Section B.7 Application of the J_m Theorem to a round wire with uniform J_z. 1
(a) The $A_z^{(c)}$ term 1
(b) The $A_z^{(m)}$ term 3
(c) Adding the two terms and checking boundary conditions 4
(d) Plots of A_z and B_θ and H_θ 5

Section B.7 Application of the J_m Theorem to a round wire with uniform J_z.

Assuming the transmission line limit of small β so $e^{-j\beta R} \approx 1$, we start with (B.6.1)

$$A_z(\mathbf{x}) = \int_C dV' \; [\mu_2 J_{cz}(\mathbf{x}') + \mu_0 \; J_{mz}(x')] \frac{1}{4\pi R} \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad \textit{(B.6.1)} \qquad \text{(B.7.1)}$$

but we go at once to the 2D solution $[\nabla^2_{2D}\mathbf{A}(\mathbf{x}) = -\mu_2 \mathbf{J_c(x)}]$ limit to get

$$-A_z(\mathbf{x}) = \int_C dA' \; [\mu_2 J_{cz}(\mathbf{x}') + \mu_0 \; J_{mz}(x')] \frac{\ln(R^2)}{4\pi} \qquad // \; \frac{1}{2\pi}\ln(1/R) = -\frac{\ln(R^2)}{4\pi}$$

The two currents are given by

$$J_{cz}(\mathbf{x}') = J_{cz} = I/(\pi a^2) \qquad // \text{ uniform} \qquad \textit{(B.1.10)}$$

$$J_{mz}(x') = K_z \, \delta(r-a) \quad \text{with} \qquad K_z = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right) H_\theta \qquad \text{(B.7.2)}$$

(a) The $A_z^{(c)}$ term

The first term in ** is then

$$-A_z^{(c)}(r,\theta) = \frac{\mu_2 I}{\pi a^2} \int_C dA' \frac{\ln(R^2)}{4\pi} = \frac{\mu_2 I}{\pi a^2} \int_0^a r' \, dr' \int_{-\pi}^{\pi} d\theta' \; \frac{\ln(R^2)}{4\pi}$$

$$= \frac{\mu_2 I}{4\pi^2 a^2} \int_0^a r' \, dr' \int_{-\pi}^{\pi} d\theta' \; \ln(r'^2 + r^2 - 2rr'\cos(\theta-\theta'))$$

$$= \frac{\mu_2 I}{2\pi^2 a^2} \int_0^a r' \, dr' \int_0^{\pi} dx \; \ln(r'^2 + r^2 - 2rr'\cos x)$$

$$= \frac{\mu_2 I}{2\pi^2 a^2} \int_0^a r' \, dr' \, Q(r',r) \qquad \text{(B.7.3)}$$

where we have defined the integral

$$Q(r',r) \equiv \int_0^{\pi} dx \ \ln [r'^2 + r^2 - 2rr' \cos x] \qquad \text{(B.7.4)}$$

The round wire geometry is shown in this drawing, where R^2 comes from the law of cosines,

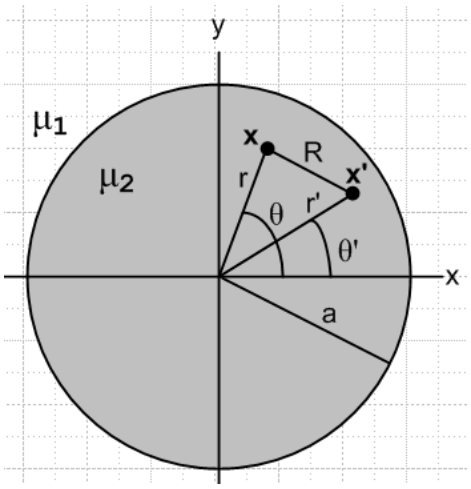

Fig B.7

The integral Q(r',r) may be evaluated using GR7 p 531 4.224,

$$9.^8 \qquad \int_0^{\pi} \ln (a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

with $a = r'^2 + r^2$ and $b = -2rr'$ and $a^2 - b^2 = (r'^2 - r^2)^2$ so that $\sqrt{a^2 - b^2} = | r'^2 - r^2 |$ .The condition $a > |b| > 0$ is met since $(r \pm r')^2 > 0 \Rightarrow r^2 + r'^2 > \pm 2rr'$ which says $a > \pm b$ so $a > |b|$. Thus,

$$Q(r',r) = \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x)] = \pi \ln \left[\frac{(r'^2 + r^2) + | r'^2 - r^2 |}{2} \right] = \begin{cases} \pi \ln [r'^2] & r' > r \\ \pi \ln [r^2] & r' < r \end{cases}$$

$$= 2\pi \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases} . \qquad \text{(B.7.5)}$$

We then have

$$-A_z^{(c)}(r,\theta) = \frac{\mu_2 I}{2\pi^2 a^2} \int_0^a dr'\, r'\, Q(r',r) = \frac{\mu_2 I}{\pi a^2} \int_0^a dr' r' \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases}$$

$$= \frac{\mu_2 I}{\pi a^2} \int_0^a dr'\, r' \{ \ln r'\, \theta(r'>r) + \ln r\, \theta(r'<r) \}$$

$$= \frac{\mu_2 I}{\pi a^2} [\, \theta(r<a) \int_r^a dr'\, r' \ln r' + \ln r \int_0^{\min(a,r)} dr'\, r' \,]$$

$$= \frac{\mu_2 I}{\pi a^2} \{ \ \theta(r<a)\, (1/2)\{a^2 \ln a - r^2 \ln r - (a^2 - r^2)/2\} + (1/2) \ln r\, [\min(a,r)]^2 \} \qquad \text{(B.7.6)}$$

where Maple says

```
int(x*ln(x),x=r..a);
```

$$\frac{1}{2}a^2\ln(a)-\frac{1}{4}a^2-\frac{1}{2}r^2\ln(r)+\frac{1}{4}r^2$$

.

We then write out $A_z^{(c)}(r)$ in its two regions

$$A_z^{(c)}(r>a) = -\frac{\mu_2 I}{\pi a^2}\ \{\ (1/2)\ a^2 \ln r\ \} = -\frac{\mu_2 I}{2\pi}\ln r \qquad \text{(B.7.7)}$$

$$A_z^{(c)}(r<a) = -\frac{\mu_2 I}{\pi a^2}\ \{\ (1/2)\{a^2\ln a - r^2\ln r - (a^2-r^2)/2\} + (1/2)\ r^2\ \ln r\} = \frac{\mu_2 I}{2\pi a^2}\ \{\ (a^2-r^2)/2 - a^2\ln a\ \}$$

(b) The $A_z^{(m)}$ term

From (B.7.1) and (B.7.2),

$$-A_z^{(m)}(\mathbf{x}) = \mu_0 \int_C dA'\ J_{mz}(x')]\ \frac{\ln(R^2)}{4\pi}$$

$$= \frac{\mu_0}{4\pi}\ \int_0^a r'\ dr' \int_{-\pi}^{\pi} d\theta'\ [-\ (\frac{\mu_2}{\mu_0}\ -\ \frac{\mu_1}{\mu_0}\ )\ H_\theta(r')\ ]\ \delta(r'-a)\ \ln(R^2)$$

$$= -\frac{a}{4\pi}\ H_\theta(a)\ (\mu_2-\mu_1) \int_{-\pi}^{\pi} d\theta'\ \ln(a^2+r^2-2ra\cos(\theta-\theta'))$$

$$= -\frac{a}{2\pi}\ H_\theta(a)\ (\mu_2-\mu_1)\ \int_0^{\pi} dx\ \ln(a^2+r^2-2ra\cos(x))$$

$$= -\frac{a}{2\pi}\ H_\theta(a)\ (\mu_2-\mu_1)\ Q(a,r)$$

$$= -\frac{a}{2\pi}\ H_\theta(a)\ (\mu_2-\mu_1)\ 2\pi \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases}$$

$$= a\ H_\theta(a)\ (\mu_1-\mu_2) \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases} \qquad \text{(B.7.8)}$$

From Ampere's law (1.1.37) we have (ignoring displacement current inside the conductor)

$$\oint \mathbf{H} \bullet d\mathbf{s}\ =\ 2\pi a H_\theta(a) = \int_S \mathbf{J} \bullet d\mathbf{S}\ \ = I \quad \Rightarrow \quad H_\theta(a) = \frac{I}{2\pi a}$$

and then

$$-A_z^{(m)}(\mathbf{x}) = \frac{I}{2\pi}\ (\mu_1-\mu_2) \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases} \ . \qquad \text{(B.7.9)}$$

Then

$$A_z^{(m)}(r>a) = \frac{I}{2\pi}(\mu_2-\mu_1)\ln r$$
$$A_z^{(m)}(r<a) = \frac{I}{2\pi}(\mu_2-\mu_1)\ln a \qquad \text{(B.7.10)}$$

(c) Adding the two terms and checking boundary conditions

$$A_z(r>a) = -\frac{\mu_2 I}{2\pi}\ln r + \frac{I}{2\pi}(\mu_2-\mu_1)\ln r = -\frac{\mu_1 I}{2\pi}\ln r$$

$$A_z(r<a) = \frac{\mu_2 I}{2\pi a^2}\{(a^2-r^2)/2 - a^2\ln a\} + \frac{I}{2\pi}(\mu_2-\mu_1)\ln a$$
$$= \frac{\mu_2 I}{2\pi}\{(a^2-r^2)/(2a^2) - \ln a\} + \frac{I}{2\pi}(\mu_2-\mu_1)\ln a$$
$$= -\frac{I}{2\pi}\{\mu_2 (r^2-a^2)/(2a^2) + \mu_1 \ln a\}$$

Here then are the finally results for the potential $A_z = A_z^{(c)} + A_z^{(m)}$,

$$A_z(r>a) = -\frac{I}{2\pi}\mu_1 \ln r$$
$$A_z(r<a) = -\frac{I}{2\pi}\{\mu_1 \ln a + \mu_2 \frac{r^2-a^2}{2a^2}\} \qquad \text{(B.7.11)}$$

As a check, we calculate the **B and H fields** implied by these potentials

$$\mathbf{B} = \text{curl } \mathbf{A} = \hat{\mathbf{r}}[\, r^{-1}\partial_\theta A_z - \partial_z A_\theta] + \hat{\boldsymbol{\theta}}[\partial_z A_r - \partial_r A_z] + \hat{\mathbf{z}}[\, r^{-1}\partial_r(rA_\theta) - r^{-1}\partial_\theta A_r\,]$$

$$= \hat{\boldsymbol{\theta}}[-\partial_r A_z] \qquad \Rightarrow \qquad B_\theta = -\partial_r A_z(r)$$

Then

$$B_\theta(r>a) = \frac{I}{2\pi}\mu_1 (1/r)$$
$$B_\theta(r<a) = \frac{I}{2\pi}\mu_2 \frac{2r}{2a^2} = \frac{I}{2\pi}\mu_2 (r/a^2) \qquad \text{(B.7.12)}$$

so the H fields are then

$$H_\theta(r>a) = \frac{I}{2\pi}(1/r)$$
$$H_\theta(r<a) = \frac{I}{2\pi}(r/a^2) \qquad \text{(B.7.13)}$$

which agrees with Ampere's Law applied in these two regions:

$$2\pi r\, H_\theta(r>a) = I \quad \Rightarrow \quad H_\theta(r>a) = \frac{I}{2\pi}(1/r)$$

$$2\pi r\, H_\theta(r<a) = I(\pi r^2/\pi a^2) \quad \Rightarrow \quad H_\theta(r>a) = \frac{I}{2\pi}(r/a^2)\ . \qquad \text{(B.7.14)}$$

Next, we check our two **boundary conditions** required by (B.6.0) :

value at r = a:

$$A_z(r>a)|^{r=a} - A_z(r<a)|^{r=a} = -\frac{I}{2\pi}\,\mu_1 \ln a - \left(\frac{I}{2\pi}\left[\mu_2 \frac{a^2-a^2}{2a^2} - \mu_1 \ln a\right]\right) = 0 \qquad \text{OK}$$

slope at r = a: [$\partial_r A_z = -B_\theta$ so use(B.7.12)]

$$\partial_r A_z(r>a)|^{r=a} = -\frac{I}{2\pi}\mu_1 (1/a)$$

$$\partial_r A_z(r<a)|^{r=a} = -\frac{I}{2\pi}\mu_2 (a/a^2) = -\frac{I}{2\pi}\mu_2 (1/a)$$

$$\frac{1}{\mu_1}\partial_r A_z(r>a)|^{r=a} - \frac{1}{\mu_2}\partial_r A_z(r<a)|^{r=a} = -\frac{I}{2\pi}(1/a) - \left[-\frac{I}{2\pi}(1/a)\right] = 0 \qquad \text{OK}$$

Thus we have shown for the round conductor with uniform J_z that "the J_m theorem" works. By adding the bogus surface current term, we generate the correct homogeneous solution which when added to the Helmholtz integral provides the correct total solution which meets both boundary conditions.

(d) Plots of A_z and B_θ and H_θ

Maple provides plots of A_z from (B.7.11), B_θ from (B.7.12) and H_θ from (B.7.13) for this round wire situation. Parameters are set to I = 1, a = 2, $\mu_1 = 2$ (dielectric), $\mu_2 = 3$ (wire).

```
restart;alias(I=I);
Az := -(I/2*Pi)*piecewise(r>a,mu1*log(r),r<=a,mu1*log(a) + mu2*(r^2-a^2)/(2*a^2));
```

$$Az := -\frac{1}{2} I \pi \left(\begin{cases} \mu 1 \ln(r) & a < r \\ \mu 1 \ln(a) + \frac{1}{2} \frac{\mu 2 \left(r^2 - a^2\right)}{a^2} & r \le a \end{cases} \right)$$

```
B := (I/2*Pi)*piecewise(r>a,mu1/r,r<=a,mu2*r/(a^2));
```

$$B := \frac{1}{2} I \pi \left(\begin{cases} \frac{\mu 1}{r} & a < r \\ \frac{\mu 2 \, r}{a^2} & r \le a \end{cases} \right)$$

```
H := (I/2*Pi)*piecewise(r>a,1/r,r<=a,r/(a^2));
```

$$H := \frac{1}{2} I \pi \left(\begin{cases} \frac{1}{r} & a < r \\ \frac{r}{a^2} & r \le a \end{cases} \right)$$

```
I := 1: a := 2: mu1 := 2: mu2 := 3:
plot([Az,B,H],r=0..8, color=[black,red,blue]);
```

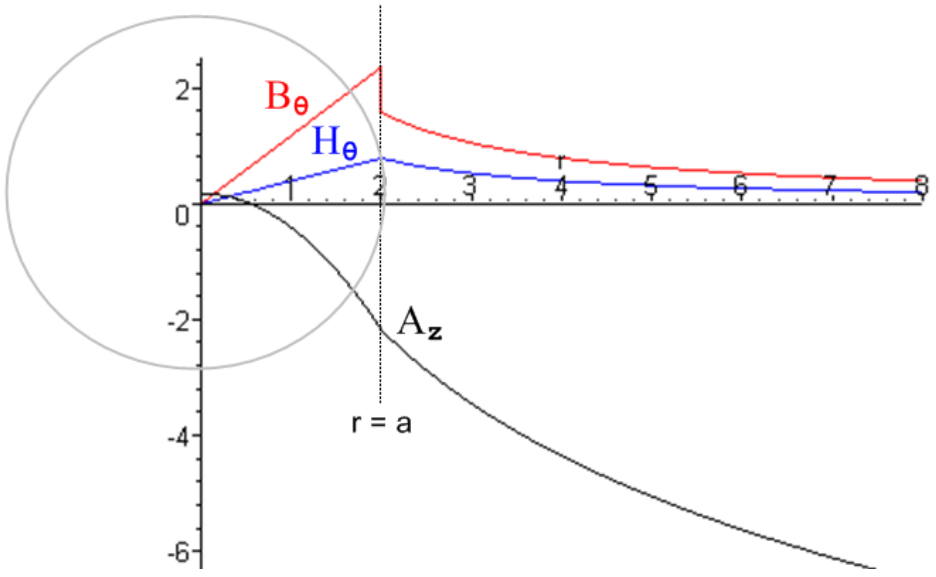

Fig B.8

A_z wanders down as ~ -ln(r) for large r; B_θ jumps down at r = a while H_θ is continuous there.