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Section B.6 Modification of King's Helmholtz integral solution when $\mu_1 \neq \mu_2$

(a) General Discussion

In Section 4.7 we wrote the vector potential for transmission line conductor C in this manner,

$$A_z(\mathbf{x},\omega) = \frac{1}{4\pi}\int_C \mu_2\, J_{zc}(x',y',z',\omega)\frac{e^{-j\beta R}}{R}\,dx'dy'dz' \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (4.7.2)$$

where we have made a notational change to be consistent with previous sections of Appendix B. Here we shall use μ_2 to refer to the permeability of the conductor, and μ_1 to be that of the dielectric (these are called μ_1 and μ in Section 4). The above "Helmholtz integral" is only the "particular solution" of the Helmholtz equation $(\nabla^2 + \beta_1^2)A_z = -\mu_2 J_{cz}$. When $\mu_1 \neq \mu_2$, it turns out that one must add a homogeneous solution $A_z^{(homo)}$ [that is, $(\nabla^2 + \beta_1^2)\, A_z^{(homo)} = 0$] to the Helmholtz solution shown above in order to meet boundary conditions. Appendix M.4 provides a very detailed study of just how this works for a round conductor with a uniform current distribution.

To avoid this massive complication, we limited the analysis of Chapter 4 to the case that $\mu_1 = \mu_2$. This means, for example, that Chapters 4,5,6 are only applicable for non-magnetic conductors in air.

With the reader's permission, we replicate the following comments from below (4.7.6), making a few small changes:

Comments regarding μ

This is a subtle subject and is not discussed in King's transmission line theory book.

If the dielectric and conductor have the same permeability so that $\mu_1 = \mu_2$, then there exists no "magnetic boundary" between the conductor and dielectric. The solution (4.7.2) is then smooth at this boundary, and so $A_z(x,y,z)$ "naturally" satisfies these two boundary conditions,

$$A_z(\mathbf{x}+) = A_z(\mathbf{x}-)$$
$$(1/\mu_1)\partial_n A_z(\mathbf{x}+) = (1/\mu_2)\,\partial_n A_z(\mathbf{x}-) \qquad (4.7.7) \qquad (B.6.0)$$

where $\mathbf{x}+$ is just outside the conductor surface and $\mathbf{x}-$ is just inside. The second equation here is just (1.1.46) in the case that there is no free surface current K_{free} flowing on the boundary, and indeed in our example at hand there is no such *free* surface current. Since we have assumed that $\mu_1 = \mu_2$, this second boundary condition just says $\partial_n A_z(\mathbf{x}+) = \partial_n A_z(\mathbf{x}-)$. Since there is no magnetic boundary at the

conductor/dielectric interface, the solution (4.7.2) is continuous and all its derivatives are also continuous at the boundary, since nothing special happens at that boundary. Thus, the Helmholtz integral solution provides the *whole* solution for A_z since it automatically meets both "boundary conditions" at this pseudo boundary.

If on the other hand we have $\mu_1 \neq \mu_2$, then there *is* a magnetic boundary between conductor and dielectric which we have to worry about. In this case, (4.7.2) cannot possibly satisfy the second boundary condition of (B.6.0) since, as already noted, the A_z of (4.7.2) satisfies $\partial_n A_z(\mathbf{x}+) = \partial_n A_z(\mathbf{x}+)$. Thus, in this case (4.7.2) is not the full solution for A_z. One must add a homogeneous Helmholtz equation solution to (4.7.2) in order to have a proper solution for A_z that satisfies both equations in (B.6.0).

It *turns out* that the correct total A_z solution can be generated by adding a certain fictitious surface current to J_z in (4.7.6). Since such a surface current vanishes on both sides of the boundary between μ_1 and μ_2, the Helmholtz solution due just to this surface current is in fact *a homogeneous solution* to the Helmholtz equation in both the conductor and dielectric regions, away from that boundary. It turns out moreover that the correct fictitious surface current to add is in fact the magnetization surface current J_m which is created at the boundary between $\mu_1 \neq \mu_2$. Adding this surface current is just a "trick" in order to generate the correct homogeneous adder solution so that the resulting total A_z satisfies both boundary conditions in (B.6.0). Formally speaking, the **J** appearing in (1.5.3) and then J_z in (4.7.2) should *not* include such magnetization currents since this **J** is really the **J** in Maxwell's equation curl **H** = ∂_t**D** + **J,** and this **J** does not include magnetization currents -- it includes only normal conduction currents.

Here we wish to prove the claim that adding the surface magnetization current (only!) to the conduction current does in fact make the boundary conditions work. After doing this proof, we will show in Section B.7 just how this works out in the case of a round conductor.

We stress that only the surface part of J_m gets added in. In general J_m will also have a "bulk" component in the dielectric and conductor. If we were to include this bulk component, we would not be adding a homogeneous solution to the particular solution, and we would in fact be creating a non-solution! In (M.3.4) we show the complete $\mathbf{J_m}$ for a round wire carrying a uniform current, and it does have both bulk and surface components.

(b) Statement and Proof of the J_m Theorem

The J_m Theorem. If J_{mz} represents the surface component of magnetization current density for a transmission line conductor of μ_2 with conduction current density J_{cz} , embedded in a dielectric medium of μ_1, then if we write

$$A_z(\mathbf{x}) = \frac{1}{4\pi}\int_C [\mu_2 J_{cz}(\mathbf{x}') + \mu_0 J_{mz}(x')]\frac{e^{-j\beta R}}{R}dV' \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \tag{B.6.1}$$

we obtain the full solution which meets boundary conditions (B.6.0) shown above.

Comment: Our proof below really applies in the "transmission line limit" of Sections 4.3 and 4.9 which is essentially a long wavelength and small β limit. In this limit, we can replace our various Helmholtz

propagators below with Laplace propagators. Nevertheless, we maintain the Helmholtz forms in the hope that the above theorem is valid for reasonably moderate (but not huge) β values. At very large β values the whole transmission line framework collapses anyway, transverse $A_{\mathbf{x}}$ and $A_{\mathbf{y}}$ components are no longer small, and the line picks up transverse waveguide activity.

(1) **Preliminaries.** We first quote a key result from Stakgold concerning boundary layer a:

$$u(x) = \int_{\sigma} dS_{\xi}\, a(\xi)\, E(x|\xi)$$

$$u(s) = \int_{\sigma} dS_{\xi}\, a(\xi)\, E(s|\xi) \qquad \text{(B.6.2)}$$

$$\partial_{\nu} u(x) = \int_{\sigma} dS_{\xi}\, a(\xi)\, \partial_{\nu} E(x|\xi)$$

$$\partial_{\nu} u(s) \;=\; [\partial_{\mathbf{n}} u(x)]^{\mathbf{x}\to\mathbf{s}\pm} = \int_{\sigma} dS_{\xi}\, a(\xi)\, \partial_{\nu} E(x|\xi) \;\mp\, a(s)/2 \qquad \text{// extra term !} \qquad \text{(B.6.3)}$$

This is a tricky subject and some words are certainly in order. In the Stakgold world, σ is a surface of n-1 dimensions existing in an n dimensional space. E(x|ξ) is the free-space propagator in that n dimensional space (the "fundamental solution"). The integrals shown above are over the surface σ, and ξ represents the n-1 dimensional coordinate of a point on the surface σ, while dS_{ξ} is a piece of "area" on the surface. (Stakgold does not write vectors in bold font as we do in this document.) Function a(ξ) is defined on the surface and is called a simple or monopole layer. Stakgold also deals with dipole layers (as in a cell membrane), but we don't care about them right now.

The question at hand is this: What happens as a point x away from the surface approaches the surface where it becomes point s? We are interested in the limit x → s. As shown in the first pair of equations, nothing unusual happens for the function u(x) defined as shown by the integral. One then says that u(x) is "continuous" at x = s. But something very unusual happens for the function $\partial_{\nu} u(x)$ where ∂_{ν} denotes a derivative locally normal to the surface at s. As x → s an "extra term" appears as shown above having value ∓a(s)/2. If normal ν points "out" from the surface then as one approaches from the outside (call it the + side), the extra term is -a(s)/2, but if the approach is from the inside (- side), the extra term changes sign. Here is a picture illustrating the geometry of the above equations: (n is normal at ξ , ν is normal at s)

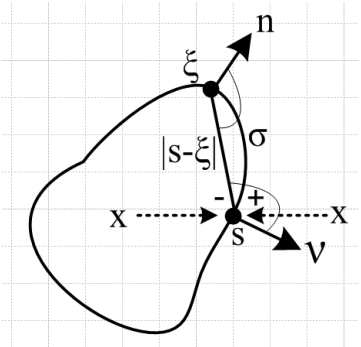

Fig B.5

The reason the extra term appears has to do with the nature of the dξ integration when ξ is very close to s which is somewhat of a singular situation since R ≡ |s-ξ| → 0. Stakgold treats surface layers in Section 6.4

of his Volume II, pages 110-120, and his treatment involves a *lot* of detail. The claims shown above appear on pages 118 and 119, though the conclusions are a bit obscured in the detail. Stakgold works in 3D with $E(x|\xi) = (1/4\pi|x-\xi|) = 1/4\pi R$ and often uses these quantities,

$$k(\mathbf{s},\xi) = \cos(\mathbf{s} \hat{-} \xi, \hat{n})/ [4\pi|\mathbf{s}-\xi|^2] \qquad \text{// cos(upper marked angle)}$$
$$k(\xi,\mathbf{s}) = \cos(\xi \hat{-} \mathbf{s}, \hat{\nu})/[4\pi|\mathbf{s}-\xi|^2] = \partial_\nu E(s|\xi) \ . \qquad \text{// cos(lower marked angle)}$$

Later in his Problem 6.18 through 6.20 Stakgold has the reader verify that the results are also valid in 2D where surface σ is then just a curve. These are the results we shall use. Although he does not state it outright, we think his results are probably valid for σ being a surface of any number of dimensions, but our only interest will be the 2D case.

(2) **Outline of Proof**

We break up (B.6.1) into these two terms:

$$A_z^{(c)}(\mathbf{x}) = \frac{1}{4\pi} \int_C [\mu_2 J_{cz}(\mathbf{x}')] \frac{e^{-j\beta R}}{R} dV' \ . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad \text{(B.6.4)}$$

$$A_z^{(m)}(\mathbf{x}) = \frac{1}{4\pi} \int_C [\mu_0 \ J_{mz}(x')] \frac{e^{-j\beta R}}{R} dV' \ . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad \text{(B.6.5)}$$

$$A_z(x) = A_z^{(c)}(\mathbf{x}) + A_z^{(m)}(\mathbf{x}) \qquad \text{two terms} \qquad \text{(B.6.6)}$$

As noted earlier, the first term is smooth at a point **s** on a conductor surface and satisfies the two boundary conditions,

$$A_z^{(c)}(\mathbf{s}+) = A_z^{(c)}(\mathbf{s}-)$$
$$\partial_n A_z^{(c)}(\mathbf{s}+) = \partial_n A_z^{(c)}(\mathbf{s}-) \ . \qquad \text{(B.6.7)}$$

For this term, which is the Helmholtz "particular" integral, we may then calculate,

$$\frac{1}{\mu_1} \partial_n A_z^{(c)}(\mathbf{s}+) \ - \ \frac{1}{\mu_2} \partial_n A_z^{(c)}(\mathbf{s}-) \ = \ + [\frac{1}{\mu_1} - \frac{1}{\mu_2}] \ \partial_n A_z^{(c)}(\mathbf{s}) \ . \qquad \text{(B.6.8)}$$

Since this is non-zero, the term $A_z^{(c)}$ on its own does not meet the required slope boundary condition at an interface between μ_1 and μ_2, and that is precisely why we need the $A_z^{(m)}$ term. Our goal is to show that

$$\frac{1}{\mu_1} \partial_n A_z^{(m)}(\mathbf{s}+) \ - \ \frac{1}{\mu_2} \partial_n A_z^{(m)}(\mathbf{s}-) \ = \ - \ [\frac{1}{\mu_1} - \frac{1}{\mu_2}] \ \partial_n A_z^{(m)}(\mathbf{s}) \ . \qquad \text{(B.6.9)}$$

so that when we add the two terms we will get

$$\frac{1}{\mu_1}\,\partial_n A_z(\mathbf{s}+) \;-\; \frac{1}{\mu_2}\,\partial_n A_z(\mathbf{s}-) \;=\; 0 \qquad \text{(B.6.10)}$$

as required by (1.1.46). The concludes our proof outline, and it remains then to demonstrate (B.6.9).

(3) Verification of (B.6.9)

We start with (B.6.5) where $\int dV'$ is over the entire transmission line conductor C,

$$A_z^{(m)}(\mathbf{x}) = \frac{1}{4\pi}\int_C dV'\,[\mu_0\; J_{mz}(\mathbf{x}')]\,\frac{e^{-j\beta R}}{R}$$

$$= \int_C dV'\,[\mu_0\, J_{mz}(\mathbf{x}')\;\; E_3(\mathbf{x}|\mathbf{x}') \qquad \text{(B.6.11)}$$

where

$$E_3(x|x') = \frac{e^{-j\beta R}}{4\pi R} \;\rightarrow\; \frac{1}{4\pi R} \quad \text{as } \beta\rightarrow 0 \quad . \qquad \text{(B.6.12)}$$

We know from Appendix C and elsewhere in this document that in the transmission line limit we can do the dz' integration in dV' and arrive at a 2D-propagator expression for the above potential, where the integral is now over the cross section area of the conductor C,

$$A_z^{(m)}(\mathbf{x}) = \int_C dA'\,[\mu_0\, J_{mz}(\mathbf{x}')]\;\; E_2(\mathbf{x}|\mathbf{x}') \qquad \text{(B.6.13)}$$

where

$$E_2(x|x') = (j/4)\,H_0^{(1)}(kR) \;\rightarrow -\;\frac{1}{2\pi}\ln(R^2)\;\; \beta\rightarrow 0 \quad . \qquad \text{(B.6.14)}$$

We are only using that portion of J_{mz} which is a surface current on the perimeter of C, so we rewrite the above as

$$A_z^{(m)}(\mathbf{x}) = \oint_C ds'\,[\mu_0\;\; K_z(\mathbf{x}')]\;\; E_2(\mathbf{x}|\mathbf{x}') \qquad \text{(B.6.15)}$$

where $K_z(\mathbf{x}')$ is the magnetization surface current (amps/m) discussed in Section B.1. Stakgold's surface integral over σ is now just a line integral around the perimeter of the conductor C cross section. Recall from (B.1.10) that the magnetization surface current is given by,

$$K_z = -\;\left(\frac{\mu_2}{\mu_0} \;-\; \frac{\mu_1}{\mu_0}\right) H_\theta \qquad \textit{(B.1.10)}$$

where H_θ is the H field tangent to the cross section surface. Inserting this K_z into (B.6.15) gives

$$A_z^{(m)}(\mathbf{x}) = \oint_C ds' \, [(\mu_1-\mu_2)\, H_\theta(\mathbf{x}')] \;\; E_2(\mathbf{x}|\mathbf{x}') \quad . \tag{B.6.16}$$

We now identify this with the first of Stakgold's equations (B.6.2) and we know we can take x→s with no surprises. If we now replace Stakgold's normal direction ν with our usual normal symbol n, we can write (B.6.3) as

$$\partial_n A_z^{(m)}(\mathbf{x}) = \oint_C ds' \, [(\mu_1-\mu_2)\, H_\theta(\mathbf{x}')] \, \partial_n E_2(\mathbf{x}|\mathbf{x}') \tag{B.6.17}$$

$$\partial_n A_z^{(m)}(\mathbf{s}\pm) = \oint_C ds' \, [(\mu_1-\mu_2)\, H_\theta(\mathbf{x}')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}') \mp [(\mu_1-\mu_2)\, H_\theta(\mathbf{s})]/2 \tag{B.6.18}$$

Now since we want to prove (B.6.9), we first evaluate its left hand side using (B.6.18) twice,

$$\frac{1}{\mu_1}\partial_n A_z^{(m)}(\mathbf{s}+) \; - \; \frac{1}{\mu_2}\partial_n A_z^{(m)}(\mathbf{s}-)$$

$$= \frac{1}{\mu_1}\{ \oint_C ds' \, [(\mu_1-\mu_2)\, H_\theta(\mathbf{x}')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}') - [(\mu_1-\mu_2)\, H_\theta(\mathbf{s})]/2 \; \}$$
$$- \frac{1}{\mu_2}\{ \oint_C ds' \, [(\mu_1-\mu_2)\, H_\theta(\mathbf{x}')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}') + [(\mu_1-\mu_2)\, H_\theta(\mathbf{s})]/2 \; \}$$

$$= (\mu_1-\mu_2)\,[\frac{1}{\mu_1}-\frac{1}{\mu_2}] \oint_C ds' \, H_\theta(\mathbf{x}') \, \partial_n E_2(\mathbf{s}|\mathbf{x}') \; - \; [\frac{1}{\mu_1}+\frac{1}{\mu_2}]\,(\mu_1-\mu_2)\, H_\theta(\mathbf{s})/2 \quad .$$

Our task of showing that (B.6.9) is true then boils down to showing that the following equation is true:

$$(\mu_1-\mu_2)\,[\frac{1}{\mu_1}-\frac{1}{\mu_2}] \oint_C ds' \, H_\theta(\mathbf{x}') \, \partial_n E_2(\mathbf{s}|\mathbf{x}') \; - \; [\frac{1}{\mu_1}+\frac{1}{\mu_2}]\,(\mu_1-\mu_2)\, H_\theta(\mathbf{s})/2 \;\; = \; - \; [\frac{1}{\mu_1}-\frac{1}{\mu_2}]\,\partial_n A_z^{(c)}(\mathbf{s}) \;\; ?$$

A question mark indicates an equation that we want to show is true, but have not yet done so.
Cancelling $(\mu_1-\mu_2)$ factors the equation in question becomes

$$[\frac{1}{\mu_1}-\frac{1}{\mu_2}] \oint_C ds' \, H_\theta(\mathbf{x}') \, \partial_n E_2(\mathbf{s}|\mathbf{x}') \; - \; [\frac{1}{\mu_1}+\frac{1}{\mu_2}]\, H_\theta(\mathbf{s})/2 \;\; = \; + \; \frac{1}{\mu_1\mu_2}\,\partial_n A_z^{(c)}(\mathbf{s}) \qquad ?$$

or

$$(\mu_2-\mu_1) \oint_C ds' \, H_\theta(\mathbf{x}') \, \partial_n E_2(\mathbf{s}|\mathbf{x}') - (1/2)(\mu_2+\mu_1)\, H_\theta(\mathbf{s}) \;\; = \;\; \partial_n A_z^{(c)}(\mathbf{s}) \qquad ? \tag{B.6.19}$$

The integral can be replaced using (B.6.18) with the s+ choice,

$$\partial_n A_z^{(m)}(\mathbf{s}+) = \oint_C ds' \, [(\mu_1-\mu_2)\, H_\theta(\mathbf{x}')] \, \partial_n E_2(\mathbf{s}|\mathbf{x}') - (1/2)[(\mu_1-\mu_2)\, H_\theta(\mathbf{s})$$

so

$$(\mu_1-\mu_2)\oint_C ds'\, H_\theta(\mathbf{x}')\,\partial_n E_2(\mathbf{s}|\mathbf{x}') = \partial_n A_z^{(m)}(\mathbf{s}+) + (1/2)(\mu_1-\mu_2)\,H_\theta(\mathbf{s}) \quad . \qquad \text{(B.6.20)}$$

Our equation of interest (B.6.19) is then

$$-\partial_n A_z^{(m)}(\mathbf{s}+) - (1/2)(\mu_1-\mu_2)\,H_\theta(\mathbf{s}) - (1/2)(\mu_2+\mu_1)\,H_\theta(\mathbf{s}) = \partial_n A_z^{(c)}(\mathbf{s}) \quad ?$$

or

$$-\partial_n A_z^{(m)}(\mathbf{s}+) - \mu_1\,H_\theta(\mathbf{s}) = \partial_n A_z^{(c)}(\mathbf{s}) \quad ?$$

or

$$-\mu_1\,H_\theta(\mathbf{s}) = \partial_n A_z^{(c)}(\mathbf{s}) + \partial_n A_z^{(m)}(\mathbf{s}+) \quad ?$$

or

$$-\mu_1\,H_\theta(s+) = \partial_n A_z(\mathbf{s}+) \quad ? \qquad \text{// using (B.6.6)} \qquad \text{(B.6.21)}$$

We now introduce a local cylindrical coordinate system in this manner relative to point **s**

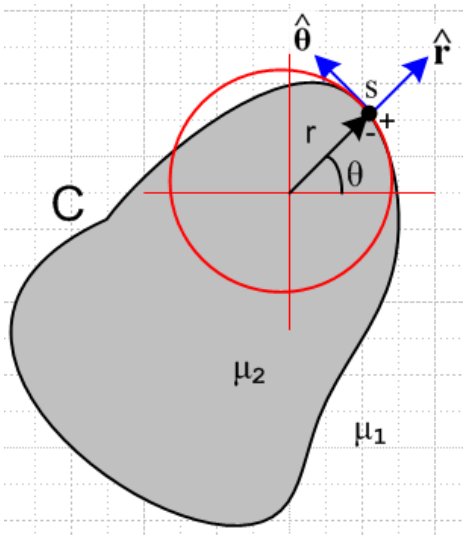

Fig B.6

Notice that $\hat{\mathbf{r}}$ is the normal vector at point **s**, so $\partial_r = \partial_n$. Then first we determine B_θ,

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \hat{\mathbf{r}}\,[\,r^{-1}\partial_\theta A_z - \partial_z A_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z A_r - \partial_r A_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rA_\theta) - r^{-1}\partial_\theta A_r\,]$$
$$= \hat{\mathbf{r}}\,[\,r^{-1}\partial_\theta A_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r A_z]$$
$$= \hat{\boldsymbol{\theta}}\,[-\partial_r A_z] \quad .$$

Here we have set $\partial_\theta A_z = 0$ according to Fact 7 of (3.8.11) which says A_z is constant on the cross section surface. The result is then,

$$B_\theta(s+) = -\partial_n A_z(s+) \quad . \qquad \text{(B.6.22)}$$

Since s+ is in the dielectric with μ_1 we then have

$$H_\theta(s+) = (1/\mu_1)\,B_\theta(s+) = -(1/\mu_1)\,\partial_r A_z(s+)$$

so

$$-\mu_1 H_\theta(s+) = \partial_r A_z(s+) \qquad \text{(B.6.23)}$$

But this last equation matches our equation in question (B.6.21), so we can then go back and erase all the question marks and we have then verified equation (B.6.9) and our proof is complete.

Comment: We noted that Stakgold's analysis is quite complicated. He uses the Laplace free-space propagators such as $E_2(x|x') = -(1/2\pi)\ln(R^2)$, but we think his analysis also applies for the Helmholtz propagators. The reason is that the Helmholtz complication does not really change the singular nature of things near R = 0. This is most obvious when comparing $e^{-j\beta R}/4\pi R$ to $1/4\pi R$. If we are wrong about this conjecture, we can regard the above theorem as proven only for small β which in fact is the transmission line limit.