Appendix B.6 section (c) PhL 2.5.14

This is my first writing of the proof of the "theorem" as opposed to the "lemma" which comes earlier. In the theorem, I worry about the effect of *other conductors* on the boundary conditions for A_z. This stuff is now all installed.

(c) Statement and Proof of The J_m Theorem

The J_m Theorem. A transmission line consists of two conductors called C_2 and C_3, so that the index 1 can refer to the dielectric. The total "particular" vector potential $A_z(x)$ due to the conduction currents in these two conductors is, according to (1.5.3),

$$\mathbf{A}(\mathbf{x})^{\mathrm{part}} = \frac{1}{4\pi} \sum_{i=2}^{3} \int_{C_i} \mu_i \mathbf{J}_{ci}(\mathbf{x}') \frac{e^{-j\beta R}}{R} dV' \ . \qquad R = |\mathbf{x}-\mathbf{x}'|$$

The theorem claims that (1) the correct total potential is given by

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \sum_{i=2}^{3} \int_{C_i} [\, \mu_i \mathbf{J}_{ci}(\mathbf{x}') + \mu_0 \mathbf{J}_{mi}(\mathbf{x}')] \frac{e^{-j\beta R}}{R} dV' \ . \qquad \text{(B.6.27)}$$

where $\mathbf{J}_{mi}(\mathbf{x}')$ represents the surface current at the surface of conductor C_i, and (2) this correct total potential satisfies the boundary conditions (B.6.0) at the surface of both conductors.

We claim the theorem is also true for a transmission line consisting of any number of conductors, but we restrict our interest to two conductors. We shall show for (B.6.27) that (B.6.0) is valid at the surface of conductor C_2 and a mirror argument then shows it is also valid at the surface of C_3.

Since we are operating in the transmission line limit, the actual claim being made is this:

$$A_z(\mathbf{x}) = \frac{1}{4\pi} \sum_{i=2}^{3} \int_{C_i} [\, \mu_i J_{czi}(\mathbf{x}') + \mu_0 J_{mzi}(\mathbf{x}')] \frac{e^{-j\beta R}}{R} dV' \ . \qquad \text{(B.6.28)}$$

To enhance clarity, we shall give each conductor its own custom integration variable. The expression above contains four terms which we now write out:

$$A_z^{(c2)}(\mathbf{x}) = \frac{1}{4\pi} \int_{C_2} [\mu_2 J_{cz2}(\mathbf{x}_2')] \frac{e^{-j\beta R2}}{R_2} dV_2' \qquad R_2 = |\mathbf{x} - \mathbf{x}_2'|$$

$$A_z^{(c3)}(\mathbf{x}) = \frac{1}{4\pi} \int_{C_3} [\mu_3 J_{cz3}(\mathbf{x}_3')] \frac{e^{-j\beta R3}}{R_3} dV_3' \qquad R_3 = |\mathbf{x} - \mathbf{x}_3'| \qquad \text{(B.6.4)'}$$

$$A_z^{(m2)}(\mathbf{x}) = \frac{1}{4\pi}\int_{C_2}[\mu_0\, J_{mz2}(\mathbf{x}_2')]\,\frac{e^{-j\beta R2}}{R_2}\,dV_2' \qquad R_2 = |\mathbf{x} - \mathbf{x}_2'|$$

$$A_z^{(m3)}(\mathbf{x}) = \frac{1}{4\pi}\int_{C_3}[\mu_0\, J_{mz3}(\mathbf{x}_3')]\,\frac{e^{-j\beta R3}}{R_3}\,dV_3' \qquad R_3 = |\mathbf{x} - \mathbf{x}_3'| \qquad \text{(B.6.5')}$$

$$A_z(\mathbf{x}) = A_z^{(c2)}(\mathbf{x}) + A_z^{(c3)}(\mathbf{x}) + A_z^{(m2)}(\mathbf{x}) + A_z^{(m3)}(\mathbf{x}) \ . \qquad \text{(B.6.6)'}$$

Without loss of generality, we shall consider $\mathbf{x} \rightarrow \mathbf{s}$ where $\mathbf{s}$ is a point on the surface of conductor C_2.

The "conduction solutions" $A_z^{(ci)}$ (that is to say, the particular solutions) are naturally smooth at point $\mathbf{s}$, as described in the text surrounding (B.6.0). Thus, we know that

$$A_z^{(ci)}(\mathbf{s}+) = A_z^{(ci)}(\mathbf{s}-)$$
$$\partial_n A_z^{(ci)}(\mathbf{s}+) = \partial_n A_z^{(ci)}(\mathbf{s}-) \ . \qquad i = 2, 3 \qquad \text{(B.6.7)'}$$

Since $\mathbf{s} = \mathbf{s}+ = \mathbf{s}-$ for these functions, we may trivially write

$$\frac{1}{\mu_1}\partial_n[A_z^{(c2)}(\mathbf{s}+) + A_z^{(c3)}(\mathbf{s}+)] - \frac{1}{\mu_2}\partial_n[A_z^{(c2)}(\mathbf{s}-) + A_z^{(c3)}(\mathbf{s}-)]$$
$$= +\left[\frac{1}{\mu_1} - \frac{1}{\mu_2}\right]\partial_n[A_z^{(c2)}(\mathbf{s}) + A_z^{(c3)}(\mathbf{s})]. \qquad \text{(B.6.8)'}$$

Since this is non-zero, the term $[A_z^{(c2)}(\mathbf{s}+) + A_z^{(c3)}(\mathbf{s}+)]$ on its own does not meet the required slope boundary condition (B.6.0) at an interface between μ_1 and μ_2, and that is precisely why we need the $A_z^{(m)}$ terms. Our goal is to show that

$$\frac{1}{\mu_1}\partial_n[A_z^{(m2)}(\mathbf{s}+) + A_z^{(m3)}(\mathbf{s}+)] - \frac{1}{\mu_2}\partial_n[A_z^{(m2)}(\mathbf{s}-) + A_z^{(m3)}(\mathbf{s}-)]$$
$$= -\left[\frac{1}{\mu_1} - \frac{1}{\mu_2}\right]\partial_n[A_z^{(c2)}(\mathbf{s}) + A_z^{(c3)}(\mathbf{s})] \ . \qquad \text{(B.6.9)'}$$

so that when we add the four terms of (B.6.6)' we will get

$$\frac{1}{\mu_1}\partial_n A_z(\mathbf{s}+) - \frac{1}{\mu_2}\partial_n A_z(\mathbf{s}-) = 0 \qquad \textit{(B.6.10)}$$

as required by (1.1.46). The concludes our proof outline, and it remains then to demonstrate (B.6.9)'.

At this point, we skip over several equations of the Lemma proof since they are all generalized simply by adding 2 or 3 subscripts in the right places. For example, the surface currents are given by,

$$K_{z2} = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right) H_{\theta 2} \qquad \text{on } C_2$$

$$K_{z3} = -\left(\frac{\mu_3}{\mu_0} - \frac{\mu_1}{\mu_0}\right) H_{\theta 3} \qquad \text{on } C_3 \quad . \qquad \textit{(B.1.10)}$$

We arrive then at (B.6.16)' as follows (recall that E_2 is the 2D Helmholtz propagator),

$$A_z^{(m2)}(\mathbf{x}) = \oint_{C2} ds_2' \, [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{x}_2')] \; E_2(\mathbf{x}|\mathbf{x}_2')$$

$$A_z^{(m3)}(\mathbf{x}) = \oint_{C3} ds_3' \, [(\mu_1-\mu_3)\, H_{\theta 3}(\mathbf{x}_3')] \; E_2(\mathbf{x}|\mathbf{x}_3') \quad . \qquad \text{(B.6.16)'}$$

A major difference appears at the next step (B.6.18)' ,

$$\partial_n A_z^{(m2)}(\mathbf{s}\pm) = \oint_{C2} ds_2' \, [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{x}')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_2') \mp [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{s})]/2 \quad .$$

$$\partial_n A_z^{(m3)}(\mathbf{s}\pm) = \oint_{C3} ds_3' \, [(\mu_1-\mu_3)\, H_{\theta 3}(\mathbf{x}')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_3') \quad . \qquad \text{(B.6.18)'}$$

The "extra Stakgold term" only appears when a point **s** lies on the surface being integrated over since it is this integration which gives rise to the singular situation. Since our **s** lies on C_2 and not on C_3, there is no "extra term" in the last equation above.

Now since we want to prove (B.6.9)', we first evaluate its left hand side using each equation of (B.6.18)' twice,

$$\frac{1}{\mu_1}\partial_n[A_z^{(m2)}(\mathbf{s}+) + A_z^{(m3)}(\mathbf{s}+)] \; - \; \frac{1}{\mu_2}\partial_n[A_z^{(m2)}(\mathbf{s}-) + A_z^{(m3)}(\mathbf{s}-)]$$

$$= \frac{1}{\mu_1}\left\{ \oint_{C2} ds_2' \, [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{x}_2')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_2') - [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{s})]/2 \right\}$$

$$- \frac{1}{\mu_2}\left\{ \oint_{C2} ds_2' \, [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{x}_2')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_2') + [(\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{s})]/2 \right\}$$

$$+ \frac{1}{\mu_1}\left\{ \oint_{C3} ds_3' \, [(\mu_1-\mu_3)\, H_{\theta 3}(\mathbf{x}_3')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_3') \right\}$$

$$- \frac{1}{\mu_2}\left\{ \oint_{C3} ds_3' \, [(\mu_1-\mu_3)\, H_{\theta 3}(\mathbf{x}_3')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_3') \right\}$$

$$= (\mu_1-\mu_2)\left[\frac{1}{\mu_1} - \frac{1}{\mu_2}\right] \oint_{C2} ds_2' \, H_{\theta 2}(\mathbf{x}_2') \, \partial_n E_2(\mathbf{s}|\mathbf{x}_2') \; - \; \left[\frac{1}{\mu_1} + \frac{1}{\mu_2}\right](\mu_1-\mu_2)\, H_{\theta 2}(\mathbf{s})/2$$

$$+ \left[\frac{1}{\mu_1} - \frac{1}{\mu_2}\right](\mu_1-\mu_3) \oint_{C3} ds_3' \, H_{\theta 3}(\mathbf{x}_3')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}_3') \qquad \text{(B.6.19)'}$$

where the last line was not present in (B.6.19). Our task of showing that (B.6.9)' is true then boils down to showing that the last expression above is equal to $-\left[\frac{1}{\mu_1}-\frac{1}{\mu_2}\right]\partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})]$. That is to say, we have to show

$$(\mu_1-\mu_2)\left[\frac{1}{\mu_1}-\frac{1}{\mu_2}\right]\oint_{C2} ds_2'\, H_{\theta 2}(\mathbf{x}_2')\,\partial_n E_2(\mathbf{s}|\mathbf{x}_2') - \left[\frac{1}{\mu_1}+\frac{1}{\mu_2}\right](\mu_1-\mu_2)\,H_{\theta 2}(\mathbf{s})/2$$
$$+\left[\frac{1}{\mu_1}-\frac{1}{\mu_2}\right](\mu_1-\mu_3)\oint_{C3} ds_3'\, H_{\theta 3}(\mathbf{x}_3')]\,\partial_n E_2(\mathbf{s}|\mathbf{x}_3') = -\left[\frac{1}{\mu_1}-\frac{1}{\mu_2}\right]\partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})] \quad ? \qquad \text{(B.6.20)'}$$

As before, a question mark indicates an equation that we want to show is true, but have not yet done so. Cancelling $(\mu_1-\mu_2)$ factors gives

$$\left[\frac{1}{\mu_1}-\frac{1}{\mu_2}\right]\oint_{C2} ds_2'\, H_{\theta 2}(\mathbf{x}_2')\,\partial_n E_2(\mathbf{s}|\mathbf{x}_2') - \left[\frac{1}{\mu_1}+\frac{1}{\mu_2}\right]H_{\theta 2}(\mathbf{s})/2$$
$$-\frac{1}{\mu_1\mu_2}(\mu_1-\mu_3)\oint_{C3} ds_3'\, H_{\theta 3}(\mathbf{x}_3')]\,\partial_n E_2(\mathbf{s}|\mathbf{x}_3') = +\frac{1}{\mu_1\mu_2}\,\partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})] \quad ?$$

or (B.6.21)'

$$(\mu_2-\mu_1)\oint_{C2} ds_2'\, H_{\theta 2}(\mathbf{x}_2')\,\partial_n E_2(\mathbf{s}|\mathbf{x}_2') - (\mu_2+\mu_1)\,H_{\theta 2}(\mathbf{s})/2$$
$$+(\mu_3-\mu_1)\oint_{C3} ds_3'\, H_{\theta 3}(\mathbf{x}_3')]\,\partial_n E_2(\mathbf{s}|\mathbf{x}_3') = +\partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})] \quad ? \qquad \text{(B.6.22)'}$$

The integrals in (B.6.22)' can be replaced using (B.6.18)' with the s+ choice,

$$\partial_n A_z^{(m2)}(\mathbf{s}+) = \oint_{C2} ds_2'\,[(\mu_1-\mu_2)\,H_{\theta 2}(\mathbf{x}_2')]\,\partial_n E_2(\mathbf{s}|\mathbf{x}_2') - (1/2)[(\mu_1-\mu_2)\,H_{\theta 2}(\mathbf{s})$$
$$\partial_n A_z^{(m3)}(\mathbf{s}) = \oint_{C3} ds_3'\,[(\mu_1-\mu_3)\,H_{\theta 3}(\mathbf{x}_2')]\,\partial_n E_2(\mathbf{s}|\mathbf{x}_3') \qquad \textit{(B.6.18)+}$$

so

$$(\mu_2-\mu_1)\oint_{C2} ds_2'\, H_{\theta 2}(\mathbf{x}_2')\,\partial_n E_2(\mathbf{s}|\mathbf{x}_2') = -\partial_n A_z^{(m2)}(\mathbf{s}+) - (1/2)[(\mu_1-\mu_2)\,H_{\theta 2}(\mathbf{s})$$
$$(\mu_3-\mu_1)\oint_{C3} ds_3'\, H_{\theta 3}(\mathbf{x}_3')]\,\partial_n E_2(\mathbf{s}|\mathbf{x}_3') = -\partial_n A_z^{(m3)}(\mathbf{s}) \quad . \qquad \text{(B.6.23)'}$$

Equation (B.6.22)' then becomes

$$-\partial_n A_z^{(m2)}(\mathbf{s}+) - (1/2)[(\mu_1-\mu_2)\,H_{\theta 2}(\mathbf{s}) - (\mu_2+\mu_1)\,H_{\theta 2}(\mathbf{s})/2$$
$$-\partial_n A_z^{(m3)}(\mathbf{s}) = \partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})] \quad ?$$

or

$$-\partial_n A_z^{(m2)}(\mathbf{s}+) - \mu_1\,H_{\theta 2}(\mathbf{s})$$
$$-\partial_n A_z^{(m3)}(\mathbf{s}) = \partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})] \quad ?$$

or

$$-\mu_1 H_{\theta 2}(\mathbf{s}) = \partial_n[A_z^{(c2)}(\mathbf{s})+A_z^{(c3)}(\mathbf{s})+A_z^{(m2)}(\mathbf{s}+)+\partial_n A_z^{(m3)}(\mathbf{s})] \quad ?$$

or

$$-\mu_1 H_{\theta 2}(\mathbf{s}) = \partial_n A_z(s+) \quad ? \quad \text{// using (B.6.6)'.} \qquad \text{(B.6.24)'}$$

But this last equation is true as shown in Fig B.6 (with $C = C_2$) and (B.6.26), so we can then go back and erase all the question marks and we have then verified equation (B.6.9)' and our proof is complete.

By then taking **s** to be a point on the surface of C_3, we would find $-\mu_1 H_{\theta 3}(\mathbf{s}) = \partial_n A_z(s+)$ for (B.6.24)' and then Fig B.6 with $C = C_3$ would verify this result as well.