstein equation (11-65) page 366, this is in Gaussian or cgs units, not SI units)

$$\mathcal{L} = (E^2 - B^2)/8\pi - \rho\varphi + \mathbf{j} \bullet \mathbf{A}/c \qquad \text{(A.6.1)}$$

where c is the speed of light, ρ is charge density, and $\mathbf{j}$ is current density. We have already shown that the term $(E^2 - B^2)$ is gauge invariant since $\mathbf{E}$ and $\mathbf{B}$ are gauge invariant. The last two terms describe the

interaction between charged matter and the electromagnetic fields and these terms are not gauge invariant since,

$$- \rho\varphi' + \mathbf{j} \bullet \mathbf{A}'/c \; = - \rho(\varphi - \partial\Lambda/\partial t) + \mathbf{j} \bullet (\mathbf{A} + \mathbf{grad}\, \Lambda)\, /c$$

$$= -\rho\varphi + j \bullet A \;\; + \; [\, \rho\, \partial\Lambda/\partial t + j \bullet (\nabla\Lambda)/c \,] \qquad . \qquad (A.6.2)$$

That is to say, the presence of the terms [...] in (A.6.2) means that the last two terms in $\mathcal{L}$ in (A.6.1) are not gauge invariant. *However*, the physics is determined by the quantity $S = \int d^3x \int dt\, \mathcal{L}$ known as the action, and we find that

$$\int d^3x \int dt \; [\rho\, \partial\Lambda/\partial t + j \bullet (\nabla\Lambda)/c] \;\; = \;\; - \int d^3x \int dt \; [(\partial\rho/\partial t) + (1/c)\, \mathrm{div}\, \mathbf{j}\,]\, \Lambda \;\; = \;\; 0 \;\; .$$

Here we have done time parts integration to move ∂_t from Λ to ρ, and spatial parts integration to move ∇ from Λ onto $\mathbf{j}$ where it becomes div. The quantity $[(\partial\rho/\partial t) + (1/c)\, \mathrm{div}\, \mathbf{j}\,]$ vanishes at every point in space because charge is conserved. Thus, the action S implied by $\mathcal{L}$ in (A.6.1) is gauge invariant.

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and here is my previous version of Section 0, now obsoleted,

0. Fact 0: The Poisson equation $-\nabla^2\varphi = \rho$ has a unique solution as stated below.

Imagine some charge distribution $\rho(\mathbf{x})$ that is constrained to a localized region within infinite space. Perhaps the charge is "glued" to some non-conducting material to keep it from moving around. There are no conductors anywhere in this space. In this idealized scenario, the solution to the Poisson equation $-\nabla^2\varphi(\mathbf{x}) = \rho(\mathbf{x})$ -- *subject to* the added condition that $\varphi(|\mathbf{x}|=\infty) = 0$ -- is unique and is given by the following expression:

$$\varphi(\mathbf{x}) \;=\; \frac{1}{4\pi} \int d^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \qquad (A.0.1)$$

where the integral is over all space. Since the charge is localized to some region, in practice the integral is over that finite region of space. We will first show that this $\varphi(\mathbf{x})$ is in fact a solution of the Poisson equation, and secondly, we will show that it is a unique solution. In an operational sense, we know the solution is unique since we put the charge there and it creates some potential and we know intuitively that there will be only one potential field possible.

To see that the above $\varphi(\mathbf{x})$ is a solution to the Poisson equation $-\nabla^2\varphi(\mathbf{x}) = \rho(\mathbf{x})$, we first consider a simpler equation $-\nabla^2 g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$. Here $g(\mathbf{x},\mathbf{x}')$ is the potential φ at an arbitrary observation point $\mathbf{x}$ which is created by a unit positive delta-function point charge located at position $\mathbf{x}'$. We know from elementary electrostatics that the potential of a point charge (in our units) is $\varphi = 1/(4\pi r)$ where r is the distance between $\mathbf{x}$ and $\mathbf{x}'$. Formally then our solution is $g(\mathbf{x},\mathbf{x}') = 1/(4\pi|\mathbf{x} - \mathbf{x}'|)$. Thus,

$$-\nabla^2 g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \qquad \Rightarrow \qquad g(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\,\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{A.0.2}$$

so

$$-\nabla^2\{\frac{1}{4\pi}\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = \delta(\mathbf{x}-\mathbf{x}') \quad \text{or} \quad -\nabla^2\{\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = 4\pi\,\delta(\mathbf{x}-\mathbf{x}')\ . \tag{A.0.3}$$

Now apply $-\nabla^2$ to the right side of (A.0.1). Since the $-\nabla^2$ variable is **x**, it passes through the (convergent) integral and hits $1/(|\mathbf{x}-\mathbf{x}'|)$ yielding $4\pi\,\delta(\mathbf{x}-\mathbf{x}')$, according to (A.0.2), so we find

$$-\nabla^2\varphi(\mathbf{x}) = \frac{1}{4\pi}\int d^3x'\,\rho(x')\,\{-\nabla^2\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\} = \frac{1}{4\pi}\int d^3x'\,\rho(x')\,4\pi\,\delta(\mathbf{x}-\mathbf{x}') = \rho(\mathbf{x}) \qquad \text{QED.}$$

Thus the $\varphi(x)$ given in (A.0.1) is a solution to Poisson's equation. The assisting function $g(\mathbf{x},\mathbf{x}')$ is called a Green's Function or fundamental solution or free-space propagator; there are many names for such an object.

Suppose there were two different Poisson solutions φ and φ_1 which both vanish on the boundary at $|x| = \infty$. Let the difference be $u = \varphi - \varphi_1$. Then we must have $-\nabla^2 u(\mathbf{x}) = 0$ with $u = 0$ at $|x| = \infty$. If we can show that the only solution of this last problem is $u(x) \equiv 0$, then any alternate solution like φ_1 must be the same as φ, so the solution φ is unique. The equation $-\nabla^2 u(\mathbf{x}) = 0$ is known as the Laplace equation, and its solutions are called harmonic functions. It is not hard to show that any harmonic function must take both is max and min values on the boundary, which here is the great sphere. Thus $u_{max} = 0$ and $u_{min} = 0$, so the only possibility is that $u(x) \equiv 0$ everywhere. Thus, the $\varphi(x)$ of (A.0.1) is unique.

Comment: In this Appendix, we shall continue to assume everything happens in free space without the presence of conductors or other "boundaries" other than the great sphere at infinity. Our analysis is complex enough without the added complication of dealing with such conductors which form "boundary conditions". Rest assured that these boundary conditions can be properly handled.

As an example, perhaps there is a piece of metal nearby our point charge which has a certain constant potential caused by a 9 volt battery connecting it to the great sphere at ∞. We can then regard our point charge as being enclosed in a region consisting of the surface of this piece of metal and the great sphere. The potential is specified everywhere on this boundary (it is 0 on the great sphere and +9 on the metal). A famous result of electrostatics (the Dirichlet Problem) is that if a point charge or any charge distribution is surrounded by a closed boundary on which the potential is prescribed, then if one first solves the problem for a Green's Function $g(\mathbf{x},\mathbf{x}')$ which vanishes everywhere on the boundary, one obtains an explicit and unique solution of the problem which consists of the original solution of (A.0.1) plus other terms which are solutions of $-\nabla^2 u(\mathbf{x}) = 0$ and which assure that the boundary conditions are met. The new Green's Function $g(\mathbf{x},\mathbf{x}')$ in this context is $1/(4\pi|\mathbf{x}-\mathbf{x}'|)$ plus a second term which solves $\nabla^2 u(\mathbf{x}) = 0$ and makes $g = 0$ on the entire boundary.

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and here is my previous version of section 7

7. The Lorentz Gauge and QED: In relativistic notation, one defines a 4-vector potential A^μ with μ = 0,1,2,3 by making the assignment $A^0 = -A_0 = (1/v)\varphi$ as the zeroth component, with **A** being the other three components. Quantity v is the speed of light in some medium. The gauge transformation of Fact 5 then takes the following form:

$$A'^\mu = A^\mu + \partial^\mu \Lambda \qquad \text{(A.7.1)}$$

In this notation, $\partial_0 = -\partial^0 = (1/v)\partial/\partial t$, but $\partial_i = \partial^i = \partial/\partial x_i$ for i = 1,2,3. The freedom indicated in (A.7.1) allows one to choose a potential A'_μ such that

$$\partial_\mu A'^\mu = \partial \bullet A' = 0 \quad \Rightarrow \quad \text{div } \mathbf{A'} + (1/v)^2 \, d\varphi'/dt = 0 \qquad \text{(A.7.2)}$$

where $(1/v)^2 = \mu\mu_0\varepsilon\varepsilon_0 = \mu\varepsilon(1/c)^2$, c being the speed of light in vacuum. Once more:

$$\text{div } \mathbf{A'} = -\mu\mu_0\varepsilon\varepsilon_0 \, d\varphi'/dt \ . \qquad \text{(A.7.3)}$$

This is the Lorentz Gauge, so called because equation (A.7.2) is Lorentz covariant, having the transformation properties of a Lorentz scalar. This is akin to things like **k•r** being invariant under rotations in 3 space.

In relativistic quantum field theory (aka quantum electrodynamics, or QED), A^μ is interpreted as the quantum field of a vector particle called the photon. In the Lagrangian density which describes the interaction between the photon and charged particles, the term $J_\mu A^\mu$ appears,

$$\mathcal{L} = \ldots - J_\mu A^\mu \qquad J_\mu = e_0 \, \bar{\psi} \textbf{ Error! Bookmark not defined.} \gamma_\mu \, \psi \qquad \text{(A.7.4)}$$

where J_μ is the electric current, an operator built from the quantum fields of charged particles. Because of gauge invariance (A.7.1), one can always pick up an extra term $J_\mu \partial^\mu \Lambda$ in the Lagrangian. Using parts integration, the derivative can be moved over onto J_μ. In order for this to have no effect on the theory, it must be true that $\partial^\mu J_\mu = 0$. But this says that the electric current is conserved, which really means that electric charge is conserved. One is perhaps used to seeing this in the more familiar form ("equation of continuity"), where $J^0 = v\rho$:

$$\partial^\mu J_\mu = \partial_\mu J^\mu = 0 \qquad \Rightarrow \qquad \text{div } \mathbf{J} + d\rho/dt = 0. \qquad \text{(A.7.5)}$$

We arrive at the conclusion that the reason electric charge is conserved and cannot leak away into the vacuum is due to the invariance of the QED Lagrangian under gauge transformations (A.7.1). More generally, symmetries of the Lagrangian always result in conserved quantities. Since 1949, unusual names have been given to similar conversed quantities: isospin, strangeness, color, charm, etc.