**Old Sections B.6 and B.7** PhL 1.17.14

**B.6 An Acrobatic Exercise: Compute H from J and $J_m$ via the vector potential A.**

For the round wire with a uniform $J_z$ we are aware of both the volume current $J_z$ and the surface current $K_z$. In this exercise, we insert *both* these currents into our formula for the vector potential **A**, we do the required integrals, and at the end we compute **B** = curl **A** and finally use **H** = **B**/μ. Hopefully in the end we shall recover once again the results for $H_\theta$ shown above. Here again is the picture of interest:

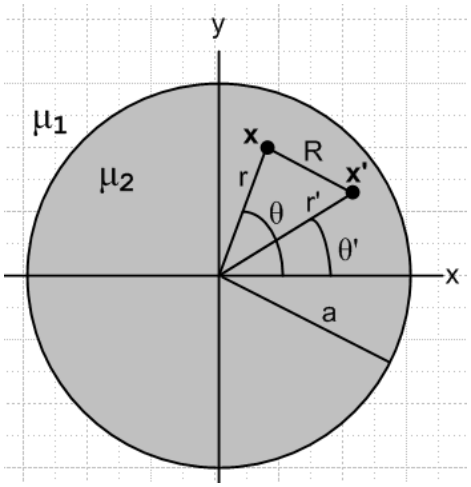


Fig B.5

Start with a 2D magnetostatics equation which is (1.5.4) with $\beta^2 = 0$,

$$\nabla^2_{2D}\mathbf{A}(\mathbf{x}) = -\mu_1\mathbf{J_m}(\mathbf{x}) - \mu_2\mathbf{J_c}(\mathbf{x}) \; . \qquad \text{(B.6.1)}$$

This 2D approximation is appropriate for a transmission line conductor in the "transmission line limit".

The particular solution from (I.1.8) is

$$\mathbf{A} = (1/2\pi) \int dV' \, [\mu_1\mathbf{J_m}(\mathbf{x'}) + \mu_2\mathbf{J_c}(\mathbf{x'})] \ln(1/R) \qquad R = |\mathbf{x}-\mathbf{x'}|$$

$$= (1/2\pi) \{ \int dS' \, [\mu_1\mathbf{K_m}(\mathbf{x'})] \ln(1/R) + \int dV' [\, \mu_2\mathbf{J_c}(\mathbf{x'})] \ln(1/R) \}$$

so

$$A_z = (1/2\pi) \{ \int dS' \, [\mu_1 K_z(\mathbf{x'})] \ln(1/R) + \int dV' [\, \mu_2 J_{cz}(\mathbf{x'})] \ln(1/R) \}$$

$$= -(1/2\pi) \{ \int dS' \, [\mu_1 K_z(\mathbf{x'})] \ln(R) + \int dV' [\, \mu_2 J_{cz}(\mathbf{x'})] \ln(R) \}$$

$$= -(1/4\pi) \{ \mu_1 \int dS' \, [K_z(\mathbf{x'})] \ln(R^2) + \mu_2 \int dV' [J_{cz}(\mathbf{x'})] \ln(R^2) \} \; . \qquad \text{(B.6.1)}$$

Here $\int dV'$ represents an integral over the "volume" of the wire slice shown in the figure (which is really a disk) and $\int dS'$ represents an integral over the "surface" of the wire slice (which is really a circle).

We now insert the uniform prescribed current $J_{cz}$ *and* our previously computed surface current $K_z$,

$$J_{cz}(\mathbf{x}') = J_{cz} = I/(\pi a^2) \tag{B.6.2}$$

$$K_z(\mathbf{x}') = K_z = (1-\mu_2/\mu_1)\,(1/2\pi a)\,I \tag{B.6.3}$$

with the result that

$$A_z = -(1/4\pi)\,\{\,\mu_1(1-\mu_2/\mu_1)\,(1/2\pi a)\,I \int dS' \ln(R^2) + \mu_2\, I/(\pi a^2)\int dV' \ln(R^2)\,\}$$

$$= -[I/(4\pi^2 a^2)]\,\{\,(\mu_1-\mu_2)\,(a/2) \int dS' \ln(R^2) + \mu_2 \int dV' \ln(R^2)\,\}\,. \tag{B.6.4}$$

The volume integral (per unit length in the z direction, so really an area integral as just noted) is,

$$\int dV' \ln(R^2) = \int_0^a r'\,dr' \int_{-\pi}^{\pi} d\theta' \ln[r'^2 + r^2 - 2rr'\cos(\theta-\theta')]$$

$$= 2\int_0^a r'\,dr' \int_0^{\pi} d\theta' \ln[r'^2 + r^2 - 2rr'\cos(\theta-\theta')]$$

$$= 2\int_0^a r'\,dr' \int_0^{\pi} dx \ln[r'^2 + r^2 - 2rr'\cos x]\,. \tag{B.6.5}$$

Define the x integral as

$$Q(r',r) \equiv \int_0^{\pi} dx \ln[r'^2 + r^2 - 2rr'\cos x] \tag{B.6.6}$$

so then

$$\int dV' \ln(R^2) = 2\int_0^a r'\,dr'\; Q(r',r)\,. \tag{B.6.7}$$

Meanwhile, our surface integral of interest (really a line integral) is

$$\int dS' \ln(R^2) = \int_{-\pi}^{\pi} [a\,d\theta'] \ln[r'^2 + r^2 - 2rr'\cos(\theta-\theta')]\,|_{r'=a}$$

$$= 2a \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x] \Big|_{r'=a} = 2a\, Q(a,r) \qquad \text{(B.6.8)}$$

so that both integrals of interest require computation of Q(r',r), which we replicate here,

$$Q(r',r) \equiv \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x] \ . \qquad \textit{(B.6.6)}$$

This integral may be evaluated using GR7 p 531 4.224,

$$9.^8 \qquad \int_0^{\pi} \ln (a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

with $a = r'^2 + r^2$ and $b = -2rr'$ and $a^2 - b^2 = (r'^2 - r^2)^2$ so that $\sqrt{a^2 - b^2} = |\, r'^2 - r^2 \,|$ .The condition $a > |b| > 0$ is met since $(r \pm r')^2 > 0 \Rightarrow r^2 + r'^2 > \pm 2rr'$ which says $a > \pm b$ so $a > |b|$. Thus,

$$Q(r',r) = \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x)] = \pi \ln \left[ \frac{(r'^2 + r^2) + |\, r'^2 - r^2 \,|}{2} \right] = \begin{cases} \pi \ln [r'^2] & r' > r \\ \pi \ln [r^2] & r' < r \end{cases}$$

$$= 2\pi \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases} \ . \qquad \text{(B.6.9)}$$

This our two integrals of interest are

$$\int dS' \ \ln(R^2) = 2a\, Q(a,r) = 4\pi a \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases} \qquad \text{(B.6.10)}$$

$$\int dV' \ln(R^2) = 2 \int_0^{a} r'\, dr' \ Q(r',r) = 4\pi \int_0^{a} r'\, dr' \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases} , \qquad \text{(B.6.11)}$$

where we still have a dr' integral to carry out for the $\int dV'$ case. For r > a we know that r > r' since r' integrates over the inside of the wire. Then

$$\int dV' \ln(R^2) = 4\pi \int_0^{a} r'\, dr' \ln(r) = 4\pi \ln(r) \left[ a^2/2 \right] = 2\pi a^2 \ln(r) \qquad r > a \qquad \text{(B.6.12)}$$

On the other hand, for r < a we have to break the integral into two parts :

$$\int dV' \ln(R^2) = 4\pi \int_0^{r} r'\, dr' \ln(r) + 4\pi \int_r^{a} r'\, dr' \ln(r')$$

$$= 4\pi \ln(r) \, (r^2/2) + 4\pi \left[ (1/2)x^2 (\ln x - 1/2) \right] \Big|_r^a$$

$$= 2\pi r^2 \ln(r) + 4\pi \left[ (1/2)a^2 (\ln a - 1/2) - (1/2)r^2 (\ln r - 1/2) \right]$$

$$= 2\pi r^2 \ln(r) + 2\pi [a^2(\ln a - 1/2) - r^2(\ln r - 1/2)]$$

$$= 2\pi r^2 \ln(r) + 2\pi a^2 \ln a - \pi a^2 - 2\pi r^2 \ln r + \pi r^2$$

$$= 2\pi a^2 \ln a + \pi(r^2 - a^2) \quad . \qquad r < a \qquad \text{(B.6.13)}$$

The result then is the following, where below it we repeat the earlier surface integral result,

$$\int dV' \ln(R^2) = \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases} \qquad \text{(B.6.14)}$$

$$\int dS' \ln(R^2) = 4\pi a \begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases} \quad . \qquad \text{(B.6.15)}$$

Now install these integrals into the $A_z$ expression (B.6.4) to get

$$A_z = -[I/(4\pi^2 a^2)] \{(\mu_1 - \mu_2)(a/2) \int dS' \ln(R^2) + \mu_2 \int dV' \ln(R^2)\} \qquad \textit{(B.6.4)}$$

$$= -[I/(4\pi^2 a^2)] \{(\mu_1 - \mu_2)(a/2) 4\pi a \begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases} + \mu_2 \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases} \} \qquad \text{(B.6.16)}$$

Then write these out for the two separate regions :

$$A_z(r>a) = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2)(a/2) \; 4\pi a \ln(r) + \mu_2 \, 2\pi a^2 \ln(r) \}$$

$$= -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) \, 2\pi a^2 \ln(r) + \mu_2 \, 2\pi a^2 \ln(r) \}$$

$$= -[I/(2\pi)] \{ \mu_1 - \mu_2) \ln(r) + \mu_2 \ln(r) \}$$

$$= -[I/(2\pi)] \{ \mu_1 \ln(r) \} \qquad \text{(B.6.17)}$$

$$A_z(r<a) = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2)(a/2) \, 4\pi a \ln(a) + \mu_2( 2\pi a^2 \ln(a) + \pi(r^2 - a^2) ) \}$$

$$= -[I/(4\pi a^2)] \{ \mu_1 - \mu_2) \, 2a^2 \ln(a) + \mu_2 2a^2 \ln(a) + \mu_2(r^2 - a^2) ) \}$$

$$= -[I/(4\pi a^2)] \{ \mu_1 \, 2a^2 \ln(a) + \mu_2(r^2 - a^2) ) \}$$

$$= -[I/(2\pi)] \{ \mu_1 \ln(a) + \mu_2(r^2 - a^2)/(2a^2) ) \} \qquad \text{(B.6.18)}$$

At the boundary r = 0 both expressions give

$$A_z(r=a) = -[I/(2\pi)] \{ \mu_1 \ln(a) \} \; . \qquad \text{(B.6.19)}$$

The next step is to compute the magnetic field **B** = curl **A** given our function $A_z(r)$. We find from the usual cylindrical coordinates curl formula (B.5.2),

$$\mathbf{B} = \text{curl } \mathbf{A} = -\ \partial_r A_z(r)\ \hat{\boldsymbol{\theta}} \quad . \tag{B.6.20}$$

First, for r > a one finds,

$$-\partial_r A_z(r) = \partial_r[I/(2\pi)\ \{\ \mu_1 \ln(r)\ \}] = (I\ \mu_1/2\pi)\ (1/r) \qquad r > a \tag{B.6.21}$$

and then for r < a,

$$-\partial_r A_z(r) = I/(2\pi)\ \partial_r\ \{\ \mu_1\ \ln(a) + \mu_2(r^2-a^2)/(2a^2)\ )\ \ \} = (I\ \mu_2/2\pi)\ \mu_2\ r/a^2 \tag{B.6.22}$$

Thus

$$\begin{aligned} &\mathbf{B} = (I\ \mu_1/2\pi)\ (1/r)\ \ \hat{\boldsymbol{\theta}} \qquad \text{for } r > a \text{ which is in the dielectric medium with } \mu_1 \\ &\mathbf{B} = (I\ \mu_2/2\pi)\ (r/a^2)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r < a \text{ which is in the conductor medium with } \mu_2 \quad . \end{aligned} \tag{B.6.23}$$

Using **B** = μ**H** in each region we then get

$$\begin{aligned} &\mathbf{H} = (I/2\pi)\ (1/r)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r > a \qquad \Rightarrow \qquad H_\theta = (I/2\pi r) \\ &\mathbf{H} = (I/2\pi)\ (r/a^2)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r < a \qquad \Rightarrow \qquad H_\theta = (Ir/2\pi a^2) \end{aligned} \tag{B.6.24}$$

Amazingly, these results agree with the elementary calculation results given in (B.4.1) and (B.4.2). The important point here is that these results only come out right when both the volume and surface currents on the round wire are included in the calculation. The elementary Ampere's Law calculation of $H_\theta$ is simple because in the Maxwell equation curl **H** = **J** (Ampere's Law), the field **H** does not "see" the surface magnetization current, so one can forget about it.

Maple provides plots of $A_z$ from (B.6.17,18), $B_\theta$ from (B.6.23) and $H_\theta$ from (B.6.24) for this round wire situation. Parameters are set to I = 1, a = 2, $\mu_1 = 2$, $\mu_2 = 3$.

```
restart;alias(I=I);
Az := -(I/2*Pi)*piecewise(r>a,mu1*log(r),r<=a,mu1*log(a) + mu2*(r^2-a^2)/(2*a^2));
```

$$Az := -\frac{1}{2} I \pi \left( \begin{cases} \mu 1 \ln(r) & a < r \\ \mu 1 \ln(a) + \frac{1}{2} \frac{\mu 2 \left(r^2 - a^2\right)}{a^2} & r \le a \end{cases} \right)$$

```
B := (I/2*Pi)*piecewise(r>a,mu1/r,r<=a,mu2*r/(a^2));
```

$$B := \frac{1}{2} I \pi \left( \begin{cases} \frac{\mu 1}{r} & a < r \\ \frac{\mu 2 \, r}{a^2} & r \le a \end{cases} \right)$$

```
H := (I/2*Pi)*piecewise(r>a,1/r,r<=a,r/(a^2));
```

$$H := \frac{1}{2} I \pi \left( \begin{cases} \frac{1}{r} & a < r \\ \frac{r}{a^2} & r \le a \end{cases} \right)$$

```
I := 1: a := 2: mu1 := 2: mu2 := 3:
plot([Az,B,H],r=0..8, color=[black,red,blue]);
```

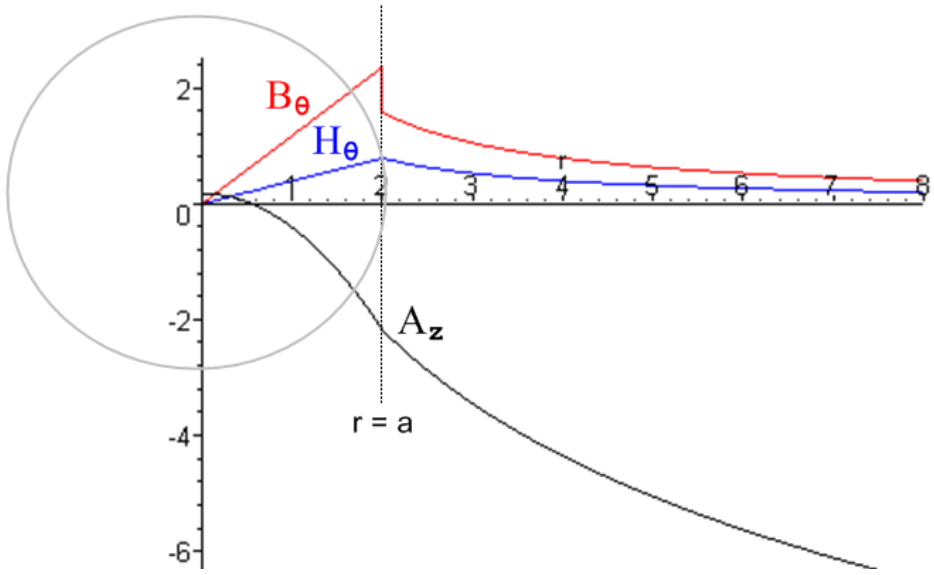


Fig B.6

$A_z$ wanders down as ~ -ln(r) for large r, $B_\theta$ jumps at r = a while $H_\theta$ is continuous there.

**B.7 Reader Exercise: Repeat the above calculation of A using a 3D analysis**

This is a guided exercise with waypoints. The starting point is this:

$\nabla^2 \mathbf{A}(\mathbf{x}) = -\mu_1 \mathbf{J}_m(\mathbf{x}) - \mu_2 \mathbf{J}_c(\mathbf{x})$ // 3D wave equation from (1.5.4) with $\beta = 0$

$\mathbf{A}(\mathbf{x}) = (1/4\pi) \int dV' \, [\mu_1 \mathbf{J}_m(\mathbf{x}') + \mu_2 \mathbf{J}_c(\mathbf{x}')] \, (1/\mathcal{R})$ $\mathcal{R} = |\mathbf{x} - \mathbf{x}'|$ . // according to (H.1.8)

(a) Insert $\mathbf{J_m}$ and $\mathbf{J_c}$ from (B.6.2) and (B.6.3) to get

$$A_z(\mathbf{x}) = (1/4\pi)\,(1/2\pi a)\,I\ \{\,(\mu_1-\mu_2)\int dS'\,(1/\mathcal{R}) + (2/a)\,\mu_2 \int dV'\,(1/\mathcal{R})\ \}$$

where now we have true surface and volume integrals. Notice that we are using currents which do not vay in z, so here we are making the same "transmission line limit" approximation made in the 2D analysis.

(b) Show these integrals may be written:

$$\int dS'\,(1/R) = \int (a\,d\theta')\ \int_{-\infty}^{\infty} dz'\,(1/\mathcal{R}) = a\ \int_{-\pi}^{\pi} d\theta'\ \int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\ )$$

$$\int dV'\,(1/R) = \int (r'd\theta')\ \int dr' \int_{-\infty}^{\infty} dz'\,(1/\mathcal{R}) = \int_{0}^{a} r'dr'\ \int_{-\pi}^{\pi} d\theta'\ \int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\ )$$

where $s^2 = (x-x')^2 + (y-y')^2 = R^2$ of the 2D problem.

(c) Install a a large cutoff $\Lambda$ to get

$$\int_{-\infty}^{\infty} dz'\,(1/\sqrt{s^2 + z'^2}\ ) \rightarrow \int_{-\Lambda/2}^{\Lambda/2} dz'\,(1/\sqrt{s^2 + z'^2}\ ) = -2\ln(s/\Lambda) = -\ln(R^2/\Lambda^2)\,.$$

(d) Show then that

$$A_z(\mathbf{x}) = = -(I/4\pi^2a^2)\ \{\,(\mu_1-\mu_2)\,[\ a^2\,Q(a,r) - \pi a^2 \ln(\Lambda^2)]$$
$$+\ 2\,\mu_2\,[\ \int_0^a r'\,dr'\ \ Q(r',r)\ -\ \ln(\Lambda^2)\,\pi\,a^2]\ \ \}$$

where Q(r',r) is the integral defined in (B.6.6).

(e) Finally, show that the $A_z(x)$ obtained here is the same as that obtained in the 2D analysis apart from the appearance here of the following extra terms

$$+\,(I/4\pi)\ \ln(\Lambda^2)\,(\mu_1+\mu_2)\,.$$

(f) Compare the 2D and 3D methods with an interpretion of these extra terms which of course have no effect on **B** = curl **A**. Physically why does the infinite term appear in the 3D analysis and not in the 2D analysis?