This original first section of Appendix A got amped up into a longer section now called A.0. The new section deals with all charges including those on pieces of metal. Here I just keep a copy of the old.

0. Lemma: The Poisson equation $\nabla^2\varphi = \rho$ has a unique solution. Needs a rewrite, Stak has lots to say.

Proof: By Poisson equation we mean $\nabla^2\varphi = \rho$, the way it usually appears in electrostatics, ignoring constants. The usual method of solving this equation is to first solve $\nabla^2 g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$ for the Green's Function $g(\mathbf{x},\mathbf{x}')$. The answer is $g(\mathbf{x},\mathbf{x}') = -1/(4\pi|\mathbf{x} - \mathbf{x}'|)$ in the case that there are no boundary conditions.

-- this solution is known as the free-space propagator, or the fundamental solution. From this fact, one concludes that the Poisson solution is given by,

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

To show this satisfies $\nabla^2\varphi = \rho$, apply ∇^2 to both sides. Inside the integral, ∇^2 hits $1/|\mathbf{x} - \mathbf{x}'| = -4\pi\, g(\mathbf{x},\mathbf{x}')$ to give $-4\pi\delta(\mathbf{x}-\mathbf{x}')$ which then picks out $-4\pi\rho(\mathbf{x})$, and the -4π's cancel. Since we have an explicit formula for the solution of the Poisson equation, it must have a unique solution [why?] . In the presence of boundary conditions, the uniqueness proof requires a bit more work. For example the Poisson equation is valid inside a closed region of space on whose bounding surface φ is specified (interior Dirichlet boundary value problem), the solution is also unique, see Stakgold Vol 2 p 102.