Rewrite of Section B.2 PhL 10.15.14

B.2 Calculation of H from the current J in a conductor

(a) An expression for H in terms of J

First, we want to clarify the connection between **H** and **J** in terms of Maxwell's equations. Since $\mathbf{J} = \sigma\mathbf{E}$, we can write the curl H equation (1.1.1) in these three equivalent ways:

a curl **H** = $j\omega\varepsilon\mathbf{E} + \mathbf{J}$ = displacement current + conduction current *(1.1.1)* (B.2.1a)

b curl **H** = $j\omega(\xi/\sigma)\,\mathbf{J}$ $\xi \equiv \varepsilon - j\sigma/\omega = \varepsilon + \sigma/j\omega$ (B.2.1b)

c curl **H** = $j\omega\xi\,\mathbf{E}$. (B.2.1c)

Using the identity curl curl = grad div $-\nabla^2$ and noting that div **H** = μ div **B** = 0 from (1.1.4) we find

curl curl **H** = $-\nabla^2\mathbf{H}$. (B.2.2)

Applying curl to the three forms above, then using (B.2.2) and doing some small algebra as shown below, one obtains these three exactly equivalent equations for **H** :

a $(\nabla^2 + \kappa^2)\mathbf{H} = -\text{curl}\,\mathbf{J}$ $\kappa^2 = \omega^2\varepsilon\mu$ // agrees with (1.5.26) (B.2.3a)

b $\nabla^2\mathbf{H} = -j\omega(\xi/\sigma)\,\text{curl}\,\mathbf{J} = -[1 + j\omega(\varepsilon/\sigma)]\,\text{curl}\,\mathbf{J}$ $\xi \equiv \varepsilon + \sigma/j\omega$ (B.2.3b)

c $(\nabla^2 + \beta^2)\mathbf{H} = 0$ $\beta^2 = \omega^2\xi\mu$ // agrees with (1.5.27) (B.2.3c)

Forms a and c make use of (1.1.2) which says curl **E** = $-j\omega\mathbf{B} = -j\omega\mu\mathbf{H}$.

Algebra:
a $-\nabla^2\mathbf{H} = \text{curl}\,(j\omega\varepsilon\mathbf{E} + \mathbf{J}) = j\omega\varepsilon(-j\omega\mu\mathbf{H}) + \text{curl}\,J = \omega^2\varepsilon\mu\mathbf{H} + \text{curl}\,\mathbf{J} = \kappa^2\mathbf{H} + \text{curl}\,\mathbf{J}$
b $-\nabla^2\mathbf{H} = \text{curl}\,(j\omega(\xi/\sigma)\,\mathbf{J}) = j\omega(\xi/\sigma)\,\text{curl}\,\mathbf{J}$
c $-\nabla^2\mathbf{H} = \text{curl}\,(j\omega\xi\,\mathbf{E}) = (j\omega\xi)(-j\omega\mu\mathbf{H}) = \omega^2\xi\mu\,\mathbf{H} = \beta^2\mathbf{H}$

We showed below (2.2.2) that for $f << 10^{18}$ Hz, we have $\omega\varepsilon/\sigma << 1$ (copper). Assuming this inequality for any practical ω, we know that $\varepsilon << \sigma/\omega$ so $\xi = \sigma/j\omega$ and $\beta^2 = \omega^2(\sigma/j\omega)\mu = -j\omega\mu\sigma$ which agrees with (1.5.1d). In this situation, since $-j\omega(\xi/\sigma) = -1$, we can write (B.2.3b) above as

$\nabla^2\mathbf{H} = -\text{curl}\,\mathbf{J}$. // copper for $f << 10^{18}$ Hz (B.2.4)

One might wonder how this last equation and (B.2.3a) can both be valid. The reason is that

$|\kappa^2/\beta^2| = \omega^2\varepsilon\mu/(-j\omega\mu\sigma) = |-\omega\varepsilon/\sigma| = \omega\varepsilon/\sigma << 1$ (B.2.5)

Then using this fact and (B.2.3c) we find

$$|\kappa^2\mathbf{H}| << |\beta^2\mathbf{H}| = |\nabla^2\mathbf{H}| \quad \text{so} \quad |\kappa^2\mathbf{H}| << |\nabla^2\mathbf{H}| \tag{B.2.6}$$

so the κ^2 term in item 1 makes no contribution.

Recalling (H.1.8),

$$-\nabla^2 f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, s(\mathbf{x}') + \text{homogeneous solutions}$$

The Poisson Equation *(H.1.8)*

one can write the Helmholtz particular integral solution of (B.2.4) as

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, \text{curl}' \, \mathbf{J}(\mathbf{x}') \tag{B.2.7}$$

where $[1/4\pi R]$ is the Poisson free-space propagator.

(b) An alternative derivation using the vector potential A

Consider a conductor with $\mu,\varepsilon,\sigma,\xi$ surrounded by a dielectric with $\mu_d,\varepsilon_d,\sigma_d,\xi_d$. Here are two equations for the vector potential associated with the conductor current **J**. The first equation is for the King gauge, while the second is for the Lorenz gauge:

$$(\nabla^2 + \beta_d^2)\mathbf{A}(\mathbf{x}) = -\mu \, \mathbf{J}(\mathbf{x}) \quad \textit{(1.5.4)} \quad \beta_d^2 = \omega^2\mu_d\xi_d \quad \text{// King gauge} \tag{B.2.8a}$$

$$(\nabla^2 + \beta_0^2)\mathbf{A}(\mathbf{x}) = -\mu \, \mathbf{J}(\mathbf{x}) \; . \quad \textit{(1.5.28)} \quad \beta_0^2 = \omega^2\mu\varepsilon \quad \text{// Lorenz gauge} \tag{B.2.8b}$$

Recalling (H.1.9),

$$-(\nabla^2+k^2) f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^3x' \, [e^{-jkR}/4\pi R] \, s(\mathbf{x}') + \text{homogeneous solutions}$$

The Helmholtz Equation *(H.1.9)*

we may write down the Helmholtz particular integral solutions to (B.2.8),

$$\mathbf{A}(\mathbf{x}) = \mu \int d^3x' \, [e^{-j\beta_d R}/4\pi R] \, \mathbf{J}(\mathbf{x}') \quad \text{King gauge} \quad R = |\mathbf{x}-\mathbf{x}'| \tag{B.2.9a}$$

$$\mathbf{A}(\mathbf{x}) = \mu \int d^3x' \, [e^{-j\beta_0 R}/4\pi R] \, \mathbf{J}(\mathbf{x}') \quad \text{Lorenz gauge} \quad R = |\mathbf{x}-\mathbf{x}'| \tag{B.2.9b}$$

where $[e^{-j\beta_0 R}/4\pi R]$ is the usual 3D Helmholtz propagator.

We know that **B** = curl **A** and therefore, using β_x to stand for either β_d or β_0,

$$\mathbf{H(x)} = (1/\mu)\ \text{curl}\ \mathbf{A(x)} = \int d^3x'\ \text{curl}\ (\ [e^{-j\beta_{\mathbf{x}R}}/4\pi R]\ \mathbf{J(x')}\ ) \qquad \text{(B.2.10)}$$

Notice that $\mathbf{J}(\mathbf{x}')$ is a constant in terms of the unprimed curl operator. A useful vector identity then is

$$\nabla \times (\varphi \mathbf{Q}) = \nabla\varphi \times \mathbf{Q} + \varphi\ (\nabla \times \mathbf{Q}) \quad = (\nabla\varphi) \times \mathbf{Q} \qquad \mathbf{Q} = \text{constant vector}\ .$$

Thus one may write

$$\mathbf{H(x)} = \int d^3x'\ \nabla[e^{-j\beta_{\mathbf{x}R}}/4\pi R] \times \mathbf{J(x')}\ .$$

Since $R = |\mathbf{x}-\mathbf{x}'|$ we know ∇ acting on any function of R is the same as $-\nabla'$ on that function, so

$$\mathbf{H(x)} = -\int d^3x'\ \nabla'[e^{-j\beta_{\mathbf{x}R}}/4\pi R] \times \mathbf{J(x')}$$

or in components (implied sum on r and s) ,

$$H_{\mathbf{i}} = -\int d^3x'\ \varepsilon_{\mathbf{irs}}\ \partial'_{\mathbf{r}}\ [e^{-j\beta_{\mathbf{x}R}}/4\pi R]\ J_{\mathbf{s}}(\mathbf{x}')\ .$$

Doing parts integration and dropping the parts (see Section A.0 (d)), we get

$$H_{\mathbf{i}} = +\int d^3x'\ \ [e^{-j\beta_{\mathbf{x}R}}/4\pi R]\ \ \varepsilon_{\mathbf{irs}}\ [\partial'_{\mathbf{r}}\ J_{\mathbf{s}}(\mathbf{x}')]$$

or

$$\mathbf{H(x)} = \int d^3x'\ [e^{-j\beta_{\mathbf{x}R}}/4\pi R]\ \text{curl}'\ \mathbf{J(x')}\ . \qquad \text{(B.2.11)}$$

We can compare this result to (B.2.7) obtained assuming $f << 10^{18}$ Hz,

$$\mathbf{H(x)} = \int d^3x'\ [1/4\pi R]\ \text{curl}'\ \mathbf{J(x')}\ . \qquad \text{3D} \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \textit{(B.2.7)}$$

We conclude that in either gauge, and for this frequency range with copper, the exponential factor $e^{-j\beta_{\mathbf{x}R}}$ in (B.2.11) may be ignored. This basically says that the main contribution to the integral comes from the region near R = 0.

At low ω and in particular at DC with ω = 0, we may assume that $\mathbf{J} = \mathbf{J}(x,y)$ with no z dependence. In this case we can do the dz' integration in (B.2.7) as shown in (J.10) which converts [1/4πR] to [-ln(s)/2π]. Renaming s to be R in 2D, we obtain

$$\mathbf{H}(x,y) = \int d^2x'\ [-\ln(R)/2\pi]\ \text{curl}'\ \mathbf{J}(x',y') \qquad \text{2D} \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \text{(B.2.12)}$$

where now [-ln(R)/2π] is the 2D Poisson free-space propagator of (I.1.8). In fact, this 2D result follows directly from (I.1.8) if we assume that $\mathbf{H} = \mathbf{H}(x,y)$ so that $\nabla^2\mathbf{H} = \nabla_{2D}^2\,\mathbf{H}$.

(c) Boundary conditions

Recall from (B.1.1) that the tangential component of H is continuous through the boundary between conductor and dielectric, even if $\mu_1 \neq \mu_2$. Consider then the transverse component of (B.2.7) at some point on the conductor surface such as the point shown in Fig 3.3. We have (t = transverse)

$$\mathbf{H}_t(\mathbf{x}) = \int d^2x' \,[1/4\pi R]\,]\,[\text{curl}'\,\mathbf{J}(\mathbf{x}')]_t \qquad\qquad R = |\mathbf{x}-\mathbf{x}'| \quad .$$

This particular integral is naturally continuous at the boundary between the media, and this agrees with the fact that $H_t(\mathbf{x})$ must have this property. Therefore, no homogeneous solutions of (B.2.4) $\nabla^2\mathbf{H}_t = 0$ need be added in, so (B.2.12) is the complete solution for $H_t(\mathbf{x})$. This solution can then be used in (B.1.10) to find the magnetization surface current.

(d) The Biot-Savart Law in 3D and 2D

Recall from above the 3D vector Helmholtz equation valid for $f << 10^{18}$ Hz, and its solution

$$\nabla^2\mathbf{H} = -\,\text{curl}\,\mathbf{J} \qquad\qquad (B.2.4)$$

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \,[1/4\pi R]\,\text{curl}'\,\mathbf{J}(\mathbf{x}') \quad . \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad (B.2.7) \qquad\qquad (B.2.13)$$

We shall now reverse the steps done in the previous section, but this time with $e^{-j\beta_x R} = 1$ based on the conclusion just drawn above. In components (B.2.7) reads,

$$H_i(\mathbf{x}) = \int d^3x' \,[\,\frac{1}{4\pi R}\,]\,\varepsilon_{ijk}\,\partial'_j J_k(\mathbf{x}') \,, \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' = \text{points to observation point } \mathbf{x} \,. \qquad (B.2.14)$$

Then move ∂_j' from J_k to (1/R) by parts integration (pick up minus sign) and throw out the parts for the usual reasons (see Section A.0 (d)),

$$H_i(\mathbf{x}) = -\int d^3x' \,\frac{1}{4\pi}\,\partial'_j\,(\frac{1}{R})\,\varepsilon_{ijk}\,J_k(\mathbf{x}') \quad . \qquad (B.2.15)$$

Then note that $\partial'_j R^{-1} = -R^{-2}\,\partial'_j R$ and

$$\partial'_j R = \partial'_j \sqrt{\Sigma_k (x'_k - x_k)^2} = (1/2)(1/R)\,2(x'_j - x_j) = R^{-1}\,(x'_j - x_j) = -\,R^{-1}\,R_j \qquad (B.2.16)$$

so that $\partial'_j R^{-1} = +R^{-3} R_j$. Only the parts minus sign remains, so

$$H_i(\mathbf{x}) = -\int d^3x' \left[\frac{1}{4\pi R^3}\right] \varepsilon_{ijk} R_j J_k(\mathbf{x}') \qquad \text{(B.2.17)}$$

or reversing the cross product order,

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \frac{1}{4\pi R^3} \mathbf{J}(\mathbf{x}') \times \mathbf{R} \ . \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \qquad \text{(B.2.18)}$$

This equation is basically the **3D Biot-Savart Law**, see for example Panofsky and Philips p 125 (7.31). For a short piece d**s**' of thin wire carrying current I, one writes $\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' = I\, d\mathbf{s}'$ so the above becomes,

$$\mathbf{H}(\mathbf{x}) = \oint \frac{1}{4\pi R^3} I\, d\mathbf{s}' \times \mathbf{R} \qquad \text{or} \qquad d\mathbf{H}(\mathbf{x}) = \frac{1}{4\pi R^3} I\, d\mathbf{s}' \times \mathbf{R} \ . \qquad \text{(B.2.19)}$$

We can apply the same process to obtain a 2D Biot-Savart Law as follows. Start with

$$\nabla^2_{2D}\, \mathbf{H}(x,y) = -\,\text{curl}\, \mathbf{J} \qquad \mathbf{J} = J_z(x,y)\hat{\mathbf{z}} \qquad \textit{(B.2.4)}$$

and its solution (B.2.5) obtained from (I.1.8)

$$\mathbf{H}(x,y) = \int d^2x' \left[\frac{1}{2\pi} \ln(1/R)\right] \text{curl}'\, \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \textit{(B.2.5)}$$

where $\frac{1}{2\pi}\ln(1/R)$ is the Poisson 2D free-space propagator. Then, inverting 1/R,

$$H_i(x,y) = -\int d^2x' \left[\frac{1}{2\pi} \ln(R)\right] \varepsilon_{ijk}\ \partial'_j J_k(\mathbf{x}') \ . \qquad \text{(B.2.20)}$$

Doing the same parts integration gives

$$H_i(x,y) = +\int d^2x' \left[\frac{1}{2\pi}\ \partial'_j \ln(R)\right] \varepsilon_{ijk}\, J_k(\mathbf{x}') \qquad \text{(B.2.21)}$$

and now using result (B.2.16) from above,

$$\partial'_j \ln(R) = R^{-1}\partial'_j R = R^{-1}\left[-R^{-1} R_j\right] = -R^{-2} R_j \qquad \text{(B.2.22)}$$

we get

$$H_i(x,y) = -\int d^2x' \left[\frac{1}{2\pi}\ R^{-2} R_j\right] \varepsilon_{ijk}\, J_k(\mathbf{x}') = -\int d^2x' \frac{1}{2\pi R^2}\ \varepsilon_{ijk} R_j\, J_k(\mathbf{x}') \qquad \text{(B.2.23)}$$

or

$$\mathbf{H}(x,y) = \int d^2x' \frac{1}{2\pi R^2}\ \mathbf{J}(\mathbf{x}') \times \mathbf{R} \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \qquad \text{(B.2.24)}$$

which is the **2D Biot-Savart Law**. It provides a way to obtain **H** from **J** in a 2D problem. In such a problem, we assume that **J** = **J**(x,y) with no z dependence.

B.3 General Method for computing the surface current J_m on a wire

Here are the steps for a wire of arbitrary cross sectional shape:

1. Compute **H** from **J**. Consider these sets of equations obtained above:

3D

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \frac{1}{4\pi R} \text{ curl}' \mathbf{J}(\mathbf{x}') \; . \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad (B.2.7)$$

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \frac{1}{4\pi R^3} \mathbf{J}(\mathbf{x}') \times \mathbf{R} \; . \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \quad \text{Biot-Savart} \qquad (B.2.18)$$

$$\mathbf{A}(\mathbf{x}) = \mu \int d^3x' \frac{1}{4\pi R} \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \mathbf{H} = (1/\mu) \text{ curl } \mathbf{A} \qquad (B.2.9)$$

2D

$$\mathbf{H}(x,y) = \int d^2x' \frac{\ln(1/R)}{2\pi} \text{ curl}' \mathbf{J}(x',y') \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad (B.2.12)$$

$$\mathbf{H}(x,y) = \int d^2x' \frac{1}{2\pi R^2} \mathbf{J}(x',y') \times \mathbf{R} \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \quad \text{Biot-Savart} \qquad (B.2.24)$$

$$\mathbf{A}(x,y) = \mu \int d^2x' \frac{\ln(1/R)}{2\pi} \mathbf{J}(x',y') \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \mathbf{H} = (1/\mu) \text{ curl } \mathbf{A} \qquad (B.3.1)$$

where the last line for **A** is the same 3D→2D resulting used for **H**. For either 3D or 2D, we thus provide three different methods for computing **H** from **J**.

2. Evaluate this **H** field at $\mathbf{x}_b = (x_b, y_b)$ for all points $\mathbf{x}_b$ on the cross section boundary.

3. Compute the component of **H** which is tangential to the boundary in the cross sectional plane. Call this component H_θ.

4. The surface current density is then given by (B.1.10),

$$K_z = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right) H_\theta \qquad \text{amps/m} \qquad (B.1.10)$$

B.4 Surface current on a round wire with uniform J

For a round wire of radius a with uniform J_z (as would be the DC case $\omega = 0$), geometric symmetry makes the calculation of **H** very easy. One need only apply Ampere's Law separately for a point r outside the wire, and for another point r inside the wire. For the outside case one finds

$$2\pi r\, H_\theta(r) = I \qquad \Rightarrow \qquad H_\theta(r) = I/(2\pi r) \qquad r \geq a \quad . \tag{B.4.1}$$

And then for the inside case the "current enclosed" is determined by a simple area fraction.

$$2\pi r\, H_\theta(r) = I\,(\pi r^2/\pi a^2) \qquad \Rightarrow \qquad H_\theta(r) = I\, r/(2\pi a^2) \qquad r \leq a \quad . \tag{B.4.2}$$

At the boundary the two expressions agree and we have

$$H_\theta = I/(2\pi a) \quad . \tag{B.4.3}$$

If this wire has magnetic permeability μ_2 and is embedded in an infinite medium of μ_1, then the surface magnetization current induced on the wire is

$$K_z = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right) H_\theta = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right) I/(2\pi a) \qquad \text{amp/m} \tag{B.4.4}$$

$$\mathbf{K}_z = K_z \hat{\mathbf{z}} \tag{B.4.5}$$

and this surface current is in the direction opposite **J** if $\mu_2 > \mu_1$. If $\mu_1 = \mu_2$, the surface current vanishes. This surface current could be expressed in volume density form as

$$\mathbf{J}_m = K_z \delta(r-a)\hat{\mathbf{z}} \qquad \text{amp/m}^2 \quad . \tag{B.4.6}$$

B.5 Computing H_θ for a round wire using the General Method of B.3

For a wire of some general cross section, symmetry is not available to allow the simple solution for H_θ outlined in the previous section. We then have to use the more general method outlined in Section B.3 above. As a check on the viability of this general method, we shall carry out "step 1" of the method and show how H_θ may be computed from **J** using the 2D formula (B.2.12).

The conduction current density in a round wire with uniform J_z is given by

$$J_z(r) = J_0\theta(a-r) \tag{B.5.1}$$

where θ is the Heaviside step function. Our first step is to compute curl **J**, and we do this in cylindrical coordinates by just staring at the cylindrical-coordinates curl formula,

$$\text{curl}\,\mathbf{J} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta J_z - \partial_z J_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z J_r - \partial_r J_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rJ_\theta) - r^{-1}\partial_\theta J_r\,] \tag{B.5.2}$$

and finding the only non-zero piece which is this (uniform J_z),

$$\text{curl}\,\mathbf{J} \;=\; [-\partial_r J_z(r)]\,\hat{\boldsymbol{\theta}} \quad . \tag{B.5.3}$$

Inserting $J_z(r)$ from above we find

$$\partial_r J_z(r) = J_0\ \partial_r\theta(a-r) = -\ J_0\ \delta(r-a) \qquad \text{(B.5.4)}$$

$$\Rightarrow\ \text{curl}\ \mathbf{J}(r) = \hat{\boldsymbol{\theta}}\ J_0\ \delta(r-a)\ . \qquad \text{(B.5.5)}$$

so we have a "ring source of curl **J**". For use in our integral for **H** we then have

$$\text{curl}'\ \mathbf{J}(r') = \hat{\boldsymbol{\theta}}'\ J_0\ \delta(r'-a)\ . \qquad \text{(B.5.6)}$$

For a current distribution which tapers off smoothly to 0 at the wire edge one would not have this singular contribution, but for a wire with prescribed uniform current, it is present, and curl **J** vanishes everywhere but on the boundary. The relevant picture is this:

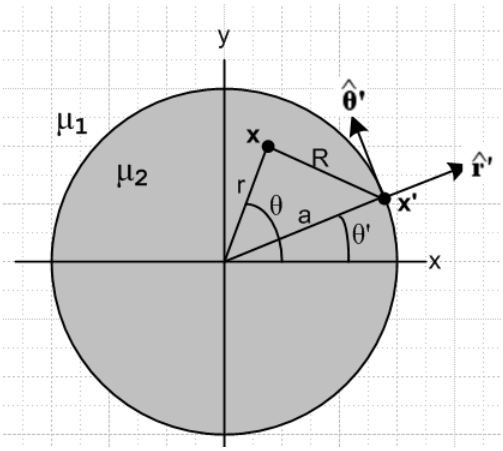

Fig B.4

From (B.2.12) the **H** field at any point **x** = (x,y) is then given by

$$\mathbf{H}(x,y) = -\frac{1}{4\pi}\int d^2\mathbf{x}'\ \ln(R^2)\ \text{curl}'\ \mathbf{J}(\mathbf{x}') = -\frac{1}{4\pi}\int d^2\mathbf{x}'\ \ln(R^2)\ \hat{\boldsymbol{\theta}}'\ J_0\ \delta(r'-a)$$

$$= -\frac{J_0 a}{4\pi}\int_{-\pi}^{\pi} d\theta'\ \ln(R^2)\big|^{r'=a}\ \hat{\boldsymbol{\theta}}'$$

or

$$\mathbf{H}(r,\theta) = -\frac{J_0 a}{4\pi}\int_{-\pi}^{\pi} d\theta'\ \ln\left[\ r^2 + a^2 - 2ar\cos(\theta'-\theta)\ \right]\hat{\boldsymbol{\theta}}'\ . \qquad \text{(B.5.7)}$$

The figure shows that

$$\hat{\boldsymbol{\theta}}' = \cos\theta'\ \hat{\mathbf{y}} - \sin\theta'\ \hat{\mathbf{x}} \qquad \text{(B.5.8)}$$

so then

$$\mathbf{H}(r,\theta) = -\frac{J_0 a}{4\pi}\int_{-\pi}^{\pi} d\theta'\ \ln\left[\ r^2 + a^2 - 2ar\cos(\theta'-\theta)\ \right]\left[\cos\theta'\ \hat{\mathbf{y}} - \sin\theta'\ \hat{\mathbf{x}}\right]\ . \qquad \text{(B.5.9)}$$

Next, let $x \equiv \theta' - \theta$. Since the $\int d\theta'$ has full range 2π, one can replace $\int_{-\pi}^{\pi} d\theta' = \int_{-\pi}^{\pi} dx$. Then

$$\mathbf{H}(r,\theta) = -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \left[\cos(x+\theta)\,\hat{\mathbf{y}} - \sin(x+\theta)\,\hat{\mathbf{x}}\right] . \qquad \text{(B.5.10)}$$

Now writing $\mathbf{H} = H_x\,\hat{\mathbf{x}} + H_y\,\hat{\mathbf{y}}$, decompose the above into two equations

$$H_x(r,\theta) = +\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \sin(x+\theta)$$

$$H_y(r,\theta) = -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos(x+\theta) \qquad \text{(B.5.11)}$$

or

$$H_x(r,\theta) = +\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \left[\sin x\cos\theta + \cos x\sin\theta \right]$$

$$H_y(r,\theta) = -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \left[\cos x\cos\theta - \sin x\sin\theta \right] . \qquad \text{(B.5.12)}$$

Since $\int_{-\pi}^{\pi} dx$ is over an even range, throw out odd integrand terms, and *then* fold the negative range into the positive adding a factor of 2 to get

$$H_x(r,\theta) = +\sin\theta\,\frac{J_0 a}{2\pi} \int_{0}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos x \quad \equiv \quad \sin\theta\,\frac{J_0 a}{2\pi}\,Q$$

$$H_y(r,\theta) = -\cos\theta\,\frac{J_0 a}{2\pi} \int_{0}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos x \quad = -\cos\theta\,\frac{J_0 a}{2\pi}\,Q \qquad \text{(B.5.13)}$$

where

$$Q \equiv \int_{0}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos x \ . \qquad \text{(B.5.14)}$$

Before evaluating this integral, we see that

$$\mathbf{H} = H_x\,\hat{\mathbf{x}} + H_y\,\hat{\mathbf{y}} = -\frac{J_0 a}{2\pi}\,Q\left[\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{x}} \right] = -\frac{J_0 a}{2\pi}\,Q\,\hat{\boldsymbol{\theta}} = H_\theta\,\hat{\boldsymbol{\theta}} \ . \qquad \text{(B.5.15)}$$

Thus we find that the resulting **H** is entirely in the $\hat{\boldsymbol{\theta}}$ direction and

$$H_\theta = -\frac{J_0 a}{2\pi}\,Q \ . \qquad \text{(B.5.16)}$$

We seek now to evaluate this integral Q,

$$Q \equiv \int_0^{\pi} dx \ln [\, r^2 + a^2 - 2ar\cos(x)\,] \cos x$$

$$= \int_0^{\pi} dx \ln [\, \{a^2\}\{ (r/a)^2 + 1 - 2(r/a)\cos(x)\}\,] \cos x$$

$$= \int_0^{\pi} dx \{ \ln (a^2) + \ln[(r/a)^2 + 1 - 2(r/a)\cos(x)] \} \cos x$$

$$= \ln (a^2)[\int_0^{\pi} dx \cos x\,] + \int_0^{\pi} dx \ln[(r/a)^2 + 1 - 2(r/a)\cos(x)] \cos x$$

$$= \ln (a^2)[0] + \int_0^{\pi} dx \ln[\alpha^2 + 1 - 2\alpha\cos(x)] \cos x \qquad \text{where} \quad \alpha \equiv r/a$$

$$= \int_0^{\pi} dx \ln[\alpha^2 + 1 - 2\alpha\cos(x)] \cos x \ . \qquad // \ = Q \qquad \text{(B.5.17)}$$

This integral is the n=1 special case of the following integral from GR7 page 589 4.379.6,

6. $$\int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right) \cos nx\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} \ln\left(1 - 2a\cos x + a^2\right) \cos nx\, dx$$

$$= -\frac{\pi}{n}a^n \qquad \left[a^2 < 1\right] \quad \text{BI (330)(11), BI (332)(5)}$$

$$= -\frac{\pi}{na^n} \qquad \left[a^2 > 1\right] \quad \text{GW (338)(13a)}$$

Therefore we find

$$Q = \begin{cases} -\pi(r/a) & r<a \\ -\pi(a/r) & r>a \end{cases} \qquad \text{(B.5.18)}$$

so

$$H_\theta = -\frac{J_0 a}{2\pi} Q = \begin{cases} (aJ_0/2)\,(r/a) & r<a \\ (aJ_0/2)\,(a/r) & r>a \end{cases} \ . \qquad \text{(B.5.19)}$$

Now the total current in the wire is $I = J_0\pi a^2$ so $(aJ_0/2) = (I/2\pi a)$ and then

$$H_\theta = \begin{cases} (I/2\pi a)\,(r/a) & r<a \\ (I/2\pi a)\,(a/r) & r>a \end{cases} = \begin{cases} I\,(r/2\pi a^2) & r<a \\ I\,(1/2\pi r) & r>a \end{cases} \ . \qquad \text{(B.5.20)}$$

Thus, we finally arrive at the same results for H_θ as obtained in (B.4.2) and (B.4.1).