Old App A facts 4 and 5 (replaced by a single fact 4) PhL 9.18.13

This is just backup storage, will be thrown out eventually.

4. Fact 4: If div **B** = 0 and curl **E** = - ∂**B**/∂t , then there exist both **A** and φ such that **B** = curl **A** and **E** = - grad φ - ∂**A**/∂t , and the quantity div **A** may be set to any function f.

Proof: We know from Fact 2 that **A** exists such that **B** = curl **A** and such that div **A** equals any arbitrary function f (we dispense with prime on **A**). Consider the vector field **E'** = **E** + ∂**A**/∂t. Apply curl to find that curl **E'** = curl **E** + ∂**B**/∂t = 0. Now apply Fact 3 to this vector **E'** to conclude that there exists φ such that **E'** = -grad φ. But this says **E** = -grad φ - ∂**A**/∂t. Thus we have expressly found **A** and φ which satisfy the requirements of Fact 4.

5. Fact 5: Assume that potentials **A** and φ have been obtained as in Fact 4. There exists an infinite set of other potentials **A'** and φ' which yield the same **E** and **B** fields. They are given by

$$\mathbf{A}' = \mathbf{A} + \text{grad } \Lambda \qquad \dim(\Lambda) = \text{amp-henry} = \text{volt-sec}$$
$$\varphi' = \varphi - \partial\Lambda/\partial t \qquad \text{(A.5.1)}$$

where Λ is an arbitrary scalar function. It is by making a particular such transformations that div **A'** can be set to an arbitrary selected function f, as discussed in Fact 2.

Proof: We already showed in Fact 2 that **A'** = **A** + grad **Λ** does not alter **B**. The second condition of (A.5.1) is required so that **E** = -grad φ - ∂**A**/∂t also remains unaltered. That is,

$$\mathbf{E}' = -\text{ grad } \varphi' - \partial\mathbf{A}'/\partial t = -\text{grad } (\varphi - \partial\Lambda/\partial t) - \partial/\partial t\ (\mathbf{A} + \text{grad } \Lambda) = -\text{ grad } \varphi - \partial\mathbf{A}/\partial t = \mathbf{E}\ .$$

The gauge transformation (A.5.1) appears in Jackson as (6.12) and (6.13).