Problems in Appendix B PhL 10.14.14

I have found some new forms of Maxwell's implied equations.

1. Allocation of E and J in the curl H equation

Consider the Maxwell equation normally written as $\xi \equiv \varepsilon - j\sigma/\omega = \varepsilon + \sigma/j\omega$

$$\text{curl } \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} = j\omega \mathbf{D} + \mathbf{J} = j\omega\varepsilon \mathbf{E} + \mathbf{J}$$

Using Ohm's Law, we can write the RHS three different ways

1 $j\omega\varepsilon \mathbf{E} + \mathbf{J}$

2 $j\omega\varepsilon(\mathbf{J}/\sigma) + \mathbf{J} = [\ j\omega(\varepsilon/\sigma) + 1]\ \mathbf{J} = (\ j\omega/\sigma)\ [\ \varepsilon + \sigma/j\omega\ ] = j\omega(\xi/\sigma)\ \mathbf{J}$

3 $j\omega(\xi/\sigma)\sigma \mathbf{E} = j\omega\xi\ \mathbf{E}$

So here are our three ways to write the curl H equation:

1 $\text{curl } \mathbf{H} = j\omega\varepsilon \mathbf{E} + \mathbf{J}$

2 $\text{curl } \mathbf{H} = j\omega(\xi/\sigma)\ \mathbf{J} =$

3 $\text{curl } \mathbf{H} = j\omega\xi\ \mathbf{E}$

I have checked with Maple to make sure the three RHS's are the same.

2. Forms of the Helmholtz equation

Consider this identity,

$$\text{curl curl } \mathbf{H} = \text{grad div} \mathbf{H} - \nabla^2 \mathbf{H} = -\nabla^2 \mathbf{H}$$

which then says

$$\nabla^2 \mathbf{H} = -\text{ curl curl } \mathbf{H}$$

This equation can then be written three ways using the three forms shown above. I will do them one at a time:

1 $\nabla^2 \mathbf{H} = -\text{ curl } [j\omega\varepsilon \mathbf{E} + \mathbf{J}] = -j\omega\varepsilon \text{ curl } \mathbf{E} - \text{curl } \mathbf{J} = -j\omega\varepsilon\ [-j\omega \mathbf{B}\ ] - \text{curl } \mathbf{J} = -\omega^2\varepsilon\mu \mathbf{H} - \text{curl } \mathbf{J}$

so that we end up with $(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{H} = -\text{curl } \mathbf{J}$

or $(\nabla^2 + k^2\ )\mathbf{H} = -\text{curl } \mathbf{J}$ with $k^2 = \omega^2\varepsilon\mu$

This form appears in lines doc here:

$(\nabla^2+k_3^2)\mathbf{E} = \mu_3 j\omega \mathbf{J}_3$ $\quad (\nabla^2+k_3^2)\mathbf{B} = -\mu_3 \text{ curl } \mathbf{J}_3$ $\quad$ // region 3 $\quad k_i^2 = \omega^2\mu_i\varepsilon_i$ $\quad$ (1.5.26)

2 $\quad \nabla^2\mathbf{H} = -\text{ curl } [j\omega(\xi/\sigma)\,\mathbf{J}] = \; -j\omega(\xi/\sigma) \text{ curl } \mathbf{J}$ $\quad \xi \equiv \varepsilon - j\sigma/\omega = \varepsilon + \sigma/j\omega$

This form does not appear anywhere in lines doc as far as I know. The ratio ξ/σ never appears.

3 $\quad \nabla^2\mathbf{H} = -\text{ curl}[j\omega\xi\,\mathbf{E}] = -j\omega\xi \text{ curl } \mathbf{E} = -j\omega\xi\,[-j\omega\mathbf{B}] = -\omega^2\xi\,\mathbf{B} = -\omega^2\xi\mu\,\mathbf{H}$

so that we end up with $\quad (\nabla^2 + \omega^2\xi\mu)\mathbf{H} = \mathbf{0}$

$(\nabla^2 + \beta^2)\mathbf{H} = \mathbf{0}$

This form appears in lines doc here.

$(\nabla^2+\beta_3^2)\mathbf{E} = 0$ $\quad (\nabla^2+\beta_3^2)\mathbf{B} = 0$ $\quad$ // region 3 $\quad \beta_i^2 = \omega^2\mu_i\xi_i$ $\quad$ (1.5.27)

Here then are the three different "Helmholtz equations" which are the same:

1 $\quad (\nabla^2 + k^2)\mathbf{H} = -\text{curl } \mathbf{J}$ $\quad$ with $\quad k^2 = \omega^2\varepsilon\mu$

2 $\quad \nabla^2\mathbf{H} = -j\omega(\xi/\sigma) \text{ curl } \mathbf{J}$ $\quad \xi \equiv \varepsilon + \sigma/j\omega \quad j\omega(\xi/\sigma) = j\omega([\varepsilon + \sigma/j\omega]/\sigma) = j\omega(\varepsilon/\sigma) + 1$

$= -[1 + j\omega(\varepsilon/\sigma)] \text{ curl } \mathbf{J}$

3 $\quad (\nabla^2 + \beta^2)\mathbf{H} = 0$ $\quad \beta^2 = \omega^2\xi\mu$

3. Copper conductor case

Question: For copper, what is the size of $j\omega(\varepsilon/\sigma)$ compared to 1 ? That is, when is $\omega\varepsilon << \sigma$? I answer this question below (2.2.2) and conclude that $\omega\varepsilon << \sigma$ as long as $f << 10^{18}$ Hz.

Therefore, in Form 2 above, we really can neglect the $j\omega(\varepsilon/\sigma)$ relative to the 1. Now go back to

$\text{curl } \mathbf{H} = \partial_t\mathbf{D} + \mathbf{J} = j\omega\mathbf{D} + \mathbf{J} = j\omega\varepsilon\mathbf{E} + \mathbf{J} = [\, j\omega(\varepsilon/\sigma) + 1]\,\mathbf{J}$

The fact that $\omega\varepsilon << \sigma$ in this range is the same as saying you can neglect the displacement current in copper relative to the conduction current. This says

$\xi \equiv \varepsilon + \sigma/j\omega \approx \sigma/j\omega$ $\quad$ all the way up to 10^{18} Hz

Then

$j\omega(\xi/\sigma) = j\omega(1/\sigma) * \sigma/j\omega = 1$ $\quad \beta^2 = \omega^2\xi\mu = \omega^2(\sigma/j\omega)\mu = -j\mu\sigma\omega$ as usual

Then the three forms shown above become, in the limit $\omega\varepsilon << \sigma$,

1 $(\nabla^2 + k^2)\mathbf{H} = -\text{curl } \mathbf{J}$ with $k^2 = \omega^2\varepsilon\mu$

2 $\nabla^2\mathbf{H} = -\text{ curl } \mathbf{J}$

3 $(\nabla^2 + \beta^2)\mathbf{H} = 0$ $\beta^2 = -j\mu\sigma\omega$

How do I explain how these can all be true at the same time? Are they if $\omega\varepsilon << \sigma$?

First consider

$\beta^2 = -j\mu\sigma\omega$ $k^2 = \omega^2\varepsilon\mu$ $| k^2/\beta^2 | = \omega\varepsilon/\sigma << 1$

so we do know that $k << \beta$ in magnitude.

Comparing 1 and 2, I would like to show that

$| k^2 \mathbf{H} | << | \nabla^2 \mathbf{H}|$

Using 3 this is the same as

$| k^2 \mathbf{H} | << | \beta^2 \mathbf{H}|$

and this is at once true since $k << \beta$. So that explains how 1 and 2 can both be true.

4. NOW take a look at Appendix B.2 (a)

I have added the above data to the start of Section B.2. I can then take two different paths. If I assume the e^{-jkz} ansatz, then $\partial_z^2 = -k^2$ and

$\nabla^2\mathbf{H} = -\text{curl } \mathbf{J}$ // copper for $f << 10^{18}$ Hz

becomes

$(\nabla^2_{2D} - k^2)\ \mathbf{H} = -\text{curl } \mathbf{J}$

but I don't know how to invert this equation very well. But I do know how to invert $-\nabla^2\mathbf{H} = \text{curl } \mathbf{J}$ from Appendix H.1.8

$-\nabla^2 f(\mathbf{x}) = s(\mathbf{x})$ => $f(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, s(\mathbf{x}')$ + homogeneous solutions

The Poisson Equation (H.1.8)

so then

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, \text{curl}' \, \mathbf{J}(\mathbf{x}')$$

I think this result is good for all ω, whereas the result I wrote

$$\mathbf{H}(x,y) = \int d^2\mathbf{x}' \, [\, \frac{1}{2\pi} \ln(1/R) \,] \, \text{curl}' \, \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \text{(B.2.5)}$$

makes the assumption that $\mathbf{J} = J_z(x,y)\hat{\mathbf{z}}$ so $e^{-jkz} \approx 1$ which means low ω. So maybe I will make the replacement above. Let's continue and see what comes next in Appendix B before doing this.

5. Take a look at Appendix B.2 (b)

Here the starting equation is

$$(\nabla^2 + \beta^2)\mathbf{A}(\mathbf{x}) = -\mu \, \mathbf{J}(\mathbf{x}) \; . \qquad \text{all of region R} \qquad \textit{(1.5.4)}$$

STOP! Who says the above equation is true inside the conductor?? I think the correct equation to be using here is this

$$(\nabla^2 - \mu\varepsilon \, \partial_t^2)\mathbf{A} = -\mu\mathbf{J} \qquad [= \text{Jackson (6.16)}] \qquad \text{(1.3.5)}$$

which uses the Lorenz gauge. I don't think the King gauge makes sense for a single conductor? Well, I show that with the large σ assumption, I do get this result

$$(\nabla^2 - \mu_1\varepsilon_1 \, \partial_t^2 - \mu_1\sigma_1\partial_t \,) \, \mathbf{A} = -\mu_3\mathbf{J}_3 \; . \qquad \text{region 3} \qquad \text{(1.3.21)}$$

and I guess I can consider region 3 as my conductor. But in the King gauge, this yields,

$$(\nabla^2 + \beta_d^2)\mathbf{A}(\mathbf{x}) = -\mu \, \mathbf{J}(\mathbf{x}) \qquad !!$$

Even though we are inside the conductor, β_d is what shows! Then it all works!

I could do a full 3D solution to this using

$$-(\nabla^2 + k^2) \, f(\mathbf{x}) = s(\mathbf{x}) \qquad \Rightarrow \qquad f(\mathbf{x}) = \int d^3x' \, [e^{-j\beta R}/4\pi R] \, s(\mathbf{x}') \; + \text{homogeneous solutions}$$

The Helmholtz Equation (H.1.9)

so the solution would be

$$\mathbf{A}(\mathbf{x}) = \mu \int d^3x' \, [e^{-j\beta R}/4\pi R] \, \mathbf{J}(\mathbf{x}')$$

Now consider

$$\mathbf{B} = \text{curl } \mathbf{A} \;\; = \mu \int d^3x' \, \text{curl} \, \{ \, [e^{-j\beta R}/4\pi R] \, \mathbf{J}(\mathbf{x}') \, \}$$

But we know that for **Q** a constant vector,

$$\nabla \text{ x } (\varphi \mathbf{Q}) = \nabla\varphi \text{ x } \mathbf{Q} + \varphi \, (\nabla \text{ x } \mathbf{Q}) \;\; = \nabla\varphi \text{ x } \mathbf{Q} \;\; = \text{grad } \varphi \text{ x } \mathbf{Q}$$

Thus

$$\mathbf{B} = \text{curl } \mathbf{A} \;\; = \mu \int d^3x' \, \text{grad} \, [e^{-j\beta R}/4\pi R] \;\; \text{x } \mathbf{J}(\mathbf{x}')$$

so

$$\mathbf{H} = \int d^3x' \, \text{grad} \, [e^{-j\beta R}/4\pi R] \;\; \text{x } \mathbf{J}(\mathbf{x}')$$

$$H_i = \int d^3x' \, [\, \text{grad} \, [e^{-j\beta R}/4\pi R] \;\; \text{x } \mathbf{J}(\mathbf{x}') \,]_i$$

$$= - \varepsilon_{irs} \int d^3x' \, [\partial'_r \, [e^{-j\beta R}/4\pi R] \;\; J_s(\mathbf{x}')$$

$$= + \varepsilon_{irs} \int d^3x' \, [e^{-j\beta R}/4\pi R] \;\; \partial'_r \, J_s(\mathbf{x}')$$

$$= + \int d^3x' \, [e^{-j\beta R}/4\pi R] \, \varepsilon_{irs} \, \partial'_r \, J_s(\mathbf{x}')$$

$$= + \int d^3x' \, [e^{-j\beta R}/4\pi R] \, [\text{curl}' \, J'(\mathbf{x}') \,]^i$$

which then says

$$\mathbf{H} = \;\; + \int d^3x' \, [e^{-j\beta R}/4\pi R] \, \text{curl}' \, \mathbf{J}'(\mathbf{x}')$$

Compare this to the other method which said

$$\mathbf{H}(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, \text{curl}' \, \mathbf{J}(\mathbf{x}')$$

How do I then argue that both these results are the same for a good conductor?