Appendix A is now stable, so I don't think this doc is needed long term.
Perhaps the doc title refers to appendix A section 6 which maybe is now section 7.

7. The Lorentz Gauge and QED: This section is certainly off the transmission-lines beaten path, but the author thought the reader might find it interesting. It is true that the nature of a transmission line results from photons "jumping back and forth" between the conductors. Unlike elsewhere in this document, everything is not fully explained in the following quick outline.

In relativistic notation one uses 4-vectors which have one time component and three spatial components such as x^μ = (ct,x,y,z) which denotes a point in "spacetime". The time component t is multiplied by the speed of light c so that all four components have the same units -- distance L. Often people measure distance in light-seconds instead of meters so in such units c = 1, but we shall display the c to keep track of units. This x^μ is a contravariant 4-vector and the corresponding covariant 4-vector is x_μ = (ct,-x,-y,-z). Thus, one has $x^0 = x_0$ (= ct) but $x^i = -x_i$. We are assuming here the "Bjorken-Drell metric" $g_{\mu\nu}$ = diag(1,-1,-1,-1). The gradient operator ∂_i "transforms as" the spatial part of the covariant 4-vector ∂_μ, and one can write $\partial^i = -\partial_i$ just as $x^i = -x_i$ for i = 1,2,3. This four-vector gradient operator can be written $\partial_\mu = (\partial_0, \partial_i)$ and $\partial^\mu = (\partial^0, \partial^i) = (\partial_0, -\partial_i)$ where $\partial^0 = \partial_0 = \frac{1}{c}\partial_t = \frac{1}{c}\frac{\partial}{\partial t}$. The four components of ∂^μ all have dimension L^{-1}. The Laplacian is $\nabla^2 = \partial_i\partial_i = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$ (implied sum on i) while the corresponding object $\Box \equiv \partial_\mu\partial^\mu = \frac{1}{c^2}\partial_t^2 - \nabla^2$ is the D'Alembertian which appears in wave equations.

Consider then the gauge transformation (A.5.1) which in relativistic notation is

$$\begin{aligned} A'^i &= A^i + \partial_i\Lambda = A^i - \partial^i\Lambda \qquad i = 1,2,3 \\ \varphi' &= \varphi - \partial_t\Lambda = \varphi - c\,\partial^0\Lambda \end{aligned} \tag{A.7.1}$$

The components of a classical vector like **A**, normally written as A_i, are in fact the contravariant components A^i in relativistic notation. If we now define $A^0 \equiv \frac{1}{c}\varphi$ we can combine the two gauge transformation equations into a single equation involving three 4-vectors (one of which is $\partial^\mu\Lambda$),

$$A'^\mu = A^\mu - \partial^\mu\Lambda \qquad \mu = 0,1,2,3. \tag{A.7.2}$$

Suppose we want $\partial_\mu A'^\mu = 0$ (implicit sum on μ = 0,1,2,3). If we could find a potential A'^μ with this property, that would be very convenient for the following reason: In general $a_\mu b^\mu$ (= $a^\mu b_\mu$ = a • b) is the same in all frames of reference related by Lorentz Transformations. If $\partial_\mu A'^\mu = 0$ in one frame, it is 0 in all frames, and that makes computational life simple. For example, let S and S" be two frames of reference related by a Lorentz transformation. Then the implication is that

$$\underset{\text{frame S observer}}{\partial_\mu A'^\mu(x^\nu) = 0} \quad \Rightarrow \quad \underset{\text{frame S" observer}}{\partial''_\mu A'^\mu(x''^\nu) = 0} \qquad \text{where} \quad \partial_\mu \equiv \partial/\partial x^\mu \text{ and } \partial''_\mu \equiv \partial/\partial x''^\mu$$

So, is it possible to have $\partial_\mu A'^\mu = 0$? Writing this out we get

$$\partial_0 A'^0 + \partial_i A'^i = 0 \quad \Rightarrow \frac{1}{c}\partial_t\left[\frac{1}{c}\varphi\right] + \text{div } \mathbf{A}' = 0 \qquad \Rightarrow \frac{1}{c^2}\partial_t\varphi + \text{div } \mathbf{A}' = 0$$

so

$$\text{div } \mathbf{A}' = -\frac{1}{c^2}\partial_t\varphi'. \tag{A.7.3}$$

But we showed in Fact 2 that given any **A**, we can find an E-B-fields-equivalent **A'** which has div **A'** = any f(**x**) we want, so we just select f(x) = $-(1/c^2)\,\partial\varphi'/\partial t$. By selecting this f(x), we are **selecting the Lorentz Gauge**. In this gauge (now dropping the prime on A), we have $\partial_\mu A^\mu = 0$. As just noted, this gauge is so named because the resulting equation $\partial_\mu A^\mu = 0$ is "covariant" under all Lorentz transformations in that both sides of this equation transform as the same tensor object, in this case a "scalar". That means the equation has the same form in all frames of references which are related by Lorentz transformations (which include rotations, velocity transformations, and combinations of same). One can interpret $\partial_\mu A^\mu = 0$ as $\partial \bullet A = 0$ where ∂ is a 4-divergence operator. Thus, in the Lorentz gauge, the 4-divergence of A^μ is always exactly 0 at every point in spacetime.

In relativistic quantum field theory (aka quantum electrodynamics, or QED), the potential A^μ is interpreted as the quantum field of a massless vector particle called the photon. The potentials φ and **A** are thus promoted from being mere "helper functions" to having their own particle interpretation. In the Lagrangian density for the photon-electron system an interaction term - $J_\mu A^\mu$ appears,

$$\mathcal{L} = \ldots - J_\mu A^\mu \qquad J_\mu = e_0\,\bar{\psi}\,\gamma_\mu\,\psi \tag{A.7.4}$$

where J_μ is the electric current, an operator built from the quantum field ψ of the electron. The number e_0 is the so-called bare (unrenormalized) charge of the electron. Because of gauge invariance (A.7.2), a gauge transformation creates a new term - $J_\mu\,\partial^\mu\Lambda$ in the Lagrangian density. In Lagrangian dynamics, the physics of QED is determined by $S = \int d^4x\,\mathcal{L} = \int d^3x \int dt\,\mathcal{L}$ which is called the action. If we insert the gauge term $-J_\mu\partial^\mu\Lambda$ into the action and do parts integration to move ∂^μ from Λ to J_μ, we end up with an action change $\Delta S = \int d^4x\,(\partial^\mu J_\mu)\Lambda$. But at every point in spacetime, we know that $\partial^\mu J_\mu = 0$ (shown in a moment) so we find that $\Delta S = 0$ which means the action S is invariant under a gauge transformation. The reason $\partial^\mu J_\mu = \partial_\mu J^\mu = 0$ is because $J^\mu = (c\rho, J^i)$ where ρ is charge density and J^i is electric current, and then the statement $\partial_\mu J^\mu = 0$ says that $\frac{1}{c}\partial_t(c\rho) + \partial_i J^i = 0$ or div **J** = $-\partial\rho/dt$. This is the equation of continuity (1.1.8) which says that if there is a current flowing out of a tiny volume of space, the charge density in that volume must be correspondingly decreasing. In other words, charge is "conserved". We can reverse our logic to conclude that the reason electric charge is conserved and cannot "leak away into the vacuum" is due to the invariance of the QED action under gauge transformations (A.7.2). More generally, symmetries (invariances) of the action always result in conserved quantities. Since 1949, unusual names have been given to similar conversed quantities: isospin, strangeness, color, charm, etc. The association of a conserved quantity with a differential symmetry of the action is known as Noether's Theorem, in honor of Emmy Noether who first showed this connection in 1915.

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4. Fact 4: If div **B** = 0 and curl **E** = - ∂**B**/∂t , then there exist both **A'** and φ' such that **B** = curl **A'** and **E** = - grad φ' - ∂**A'**/∂t , *and* the quantity div **A'** may be set to any function f.

Proof: We know from Fact 2 that **A'** exists such that **B** = curl **A'** and such that div **A'** equals any arbitrary function f. If we start with some arbitrary **A** and φ, the successful **A**' from (A.2.1) is **A**' = **A** + grad Λ where Λ is given by (A.2.3) as an integral over f. What is the corresponding φ' ? Since the **E** field corresponding to (**A**,φ) and (**A**',φ') must be the same, we must have **E** = **E**' or

$$- \text{grad } \varphi - \partial_t \mathbf{A} = - \text{grad } \varphi' - \partial_t \mathbf{A'} .$$

Since **A**' = **A** + grad Λ, this says that - grad φ = - grad φ' - grad ∂_t Λ which is satisfied by φ' = φ - ∂_tΛ. Thus, the successful potential pair giving div **A**' = f is this:

$$\mathbf{A'} = \mathbf{A} + \text{grad } \Lambda$$
$$\varphi' = \varphi - \partial_t \Lambda \qquad \text{(A.4.1)}$$

where

$$\Lambda(\mathbf{x}) = -\frac{1}{4\pi} \int d^3\mathbf{x'} \frac{f(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|}$$

(A.2.3)

The pair of equations (A.4.1) is called a **gauge transformation** and we have just seen in Facts 2 and 4 that a gauge transformation preserves both **E** and **B**. Each possible choice f defines a function Λ which then gives the transformation. There are an infinite set of f and corresponding Λ, so there are an infinite number of gauge transformations which leave the electric and magnetic fields invariant. We are free to **choose a gauge** such that div **A**' = f for any f we like.