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This Appendix B has been deleted from lines doc. It was on a subject that did not have to be in an Appendix since it was treated in full in the main lines doc. I then wrote a completely unrelated Appendix B on a different subject (fields for round wire in partial waves).

Appendix B: The Complex Dielectric Constant and Free Charge in a Medium

B.1 A conducting parallel plate capacitor
B.2 Complex ε and the loss tangent of a dielectric
B.3 Continuity and free charge in a medium

B.1 A conducting parallel plate capacitor

In our transmission line analysis we follow the general approach used in King. One notion that King finds useful is that of a complex dielectric constant ξ which incorporates the conductivity σ of a medium along with the normal dielectric constant ε. Associated with this complex dielectric constant is a so-called "effective" surface charge $n_{\mathbf{eff}}$.

The term "dielectric constant" is somewhat a misnomer. As noted below, ε and ξ are not constants but in fact are strong functions of frequency ω. When small ranges of ω are considered, ε appears to be constant if all medium resonances are far away.

A good way to understand ξ and $n_{\mathbf{eff}}$ is to consider a simple "device": a parallel plate capacitor (area A) filled with a conducting dielectric (σ and ε). The plate spacing s is assumed small so edge effects can be ignored. One can regard this device as a parallel RC circuit.

Let us first imagine that the R portion is turned off and we have just the C part. Then, assuming the usual exp(jωt) time dependence, we can say that Q = CV = nA, where A is the area of a plate and n is the surface charge density per unit area. Doing a time derivative and then converting everything to the frequency domain (see (1.5.5) and Section 1.6) one gets I(ω) = jωn(ω)A, so we have a simple relationship between the current I and the area surface charge density n (arguments ω are now suppressed),

$$I = j\omega nA \quad . \tag{B.1}$$

If we now turn on the R part of our device, we get an extra contribution to the current, so now:

$$I = j\omega nA + V/R \qquad R = s/(\sigma A) \qquad \text{// since } R = V/I_{\mathbf{R}} = Es/JA = (s/A)(E/J) = (s/A)(1/\sigma) = s/(A\sigma)$$

or

$$I = j\omega nA + VA\sigma/s \; . \tag{B.2}$$

As noted, s is the spacing between the plates and σ is the dielectric conductivity. We know that

$$V = Es \tag{B.3}$$

where E is the electric field between the plates, and we also know that E = n/ε. This last fact follows from (1.1.3) that div**D** = ρ applied to a thin gaussian box containing one of the plate surfaces:

$\int_V dV \, \text{div}\, \mathbf{D} = \int_S d\mathbf{S} \bullet \mathbf{D}$ // divergence theorem = Gauss's theorem, d**S** = surface patch

$$\Rightarrow \int_V dV \, \rho = D\,A \quad \Rightarrow \quad nA = (\varepsilon E)A \quad \Rightarrow \quad E = n/\varepsilon \,. \tag{B.4}$$

From (B.3) and (B.4) we find that $VA\sigma/s = (Es)A\sigma/s = EA\sigma = nA\sigma/\varepsilon$ so (B.1) says

$$I = j\omega nA + nA\sigma/\varepsilon = j\omega nA \left[1 + \frac{nA\sigma/\varepsilon}{j\omega nA} \right] = j\omega nA \left[\varepsilon + \frac{\sigma}{j\omega} \right] / \varepsilon$$

or

$$I = j\omega An(\xi/\varepsilon) \qquad \text{where } \xi \equiv [\varepsilon + \sigma/(j\omega)]$$

or

$$I = j\omega An_{\mathbf{eff}} \qquad \text{where } n_{\mathbf{eff}} \equiv n\,(\xi/\varepsilon) \,. \tag{B.5}$$

By defining $n_{\mathbf{eff}}$ in this manner, we are able to incorporate the effect of the R part of our device into the C part.

In solving a problem, we can of course work either with the true surface charge n, or with the effective quantity $n_{\mathbf{eff}}$. Notice that,

$$n_{\mathbf{eff}}/\xi = n/\varepsilon = E \tag{B.6}$$

The advantage of using $n_{\mathbf{eff}}$ is its simple relationship (B.5) to the total current I in a situation where both R and C effects are present.

Main Point: For our parallel plate RC device, $I = j\omega n_{\mathbf{eff}}A$, which mimics the result $I = j\omega nA$ when there is no resistive loss.

The admittance of our conducting capacitor is given by

$$Y \equiv \frac{I}{V} = \frac{j\omega An(\xi/\varepsilon)}{ns/\varepsilon} = \frac{j\omega A\xi}{s} = j\omega C' \qquad \text{where } C' \equiv \xi\,(A/s) = \text{complex effective capacitance}$$

$$= (j\omega A/s)\ [\varepsilon + \sigma/(j\omega)] = (\sigma A/s) + j\omega\,(\varepsilon A/s) \equiv G + j\omega C$$

where (B.7)

$G = \sigma A/s$ = conductance = 1/R for R shown in (B.1)

$C = \varepsilon A/s$ = capacitance

Again, using the definition of ξ, we can absorb the R effect into the C part of our device by replacing C with the complex capacitance $C' = (\xi/\varepsilon)C$. Finally,

$$\frac{I}{V} = j\omega C' \quad => \qquad j\omega C' = \frac{j\omega A n_{\mathbf{eff}}}{V} \qquad => \qquad C' = \frac{A n_{\mathbf{eff}}}{V} \equiv \frac{q_{\mathbf{eff}}}{V} \tag{B.8}$$

which relates complex capacitance C' to the total effective charge $q_{\mathbf{eff}}$ and the voltage V.

B.2 Complex ε and the loss tangent of a dielectric.

The normal dielectric constant in fact has a real and imaginary part, usually expressed as:

$$\varepsilon = \varepsilon' - j\varepsilon'' \quad . \tag{B.9}$$

Thus, we can write our complex dielectric constant ξ as

$$\xi = [\varepsilon - j\sigma/\omega] = [\ \varepsilon'\varepsilon_0 - j(\sigma + \omega\varepsilon''\varepsilon_0)/\omega\ ] \tag{B.10}$$

The cause of ε" is a time-lag loss mechanism in the dielectric. In polar dielectrics, this is due to friction which keeps the polar molecules from instantly following the **E** field at high frequencies. In non-polar dielectrics (those used in coax cables) this is mainly due to the classical resonance absorption mechanisms, such as molecular vibrational modes. The standard classical (non-quantum) theory for ε due to vibrating electrons in a medium yields this general form for ε(ω): [Jackson p 310 (7.51)]

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0} \sum_{\mathbf{n}} \frac{f_{\mathbf{n}}}{\omega_{\mathbf{n}}^2 - \omega^2 - j\omega\gamma_{\mathbf{n}}} \quad = \quad 1 + \frac{Ne^2}{m\varepsilon_0} \sum_{\mathbf{n}} \frac{f_{\mathbf{n}}[(\omega_{\mathbf{n}}^2-\omega^2) + j\omega\gamma_{\mathbf{n}}]}{(\omega_{\mathbf{n}}^2 - \omega^2)^2 + (\omega\gamma_{\mathbf{n}})^2} \tag{B.11}$$

where $\Sigma_{\mathbf{n}}$ is a sum over the possible resonances of electrons in the medium, $\omega_{\mathbf{n}}$ are the resonant frequences, $\gamma_{\mathbf{n}}$ the corresponding damping factors, and $f_{\mathbf{n}}$ the relative "oscillator strengths". Here m is the electron mass, e the electron charge, and N is the density of electrons per m^3. Each resonance is being modeled by a damped driven harmonic oscillator. The main point of (B.11) is that in general ε(ω) <u>always</u> has an imaginary part, which comes from the $j\gamma_{\mathbf{n}}$ damping term. In general ε" is small except in the region of a resonance (in the above, at ω such that $\omega \approx \omega_{\mathbf{n}}$).

We can conveniently incorporate both loss mechanisms (conduction of dielectric and absorption) by defining $\sigma_{\mathbf{eff}}$ to be the grouping which appears in (B.10),

$$\sigma_{\mathbf{eff}} \equiv (\sigma + \omega\varepsilon''\varepsilon_0) \tag{B.12}$$

so then

$$\xi = [\varepsilon'\varepsilon_0 - j\sigma_{\mathbf{eff}}/\omega] = \varepsilon'\varepsilon_0\ [\ 1 - j\sigma_{\mathbf{eff}}/(\omega\varepsilon'\varepsilon_0)] = \varepsilon'\varepsilon_0(\ 1 - j\ \tan_{\mathbf{L}}) \tag{B.13}$$

where we have defined $\tan_{\mathbf{L}}$ to be

$$\tan_{\mathbf{L}} \equiv \frac{\sigma_{\mathbf{eff}}}{\omega\varepsilon'\varepsilon_0} = \frac{\sigma + \omega\varepsilon''\varepsilon_0}{\omega\varepsilon'\varepsilon_0} \quad . \tag{B.14}$$

This last quantity $\tan_L$ is called the "loss tangent" (or dissipation factor) of the dielectric because it is (minus) the ratio of the imaginary and real parts of ξ, as (B.13) shows, and there is a "loss angle" in the complex ξ plane which is $\theta_L = \tan^{-1}(\tan_L) = \tan^{-1}[\sigma_{eff}/(\omega\varepsilon'\varepsilon_0)]$. If $\theta_L = 0$, there is no loss. This $\tan_L$ is a function of ω explicitly as shown, and implicitly since in general σ, ε' and ε'' are functions of ω.

B.3 Continuity and free charge in a medium

This section is unrelated to the above two sections and is just placed here for convenience.

Consider the continuity equation (1.1.8) inside a medium, div $\mathbf{J} = -\partial\rho/\partial t$, where $\mathbf{J} = \sigma\mathbf{E}$ is the conduction current in the medium, and ρ is the free charge. Using (1.1.7, 6 and 3), div $\mathbf{J} = \sigma$ div$\mathbf{E} = \sigma(\rho/\varepsilon)$ so we get:

$$\frac{\partial\rho(t)}{\partial t} + \frac{\sigma}{\varepsilon}\,\rho(t) = 0 \quad \Leftrightarrow \qquad [j\omega + \frac{\sigma}{\varepsilon}]\,\rho(\omega) = 0 \ \text{ for } \rho = \rho_0\, e^{j\omega t} \text{ time dependence} \tag{B.15}$$

The ω-domain equation says that the only monochrome $e^{j\omega t}$ solution is $\rho(\omega) = 0 \Rightarrow \rho(t) = 0$. The t-domain equation has this transient solution

$$\rho(t) = \rho(0)\exp(-t/\tau) \qquad \tau = \varepsilon/\sigma \quad . \tag{B.16}$$

This says that if there is free charge out in the middle of a conducting medium, it will run away and park on the boundaries. In our conducting capacitor, this happens with time constant τ despite the fact that the capacitor plates are driving the medium with $e^{j\omega t}$ time dependence. Any existing charge density will dissipate in time τ.

Examples:	copper	$\sigma = 5.81 \times 10^{7}$ mho/m	$\varepsilon \approx \varepsilon_0 = 8.85 \times 10^{-12}$ farad/m $\Rightarrow \ \tau \approx 1.5 \times 10^{-18}$ sec
	polyethylene	$\sigma \sim 10^{-14}$ mho/m ($\rho > 10^{16}$ ohm-cm)	$\varepsilon \approx 2.3\,\varepsilon_0 = 8.85 \times 10^{-12}$ farad/m $\Rightarrow \ \tau \approx$ 2000 sec $\approx$ 34 minutes

To summarize, there is no steady-state sinusoidal free charge density out in the middle of a medium, conducting or otherwise. Even if free charge could somehow exist at $t = 0$ inside a medium, after $t >> \tau$ free charge is only allowed on boundaries. One can of course have a current $\mathbf{J}$ in a medium, but the charge is balanced so there is zero net charge density, $\rho = 0$.