Magnetostatics and the Round Wire PhL 11.6.13

(11.15.13) This was my breakthrough calculation showing how, when you compute **A** for a round wire *including* the surface current, you get an **A** which is linear in μ_1 outside the wire, but linear in μ_2 inside the wire. This was always a mystery to me, though I knew it somehow had to come out that way. It was not until I was aware of the surface current that I could get this result.

The order of this file is a little deceiving. I really did Section 5 first to discover the surface current (first time ever I did that), then I used that in the 3D analysis of the sections 1-4 to compute **A**, then **B**, then **H**. I was truly amazed when **H** was correct. In the last section I use the 2D method to compute A and get the same answer apart from constant Λ terms.

All the material here is now in Appendix B. There I did the 2D method, and left the 3D method as a reader exercise with waypoints in the last section of App B. Those waypoints are derived in a second short document that should not get lost.

So I think this document can be marked REVIEWED. For me it as pretty significant on my stumbling path.

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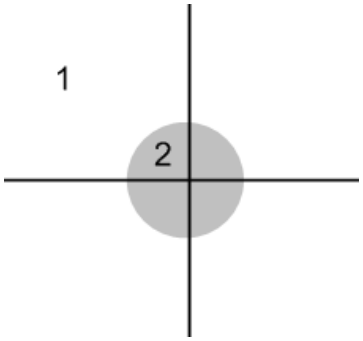
1. Setup of the Problem of finding A for a round wire.

Here is my picture of interest, end view of the round wire



I have my reliable B&B open to page 143. They quickly arrive at (using my μ scale)

$\nabla^2\mathbf{A}(\mathbf{x}) = -\mu\mathbf{J}(\mathbf{x})$ B&B (5.46) magnetostatics

where μ is that appropriate for point **x** in space. Now if I regard the surface current K discussed below to be part of region 1, I can write

$$\nabla^2\mathbf{A}(\mathbf{x}) = -\mu_1\mathbf{J}_m(\mathbf{x}) \qquad \text{x in region 1}$$

$$\nabla^2\mathbf{A}(\mathbf{x}) = -\mu_2\mathbf{J}_c(\mathbf{x}) \qquad \text{x in region 2}$$

Now I can add these equations to get an equation for region 1+2:

$$\nabla^2\mathbf{A}(\mathbf{x}) = -\mu_1\mathbf{J}_m(\mathbf{x}) - \mu_2\mathbf{J}_c(\mathbf{x}) \qquad \text{// compare to (B.6.1) which is 2D version}$$

where the currents don't exist in the same region, but that is fine. We then find

$$\mathbf{A}(\mathbf{x}) = (1/4\pi) \int dV' \, [\mu_1\mathbf{J}_m(\mathbf{x}') + \mu_2\mathbf{J}_c(\mathbf{x}')] \, (1/R) \qquad R = |\mathbf{x}-\mathbf{x}'|$$

and now I think we are getting somewhere. This is the generalization of

$$\mathbf{A} = (1/4\pi) \int dV \, (\mu\mathbf{J}) \, (1/R) \qquad \text{B\&B (5.47)}$$

so my situation. Now I can rewrite the above as

$$\mathbf{A}(\mathbf{x}) = (1/4\pi) \int dS' \, [\mu_1\mathbf{K}(\mathbf{x}')] \, (1/R) + (1/4\pi) \int dV' \, [\mu_2\mathbf{J}_c(\mathbf{x}')] \, (1/R)$$

In our case, everything is in the z direction. I should maybe have said that earlier. But in Cartesians the above is OK as stated. So

$$A_z(\mathbf{x}) = (1/4\pi) \int dS' \, [\mu_1 K_z(\mathbf{x}')] \, (1/R) + (1/4\pi) \int dV' \, [\mu_2 J_{cz}(\mathbf{x}')] \, (1/R)$$

Now we know that (round wire, DC) (the K result comes from below)

$$J_{cz}(\mathbf{x}') = J_{cz} = I/(\pi a^2)$$

$$K_z(\mathbf{x}') = K_z = (1-\mu_2/\mu_1) \, H_{t1} = (1-\mu_2/\mu_1) \, (1/2\pi a) \, I$$

Then we have for all of region 1 and 2

$$A_z(\mathbf{x}) = (1/4\pi) \, \mu_1 K_z \int dS' \, (1/R) + (1/4\pi) \, \mu_2 J_{cz} \int dV' \, (1/R) \qquad \text{// compare (B.6.4)}$$

where $\int dS'$ is over the entire wire surface, and $\int dV'$ is over the entire wire volume. Installing the above values we rewrite as

$$A_z(\mathbf{x}) = (1/4\pi)\,\mu_1(1-\mu_2/\mu_1)\,(1/2\pi a)\,I\int dS'\,(1/R) + (1/4\pi)\,\mu_2\,I/(\pi a^2)\int dV'\,(1/R)$$

or

$$A_z(\mathbf{x}) = (1/4\pi)\,(\mu_1-\mu_2)\,(1/2\pi a)\,I\int dS'\,(1/R) + (1/4\pi)\,\mu_2\,I/(\pi a^2)\int dV'\,(1/R)$$

$$= (1/4\pi)\,(1/2\pi a)\,I\;\{\,(\mu_1-\mu_2)\int dS'\,(1/R) + (2/a)\,\mu_2\int dV'\,(1/R)\;\}$$

$$= (1/4\pi)\,(1/2\pi a)\,I\;\{\,(\mu_1-\mu_2)\;I_1 + (2/a)\,\mu_2 I_2\,\}$$ // defines two integrals

2. Evaluation of the two integrals

The Integrals (this is all just in cross section)

$$I_1 \equiv \int_{surf} dS'\,(1/R) = \int (a\,d\theta')\int_{-\infty}^{\infty} dz'\,(1/R)$$

$$I_2 \equiv \int dV'\,(1/R) = \int (r'\,d\theta')\int dr'\int_{-\infty}^{\infty} dz'\,(1/R)$$

We now access information in some other docs. In cyl coords we know that

$$|\mathbf{x}-\mathbf{x}'|^2 = R^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta') + (z-z')^2$$

If we choose to observe things from z = 0 we get

$$R^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta') + z'^2 \quad = s^2 + z'^2 \qquad\qquad s^2 = (x-x')^2 + (y-y')^2$$

$$R = \sqrt{s^2 + z'^2}$$

Then we have

$$I_1 \equiv \int (a\,d\theta')\int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\,) = a\int_{-\pi}^{\pi} d\theta'\int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\,)$$

$$I_2 \equiv \int (r'\,d\theta')\int dr'\int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\,) = \int_{0}^{a} r'\,dr'\int_{-\pi}^{\pi} d\theta'\int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\,)$$

and we have the same $d\theta' dz'$ integral in both cases. I show in appendix C that

$$\int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\,) = -2\ln(s/\Lambda) = -2\ln(\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}\,/\Lambda)$$

First work on I_2:

$$I_2 = \int_0^a r'dr' \int_{-\pi}^{\pi} d\theta' \, [-2\ln(\sqrt{r'^2+r^2- 2rr' \cos\theta'} /\Lambda)]$$

$$= - 2 \int_0^a r'dr' \, [\int_0^{\pi} d\theta' \ln[(r'^2+r^2- 2rr' \cos\theta')/\Lambda^2]]$$

where I just drop the Λ stuff since it just makes a constant which we drop in **B** = curl **A**. So

$$I_2 = - 2 \int_0^a r'dr' \int_0^{\pi} d\theta' \ln(r'^2+r^2- 2rr' \cos\theta')$$

I then show in verifications that

$$\int_0^{\pi} d\theta' \ln(r'^2+r^2- 2rr' \cos\theta') = \pi \ln \{ [(r'^2+r^2) + | r'^2- r^2|] /2 \}$$

so that

$$I_2 = - 2\pi \int_0^a r'dr' \ln \{ [(r'^2+r^2) + | r'^2- r^2|] /2 \}$$

I show in verifications that

$$\int_0^a r'dr' \ln \{ [(r'^2+r^2) + | r'^2- r^2|] /2 \} = a^2 \ln(r) \qquad \text{for } r > a \text{ which means } r' < r$$

$$\int_0^a r'dr' \ln \{ [(r'^2+r^2) + | r'^2- r^2|] /2 \} = (1/2)r^2 \qquad \text{for } r < a$$

where we always drop constants since they don't affect **B** = curl **A**.

So we conclude that

$$I_2 = - 2\pi [a^2\ln(r) \theta(r>a) + (1/2)r^2\theta(r<a)]$$

Now what about the I_1 integral?

$$I_1 \equiv \int (ad\theta') \int_{-\infty}^{\infty} dz' \, (1/\sqrt{s^2 + z'^2}) = a \int_{-\pi}^{\pi} d\theta' \int_{-\infty}^{\infty} dz' \, (1/\sqrt{s^2 + z'^2})$$

$$= 2a \int_0^{\pi} d\theta' \int_{-\infty}^{\infty} dz' \, (1/\sqrt{s^2 + z'^2})$$

As before, the dz' integral gives $-2\ln(\sqrt{r'^2+r^2- 2rr' \cos\theta'}\,/\Lambda)$ so we then have

$$I_1 = 2a \int_0^{\pi} d\theta' \,[-2\ln(\sqrt{r'^2+r^2- 2rr' \cos\theta'}\,/\Lambda)]$$

$$= -2a \int_0^{\pi} d\theta' \ln[(r'^2+r^2- 2rr' \cos\theta')/\Lambda^2]$$

$$= -2a \int_0^{\pi} d\theta' \ln[(r'^2+r^2- 2rr' \cos\theta')]$$

Now I showed above that

$$\int_0^{\pi} d\theta' \ln(r'^2+r^2- 2rr' \cos\theta') = \pi \ln \{ [(r'^2+r^2) + | r'^2- r^2|] /2 \}$$

so it seems then that we have

$$I_1 = -2\pi a \ln \{ [(r'^2+r^2) + | r'^2- r^2|] /2 \}$$

Now the I_1 is in our surface integral and there we have r' = a, so this says

$$I_1 = -2\pi a \ln \{ [(a^2+r^2) + | a^2- r^2|] /2 \}$$

So this has two regions as well. If a > r then

$$I_1 = -2\pi a \ln \{ [(a^2+r^2) + (a^2- r^2)] /2 \} = -2\pi a \ln(a^2) \qquad r < a$$

$$I_1 = -2\pi a \ln \{ [(a^2+r^2) - (a^2- r^2)] /2 \} = -2\pi a \ln(r^2) \qquad r > a$$

So I find that

$$I_1 = -2\pi a \{ \ln(a^2)\theta(r<a) + \ln(r^2)\theta(r>a) \}$$

so backtrack now to

$$A_z(r) = (1/4\pi) (1/2\pi a) I \{ (\mu_1-\mu_2) I_1 + (2/a) \mu_2 I_2 \}$$

where we have so found above that

$$I_1 = -2\pi a \{ \ln(a^2)\theta(r<a) + \ln(r^2)\theta(r>a) \}$$

$$I_2 = - 2\pi [a^2 \ln(r) \theta(r>a) + (1/2) r^2 \theta(r<a)]$$

Rewrite I_2 so same θ:

$I_1 = -2\pi a [\ln(a^2)\theta(r<a) + \ln(r^2)\theta(r>a)]$

$I_2 = -2\pi [(1/2)r^2\theta(r<a) + a^2\ln(r)\,\theta(r>a)]$

3. Install integral results into the A formula

We then get two results for $A_z(r)$ depending on the size of r. Installing

$A_z(r) = (1/4\pi)(1/2\pi a) I \{ (\mu_1-\mu_2) I_1 + (2/a)\mu_2 I_2 \}$

For r > a (outside) we get

$A_z(r) = (1/4\pi)(1/2\pi a) I \{ (\mu_1-\mu_2) [-2\pi a \ln(r^2)] + (2/a)\mu_2[-2\pi a^2\ln(r)] \}$

$= -(1/4\pi)(1/2\pi a) I\, 2a \{ (\mu_1-\mu_2) [\pi \ln(r^2)] + 2\pi\mu_2[\ln(r)] \}$

$= -(1/4\pi)(1/2\pi a) I\, 2a \{ (\mu_1-\mu_2) [2\pi \ln(r)] + 2\pi\mu_2[\ln(r)] \}$

$= -(1/2)(1/2\pi a) I\, 2a \{ (\mu_1-\mu_2) \ln(r) + \mu_2\ln(r)\}$

$= -(1/2\pi) I \{ (\mu_1\ln(r)\}$

$= -(1/2\pi) I \mu_1\ln(r)$ r > a outside

On the other hand, for r < a we get

$A_z(r) = (1/4\pi)(1/2\pi a) I \{ (\mu_1-\mu_2) [-2\pi a \ln(a^2)] + (2/a)\mu_2[-2\pi(1/2)r^2] \}$

$= -(1/2)(1/2\pi a) I \{ (\mu_1-\mu_2) [a \ln(a^2)] + (2/a)\mu_2[(1/2)r^2] \}$

$= -(1/2)(1/2\pi a) I \{ (\mu_1-\mu_2) [a \ln(a^2)] + (1/a)\mu_2[r^2] \}$

$= -(1/2)(1/2\pi a) I \{ (\mu_1-\mu_2) [a \ln(a^2)] + a\,\mu_2[r^2/a^2] \}$

$= -(1/2)(1/2\pi) I \{ (\mu_1-\mu_2) [\ln(a^2)] + \mu_2[r^2/a^2] \}$

Now I guess drop the constant term to get

$= -(1/4\pi) I \mu_2[r^2/a^2]$ r < a inside

So I seem to obtain these expressions for $A_z(r)$ by evaluating the Helm integral thing

$A_z(r) = -\ (1/2\pi)\ I\ \mu_1 \ln(r)$ r > a outside // same as (B.6.17)

$A_z(r) = -\ (1/4\pi)\ I\ \mu_2 [r^2/a^2]$ r < a inside // non-constant part of (B.6.18)

I may have errors, but I obtain here an important general result of great interest to me: in the outside formula, you see μ_1 appear, whereas in the inside formula you see μ_2 appear.

4. Verify the A_z results.

Now let's try to get **B** from **B** = curl **A**. It happens that

$$\mathbf{B} = \hat{\boldsymbol{\theta}}\ [\ -\partial_r A_z]$$

$$B = -\partial_r A_z$$

So we compute

$B_{1t} = (1/2\pi)\ I\ \mu_1\ (1/r)$ r > a outside

$B_{2t} = (1/4\pi)\ I\ \mu_2\ 2r/a^2 = (1/2\pi)\ I\ \mu_2\ (r/a^2)$ r < a inside

Then we compute H

$H_{1t} = (1/2\pi)\ I\ (1/r)$ r > a outside

$H_{2t} = (1/2\pi)\ I\ (r/a^2)$ r < a inside

But these are just the results you get from curl **H** = **J** Ampere's Law:

$H_{1t}\ 2\pi r = I$ outside

$H_{1r}\ 2\pi r = I\ (\pi r^2/\pi a^2) = I(r^2/a^2)$ inside

So FINALLY I see how the 3D Helmholtz type integral solution gives the desired results, and you must include the surface current indeed!

So when you compute the results of this problem using the vector potential

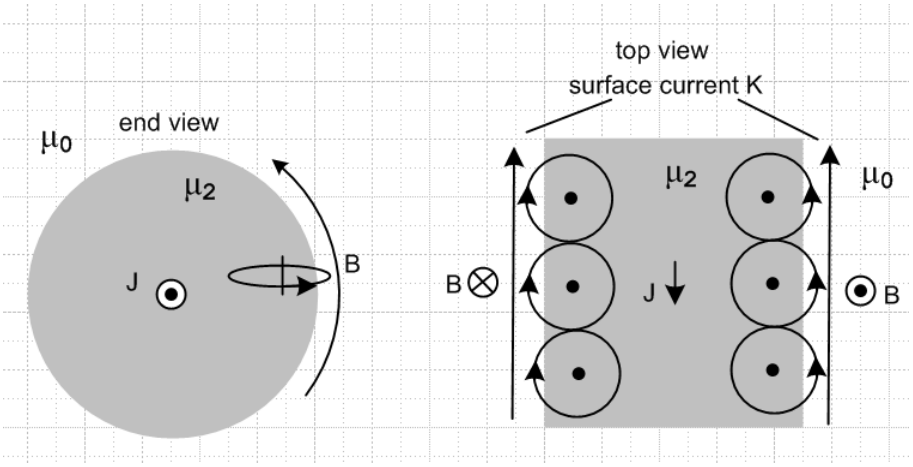
$$\mathbf{A} = (1/4\pi) \int dV\ (\mu \mathbf{J})\ (1/R)$$

the computation is NON TRIVIAL and you see all the details above.

5. The Surface Current

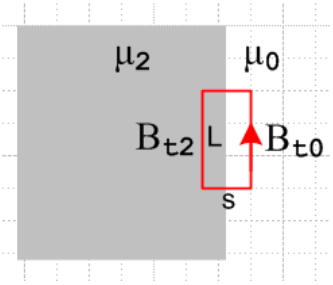
The results of this section were used in Section 1 above.

I am going to draw a picture which shows how the B field lines up the mag dipoles to create a surface current.



This clearly shows the mechanism which causes a surface current in the opposite direction to J.

What is the size of this surface current? Here is a blowup of the rightmost point in the left drawing:



so there is a surface current K going away from the observer inside the loop

Apply our integral form

$$\text{curl } \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \qquad \Leftrightarrow \qquad \oint \mathbf{H} \bullet d\mathbf{s} = \int_S [\partial_t \mathbf{D} + \mathbf{J}] \bullet d\mathbf{A} \qquad (1.1.23)$$

Now H does not "see" the surface current, so J is only the conduction current. This is a mag static problem so that $\partial_t \mathbf{D} = 0$. We are going to find of course that

$H_{t2} = H_{t0}$ at the boundary

We confirm this by directly computing H from Ampere's Law (current is uniform)

$2\pi r\, H_{t2} = (\pi r^2/\pi a^2)\, I = (r^2/a^2)\, I \quad \Rightarrow \quad H_{t2} = (1/2\pi a^2)\, I\, r \qquad r < a$

Outside the conductor, since H does not "see" K we get

$2\pi r\, H_{t0} = I \qquad \Rightarrow \qquad H_{t0} = (1/2\pi)\, I\, r^{-1} \qquad r > a$

and the values match at r = a where we have $H_{t0} = H_{t1} = H_{t2} = (1/2\pi a)\, I$.

Now how do we compute the surface current K? consider

$$\text{curl } \mathbf{B} = \mu\varepsilon\, \partial_t \mathbf{E} + \mu\mathbf{J} \quad \Leftrightarrow \quad \oint \mathbf{B} \bullet d\mathbf{s} = \mu \int_S [\varepsilon\, \partial_t \mathbf{E} + \mathbf{J}] \bullet d\mathbf{A} \qquad (1.1.24)$$

$$\text{curl } \mathbf{B} = \mu\mathbf{J} \quad \Leftrightarrow \quad \oint \mathbf{B} \bullet d\mathbf{s} = \int_S \mu\, \mathbf{J} \bullet d\mathbf{A} \qquad (1.1.24)$$

where **J** includes the volume conduction current *and* the surface current $\mathbf{J}_m = \text{curl } \mathbf{M}$. Again $\partial_t \mathbf{E} = 0$. Write the above as

$$\oint \mathbf{B} \bullet d\mathbf{s} = \int_S \mu\, [\mathbf{J}_c + \mathbf{J}_m] \bullet d\mathbf{A}$$

In the limit that s→ 0 the finite volume $\mathbf{J}_c$ makes no contribution. The $\mathbf{J}_m$ surface current is located entirely in the μ_0 region 0, as you would say for a surface charge. Thus

$$\int_S \mu\, [\mathbf{J}_c + \mathbf{J}_m] \bullet d\mathbf{A} \approx \mu_0\, K\, L$$

Meanwhile

$$\oint \mathbf{B} \bullet d\mathbf{s} = (B_{t0})L - (B_{t1})L$$

Then we find that

$$\mu_0 K = B_{t0} - B_{t2} = \mu_0 H_{t0} - \mu_2 H_{t2} = (\mu_0 - \mu_2)\, H_{t0}$$

and then the surface current density is,

$K = (1-\mu_2/\mu_0)\, H_{t0}$ // this derivation is in Appendix B now

which goes away if $\mu_2 = \mu_0$.

Mystery regarding integral the curl equations

Go back to the claim that B "sees" all the currents:

$$\text{curl } \mathbf{B} = \mu\varepsilon\, \partial_t \mathbf{E} + \mu\mathbf{J}_c + \mu\mathbf{J}_m \quad (*)$$ // wrong!

Divide by μ on the left to get

$$\text{curl } \mathbf{H} = \varepsilon \partial_t \mathbf{E} + \mathbf{J_c} + \mathbf{J_m}$$

But this says that H sees J_m which we know is wrong. What gives? Well, what gives is that I have the curl B equation written incorrectly. Start with the curl H equation

$$\text{curl } \mathbf{H} = \varepsilon \partial_t \mathbf{E} + \mathbf{J_c}$$
$$\text{curl } \mu_0 \mathbf{H} = \mu_0 \varepsilon \partial_t \mathbf{E} + \mu_0 \mathbf{J_c}$$

Then write $\mu_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}$ to get

$$\text{curl } (\mathbf{B} - \mu_0 \mathbf{M}) = \mu_0 \varepsilon \partial_t \mathbf{E} + \mu_0 \mathbf{J_c}$$

or

$$\text{curl } \mathbf{B} = \mu_0 \varepsilon \partial_t \mathbf{E} + \mu_0 \mathbf{J_c} + \mu_0 \text{ curl } \mathbf{M}$$

or

$$\text{curl } \mathbf{B} = \mu_0 \varepsilon \partial_t \mathbf{E} + \mu_0 \mathbf{J_c} + \mu_0 \mathbf{J_m} \qquad \text{// right}$$

This is the corrected version of (*) shown above. Then go do the integral form

$$(1/\mu_0)\text{curl } \mathbf{B} = \varepsilon \partial_t \mathbf{E} + \mathbf{J_c} + \mathbf{J_m}$$

$$\Leftrightarrow \qquad (1/\mu_0) \oint \mathbf{B} \bullet d\mathbf{s} = \int_S [\varepsilon \partial_t \mathbf{E} + \mathbf{J_c} + \mathbf{J_m}] \bullet d\mathbf{A}$$

NOW we can apply this thing to get

$$(1/\mu_0)\,[(B_{t0})L - (B_{t1})L] = KL$$

and then

$$K = (1/\mu_0)\,[B_{t0} - B_{t1}] = (1/\mu_0)\,[\mu_0 H_{t0} - \mu_1 H_{t1}] = (1/\mu_0)\,[\mu_0 - \mu_1]\,H_{t1}$$

$$= (1-\mu_1/\mu_0)\,H_{t1}$$

as found above, but not much more clearly! I need to go fix Chapter 1 once again ******* to get all this stuff right. But hold off on that until we finish with the current problem.

6. Surface current for a non-round wire.

When I get to Chapter 4, I will have this volume of current in the arbitrarily shaped conductor,

$$J_{z1}(x,y,z) = b_1(x,y)\, i_1(z)$$

Question: Can you compute the surface current knowing the volume current?

Consider

$$\text{curl } \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \qquad \Leftrightarrow \qquad \oint \mathbf{H} \bullet d\mathbf{s} = \int_S [\partial_t \mathbf{D} + \mathbf{J}] \bullet d\mathbf{A} \qquad (1.1.23)$$

In a DC situation or perhaps the AC situation with E = 0 inside the conductor, this says

$$\oint \mathbf{H} \bullet d\mathbf{s} = \int_S \mathbf{J} \bullet d\mathbf{A}$$

This would get you the line integral of H_t around a path just inside the surface. If not a round wire, we have no obvious way to recover H_t from this formula at some point on the boundary.

So consider instead

$$\text{curl } \mathbf{H} = \mathbf{J} \qquad \partial_x H_y(x,y) - \partial_y H_x(x,y) = J_z(x,y)$$

This is a 2D PDE where the source is given. Is this a standard equation of some sort? Hold that thought and try this. Apply curl to both sides

$$\text{curl curl } \mathbf{H} = \text{curl } \mathbf{J}$$

$$\text{grad div } \mathbf{H} - \nabla^2 \mathbf{H} = \text{curl } \mathbf{J}$$

Now if we always work inside the conductor where there is some fixed μ, then this says (div B = 0)

$$\nabla^2 \mathbf{H} = -\text{ curl } \mathbf{J}$$

and then I want to write my standard Helmholtz solution

$$\mathbf{H}(\mathbf{x}) = \frac{1}{4\pi} \int d^3\mathbf{x}' \frac{\text{curl}' \mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ . \qquad (A.1.2)$$

Now if $J = J_z$ only, we can say

$$\text{curl } \mathbf{F} = (\partial_x F_y - \partial_y F_x)\, \hat{\mathbf{z}} + (\partial_y F_z - \partial_z F_y)\, \hat{\mathbf{x}} + (\partial_z F_x - \partial_x F_z)\, \hat{\mathbf{y}}$$

Therefore

$$\text{curl } \mathbf{J} = (\partial_x J_y - \partial_y J_x)\, \hat{\mathbf{z}} + (\partial_y J_z - \partial_z J_y)\, \hat{\mathbf{x}} + (\partial_z J_x - \partial_x J_z)\, \hat{\mathbf{y}}$$

Now as $\mathbf{x}'$ moves around in the above integration (over the entire conductor), we can take our primed unit vectors be same as unprimed -- they are just constants. We also assume **J** is only a function of x and y, and furthermore we only have J_z . Then Then we can say

$$\text{curl } \mathbf{J}(\mathbf{x}) = (\partial_y J_z(x,y))\, \hat{\mathbf{x}} + (-\partial_x J_z(x,y))\, \hat{\mathbf{y}}$$

$$\text{curl}' \mathbf{J(x')} = (\partial_{\mathbf{y}'} J_{\mathbf{z}'}(x',y'))\, \hat{\mathbf{x}} + (-\partial_{\mathbf{x}'} J_{\mathbf{z}'}(x',y'))\, \hat{\mathbf{y}}$$

Then the integral becomes

$$H_{\mathbf{x}}(x,y) = (1/4\pi) \int d^3x'\ \partial_{\mathbf{y}'} J_{\mathbf{z}'}(x',y')/R \qquad R = |\mathbf{x} - \mathbf{x}'|$$

$$H_{\mathbf{y}}(x,y) = -(1/4\pi) \int d^3x'\ \partial_{\mathbf{x}'} J_{\mathbf{z}'}(x',y')/R$$

So we have at least made some progress. In theory we can compute H just inside the boundary using these expressions. We could do the z' integral in our usual manner perhaps.

$$\int_{-\infty}^{\infty} dz'\ (1/\sqrt{s^2 + z'^2}\ ) = -2\ln(s/\Lambda) = -2\ln(\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}\ /\Lambda)$$

But instead of doing that, suppose we do parts in each equation above, and $(J_{\mathbf{z}} 1/R) = 0$ at $\equiv\infty$ due to the 1/R. Then meanwhile

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$2RdR = 2(y'-y)dy' \qquad \Rightarrow \qquad \partial_{\mathbf{y}'} R^{-1} = -R^{-2}\partial_{\mathbf{y}'} R = -R^{-2}\,(y'-y)/R = -R^{-3}(y'-y)$$

So then maybe we get

$$H_{\mathbf{x}}(x,y) = -(1/4\pi) \int d^3x'\ J_{\mathbf{z}'}(x',y')[\,\partial_{\mathbf{y}'} R^{-1}] = (1/4\pi) \int d^3x'\, J_{\mathbf{z}'}(x',y')\,(y'-y)/R^3$$

So then

$$H_{\mathbf{x}}(x,y) = (1/4\pi) \int d^3x'\, J_{\mathbf{z}'}(x',y')\,(y'-y)/R^3$$

$$H_{\mathbf{y}}(x,y) = -(1/4\pi) \int d^3x'\, J_{\mathbf{z}'}(x',y')\,(x'-x)/R^3$$

so these are some kind of moments of $J_{\mathbf{z}}/R^3$.Now what about the s' integral?

$$\int_{-\infty}^{\infty} dz'\ (s^2+z'^2)^{-3/2} = 1/s^2$$

says Maple,

```
Int((s^2+z^2)^(-3/2),z = 0..infinity);
```

$$\int_0^{\infty} \frac{1}{\left(s^2+z^2\right)^{\left(\frac{3}{2}\right)}}\,dz$$

```
int((s^2+z^2)^(-3/2),z = 0..infinity);
```

$$\frac{1}{s^2}$$

Then we seem to have

$$H_x(x,y) = (1/4\pi) \int dx'dy'\, J_z(x',y')\, (y'-y)/s^2$$

$$H_y(x,y) = -(1/4\pi) \int dx'dy'\, J_z(x',y')\, (x'-x)/s^2 \qquad s \equiv \sqrt{(x-x')^2 + (y-y')^2}$$

No matter how I rewrite this, it is always going to require some integral over the volume current density and won't have some trivial form such as J_z near the boundary. So in Chapter 4 we are going to have to "carry" a whole extra term to handle the boundary current if the μ's are not equal. So go look now at Section 4.3 where I now add in a magnetization current

$$A_{z1}(\mathbf{x}) = \frac{\mu}{4\pi} \int_{C_1} [J_{z1}(\mathbf{x}') + J_{z1m}] \frac{e^{-j\beta R}}{R}\, dx'dy'dz' \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (4.3.1)$$

We then make the same assumption of separation of variables to write

$$J_{z1}(x,y,z) = b_1(x,y)\, i_1(z)$$
$$A/m^2 \qquad 1/m^2 \quad A \qquad (4.3.2)$$

where i_1 is scaled such that

$$\int_{C_1} dx\, dy\, b_1(x,y) = 1\ . \qquad (4.3.3)$$

and then I have to add

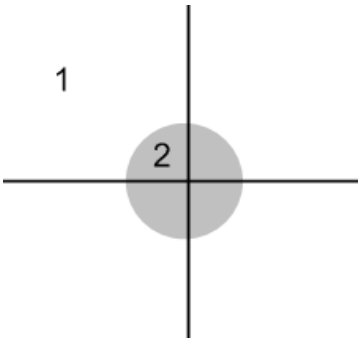
$$J_{z1m}(x,y,z) = b_{1m}(x,y)\, i_1(z)$$
$$A/m^2 \qquad 1/m^2 \quad A \qquad (4.3.2)$$

$$\int_{C_1} dx\, dy\, b_{1m}(x,y) = \lambda_1\ . \qquad (4.3.3)$$

where λ is some new constant! In (4.3.2) we have already set the sale of i_1 so it is fixed. I could calculate λ for a round wire I suppose. It is dimensionless and is a function of the conductor shape I think. Each conductor has a different λ so they don't cancel when you form W(z).

Well this was a first cut at this subject.

7. Redo this all using 2D propagator



We start with a 2D magnetostatics equation which is (1.5.4) with $\beta^2 = 0$,

$$\nabla^2_{2D}\mathbf{A}(\mathbf{x}) = -\mu_1\mathbf{J_m}(\mathbf{x}) - \mu_2\mathbf{J_c}(\mathbf{x})$$

The particular solution is obtained using the 2D propagator $g = (1/2\pi)\ln(1/R)$,

$$\mathbf{A} = (1/2\pi)\int dV' [\mu_1\mathbf{J_m}(\mathbf{x}') + \mu_2\mathbf{J_c}(\mathbf{x}')] \ln(1/R) \qquad R = |\mathbf{x}-\mathbf{x}'|$$

$$= (1/2\pi)\{\int dS' [\mu_1\mathbf{K_m}(\mathbf{x}')] \ln(1/R) + \int dV'[\mu_2\mathbf{J_c}(\mathbf{x}')] \ln(1/R)\}$$

so

$$A_z = (1/2\pi)\{\int dS' [\mu_1 K_z(\mathbf{x}')] \ln(1/R) + \int dV'[\mu_2 J_{cz}(\mathbf{x}')] \ln(1/R)\}$$

$$= -(1/2\pi)\{\int dS' [\mu_1 K_z(\mathbf{x}')] \ln(R) + \int dV'[\mu_2 J_{cz}(\mathbf{x}')] \ln(R)\}$$

$$= -(1/4\pi)\{\mu_1\int dS' [K_z(\mathbf{x}')] \ln(R^2) + \mu_2\int dV'[J_{cz}(\mathbf{x}')] \ln(R^2)\}$$

At this point we insert the uniform prescribed current J_{cz} *and* our previously computed surface current K_z,

$$J_{cz}(\mathbf{x}') = J_{cz} = I/(\pi a^2)$$

$$K_z(\mathbf{x}') = K_z = (1-\mu_2/\mu_1)(1/2\pi a) I$$

with the result that

$$A_z = -(1/4\pi)\ \{\ \mu_1(1-\mu_2/\mu_1)\ (1/2\pi a)\ I\ \int dS'\ \ln(R^2)\ + \mu_2\ I/(\pi a^2) \int dV'\ \ln(R^2)\ \}$$

$$= -[I/(4\pi^2 a^2)]\ \{\ \mu_1-\mu_2)\ (a/2)\ \int dS'\ \ln(R^2)\ + \mu_2 \int dV'\ \ln(R^2)\ \}$$

The volume integral (per unit length in the z direction, so really an area integral)

$$\int dV'\ \ln(R^2) = \int_0^a r'\ dr'\ \int_{-\pi}^{\pi} d\theta'\ \ln\ [r'^2 + r^2 - 2rr'\cos(\theta-\theta')]$$

$$= 2\ \int_0^a r'\ dr'\ \int_0^{\pi} d\theta'\ \ln\ [r'^2 + r^2 - 2rr'\cos(\theta-\theta')]$$

$$= 2\ \int_0^a r'\ dr'\ \int_0^{\pi} dx\ \ln\ [r'^2 + r^2 - 2rr'\cos x]$$

Let us define this x integral as

$$Q(r',r) \equiv \int_0^{\pi} dx\ \ln\ [r'^2 + r^2 - 2rr'\cos x]$$

so then

$$\int dV'\ \ln(R^2) = 2\ \int_0^a r'\ dr'\ Q(r',r)$$

Meanwhile, our surface integral (really a line integral) of interest is

$$\int dS'\ \ln(R^2) = \int_{-\pi}^{\pi} [a\ d\theta']\ \ln\ [r'^2 + r^2 - 2rr'\cos(\theta-\theta')]\ |^{r'=a}$$

$$= 2a\ \int_0^{\pi} dx\ \ln\ [r'^2 + r^2 - 2rr'\cos x]\ |^{r'=a} = 2a\ Q(a,r)$$

so that both integrals of interest require computation of Q(r',r), which we replicate here,

$$Q(r',r) \equiv \int_0^{\pi} dx\ \ln\ [r'^2 + r^2 - 2rr'\cos x]$$

This integral may be evaluated using GR7 *****

$$9.^8 \qquad \int_0^{\pi} \ln(a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

with $a = r'^2 + r^2$ and $b = -2rr'$ and $a^2 - b^2 = (r'^2 - r^2)^2$ so that $\sqrt{a^2 - b^2} = |\, r'^2 - r^2 \,|$ .The condition $a > |b| > 0$ is met since $(r \pm r')^2 > 0 \Rightarrow r^2 + r'^2 > \pm 2rr'$ which says $a > \pm b$ so $a > |b|$. Thus,

$$Q(r',r) = \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x)] = \pi \ln \left[\frac{(r'^2 + r^2) + |\, r'^2 - r^2 |}{2} \right] = \begin{cases} \pi \ln [r'^2] & r' > r \\ \pi \ln [r^2] & r' < r \end{cases}$$

$$= 2\pi \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases}$$

This our two integrals of interest are

$$\int dS' \ \ln(R^2) = 2a\, Q(a,r) = 4\pi a \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases}$$

$$\int dV' \ln(R^2) = 2 \int_0^a r'\, dr'\ Q(r',r) = 4\pi \int_0^a r'\, dr' \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases}$$

where we still have a dr' integral to carry out. For $r > a$ we know that $r > r'$ since r' integrates over the inside of the wire. Then

$$\int dV' \ln(R^2) = 4\pi \int_0^a r'\, dr' \ln(r) = 4\pi \ln(r) \left[a^2/2 \right] = 2\pi a^2 \ln(r) \qquad r > a$$

On the other hand, for $r < a$ we have to break the integral into two parts :

$$\int dV' \ln(R^2) = 4\pi \int_0^r r'\, dr' \ln(r) + 4\pi \int_r^a r'\, dr' \ln(r')$$

$$= 4\pi \ln(r)\, (r^2/2) + 4\pi \left[(1/2)x^2 (\ln x - 1/2) \right] \Big|_r^a$$

$$= 2\pi r^2 \ln(r) + 4\pi \left[(1/2)a^2 (\ln a - 1/2) - (1/2) r^2 (\ln r - 1/2) \right]$$

$$= 2\pi r^2 \ln(r) + 2\pi \left[a^2 (\ln a - 1/2) - r^2 (\ln r - 1/2) \right]$$

$$= 2\pi r^2 \ln(r) + 2\pi a^2 \ln a - \pi a^2 - 2\pi r^2 \ln r + \pi r^2$$

$$= 2\pi a^2 \ln a + \pi (r^2 - a^2) \qquad r < a$$

The result then is the following, where below it we repeat the earlier surface integral result,

$$\int dV' \ln(R^2) = \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases}$$

$$\int dS' \ln(R^2) = 4\pi a \begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases}$$

We can now install these integrals into our A_z expression *** to get

$$A_z = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) (a/2) \int dS' \ln(R^2) + \mu_2 \int dV' \ln(R^2) \}$$

$$= -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) (a/2) \ 4\pi a \begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases} + \mu_2 \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases} \}$$

We can then write these out for the two separate cases

$$A_z(r>a) = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) (a/2) \ 4\pi a \ln(r) + \mu_2 \ 2\pi a^2 \ \ln(r) \}$$

$$= -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) \ 2\pi a^2 \ln(r) + \mu_2 \ 2\pi a^2 \ \ln(r) \}$$

$$= -[I/(2\pi)] \{ \mu_1 - \mu_2) \ln(r) + \mu_2 \ln(r) \}$$

$$= -[I/(2\pi)] \{ \mu_1 \ln(r) \}$$

$$A_z(r<a) = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) (a/2) \ 4\pi a \ln(a) + \mu_2 (\ 2\pi a^2 \ln(a) + \pi(r^2 - a^2) \ ) \ \}$$

$$= -[I/(4\pi a^2)] \{ \mu_1 - \mu_2) \ 2a^2 \ln(a) + \mu_2 2a^2 \ln(a) + \mu_2 (r^2 - a^2) \ ) \ \}$$

$$= -[I/(4\pi a^2)] \{ \mu_1 \ 2a^2 \ln(a) + \mu_2 (r^2 - a^2) \ ) \ \}$$

$$= -[I/(2\pi)] \{ \mu_1 \ln(a) + \mu_2 (r^2 - a^2)/(2a^2) \ ) \ \}$$

At the boundary r = 0 both expressions give

$$A_z(r=a) = \ -[I/(2\pi)] \{ \mu_1 \ln(a) \}$$

The next step is to compute the magnetic field **B** = curl **A** given our function $A_z(r)$. We find

$$\mathbf{B} = \text{curl } \mathbf{A} = - \ \partial_r A_z(r) \ \hat{\boldsymbol{\theta}}$$

First, for r > a we get

$$-\partial_r A_z(r) = \partial_r [I/(2\pi) \{ \mu_1 \ln(r) \}] = (I \mu_1/2\pi) \ (1/r) \qquad r > a$$

then for r < a,

$$-\partial_r A_z(r) = I/(2\pi)\,\partial_r \{ \mu_1 \ln(a) + \mu_2(r^2-a^2)/(2a^2)) \} = (I\,\mu_2/2\pi)\,\mu_2\, r/a^2$$

Thus

$\mathbf{B} = (I\,\mu_1/2\pi)\,(1/r)\,\hat{\boldsymbol{\theta}}$ for r > a which is in the medium with μ_1

$\mathbf{B} = (I\,\mu_2/2\pi)\,(r/a^2)\,\hat{\boldsymbol{\theta}}$ for r < a which is in the medium with μ_2

Using $\mathbf{B} = \mu\mathbf{H}$ in each region we then get

$\mathbf{H} = (I/2\pi)\,(1/r)\,\hat{\boldsymbol{\theta}}$ for r > a => $H_\theta = -(I/2\pi r)$

$\mathbf{H} = (I/2\pi)\,(r/a^2)\,\hat{\boldsymbol{\theta}}$ for r < a => $H_\theta = -(Ir/2\pi a^2)$

Amazingly, these results agree with the elementary calculation results given in ***.