B.1 Relationship between surface current K and the field H at a conductor boundary

First, consider this blow up of a piece of the boundary between a conductor (medium 2) and a dielectric (medium 1). Both media extend uniformly in the z direction, so we are looking at a piece of the cross section of a transmission line at a particular point on the surface of one of the conductors.

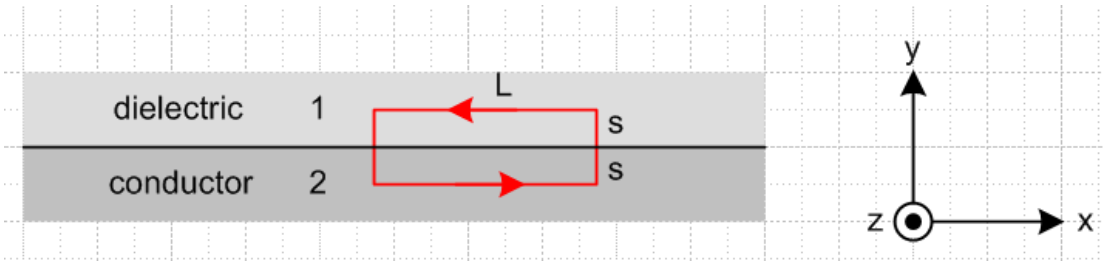

Fig B.1

We shall assume that the conduction current is positive in the z direction, so $\mathbf{J} = J_z\hat{\mathbf{z}}$ with $J_z > 0$. Since the lower medium is the conductor in the drawing, the **B** and **H** field at the boundary are in the $-\hat{\mathbf{x}}$ direction, that is to say, they point to the left due to the right hand rule relating **J** and **B** or **H** .

According to (1.1.44), the tangential component of the H field is continuous at a boundary provided the boundary does not carry a *free* surface current, which is our situation here. Therefore,

$$H_{x2} = H_{x1} \qquad (1/\mu_1)B_{x1} = (1/\mu_2)B_{x2} \quad . \tag{B.1.1}$$

Assuming $\mu_2 \geq \mu_1$ (which would be the case if $\mu_1 = \mu_0$), the right equation implies $|B_{x2}| \geq |B_{x1}|$ so the B field is larger inside the conductor. But in our picture, both B_{x2} and B_{x1} are negative, so $-B_{x2} \geq -B_{x1}$ which then says $B_{x2} \leq B_{x1}$ and finally $(B_{x2} - B_{x1}) \leq 0$. Also, $H_{x2} = H_{x1} \leq 0$. For the red loop shown in the figure one then has, as $s \rightarrow 0$,

$$\oint \mathbf{B} \bullet d\mathbf{s} \;=\; B_{x2}L - B_{x1}L \;= (B_{x2} - B_{x1})L \;\leq 0\,. \tag{B.1.2}$$

Now consider (1.1.31) and (1.1.24) which say (in the ω domain),

$$\oint \mathbf{B} \bullet d\mathbf{s} \;=\; \int_S \text{curl}\, \mathbf{B} \bullet d\mathbf{S} \;= \mu_0 \int_S [\, j\omega\varepsilon\mathbf{E} + \mathbf{J}_c + \mathbf{J}_m] \bullet d\mathbf{S} \tag{B.1.3}$$

where $d\mathbf{S} = dS\,\hat{\mathbf{z}}$. Since $\mathbf{E} \bullet d\mathbf{S}$ involves only E_z (parallel to surface), and since by (1.1.41) such E_z is continuous at the boundary, and since $E_z \approx 0$ inside the conductor, the $\varepsilon j\omega\mathbf{E}$ term makes no contribution, giving then

$$\oint \mathbf{B} \bullet d\mathbf{s} \;=\; \mu_0 \int_S [\mathbf{J}_c + \mathbf{J}_m] \bullet d\mathbf{S}. \tag{B.1.4}$$

As the distance s is taken to 0 in the red math loop above, $\int_S \mathbf{J_c} \bullet d\mathbf{S} \rightarrow 0$ because the conduction current is non-singular at the boundary. That is to say, $\int_S \mathbf{J_c} \bullet d\mathbf{S} \rightarrow \mathbf{J_c} \bullet \int_S d\mathbf{S} \rightarrow 0$. Since we shall take this limit in the end, we can then ignore the $\mathbf{J_c}$ term in (B.1.4) and write

$$\oint \mathbf{B} \bullet d\mathbf{s} = \mu_0 \int_S \mathbf{J_m} \bullet d\mathbf{S}. \qquad \text{intending to take } s \rightarrow 0 \ . \tag{B.1.5}$$

Since $\mathbf{J_c}$ flows in the $+\hat{\mathbf{z}}$ direction, the surface current $\mathbf{J_m}$ flows in the $-\hat{\mathbf{z}}$ direction (as shown below), so write

$$\mathbf{J_m} = K_z \, \delta(y) \, \hat{\mathbf{z}} \tag{B.1.6}$$

where $K_z \leq 0$ is the magnitude of the surface current. Then

$$\int_S \mathbf{J_m} \bullet d\mathbf{S} = K_z \int_0^L dx \int_{-s}^{s} dy \, \delta(y) \, \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = K_z \int_0^L dx = K_z L \ . \tag{B.1.7}$$

Thus from (B.1.2), (B.1.5) and (B.1.7) we find that

$$\mu_0 K_z = B_{x2} - B_{x1} \tag{B.1.8}$$

Compare this with (1.1.44) which says (as noted earlier, $K_z^{free} = 0$ on our boundary)

$$K_z^{free} = H_{x2} - H_{x1} \tag{1.1.44}$$

The H field does not "see" our magnetization surface current K_z, but the B field does see it.

The signs are consistent with $K_z \leq 0$ and $(B_{x2} - B_{x1}) \leq 0$ as noted above. Then from (B.1.1) we find

$$\mu_0 K_z = B_{x2} - B_{x1} = (\mu_2 H_{x2} - \mu_1 H_{x1}) = (\mu_2 - \mu_1) H_{x2}$$

and finally

$$K_z = \left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0} \right) H_{x2} \ . \tag{B.1.9}$$

As noted earlier, $H_{x2} < 0$ so $K_z \leq 0$ is consistent with $\mu_2 \geq \mu_1$.

We now rewrite this result in terms of a different picture:

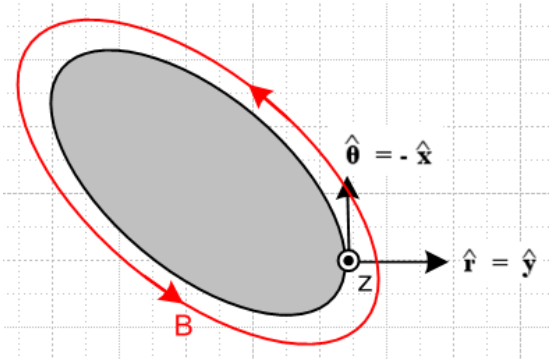

Fig B.2

This shows the cross section of the entire conductor in gray, and $\mathbf{J_c}$ is still directed toward the viewer. In this picture a point on the surface is associated with a local coordinate system for which $\hat{\mathbf{r}} = \hat{\mathbf{y}}$ is normal to the surface and $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}$ is tangent to the surface (so $H_\theta = -H_x$). We are thinking of (r,θ,z) as local cylindrical coordinates at the point shown on the conductor surface, where $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{z}}$, and the x,y,z directions of the figure match those of the previous figure where as usual $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$. Then (B.1.9) says

$$K_z = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right) H_\theta \qquad \text{amps/m} \qquad \text{(B.1.10)}$$

where $H_\theta > 0$ and $K_z \leq 0$.

Conclusion: At the surface of a conductor, if the conduction current is in the + z direction, the magnetization surface current is in the - z direction and has a magnitude given by $K_z = -(\mu_2 - \mu_1) H_\theta/\mu_0$ where H_θ is the tangential H field at the surface of the conductor (either inside or outside since both are the same). If $\mu_1 = \mu_2$ then there is no surface current since there is no imbalance of magnetization current at the boundary.

Example: For a round wire of radius a carrying an axially symmetric current distribution, we know that $2\pi a H_\theta = I$ so $H_\theta = I/(2\pi a)$ at the surface. Then

$$K_z = -\left(\frac{\mu_2}{\mu_0} - \frac{\mu_1}{\mu_0}\right)[\, I/(2\pi a)] \qquad \text{round wire of radius a and } \mu_2\text{, dielectric } \mu_1 \qquad \text{(B.1.11)}$$

Physical mechanism of the surface current. As a reminder, a surface magnetization current arises at a boundary between media with different μ values just the way surface polarization charge arises at a boundary between media with different ε. In the μ case, here is a suggestive picture :

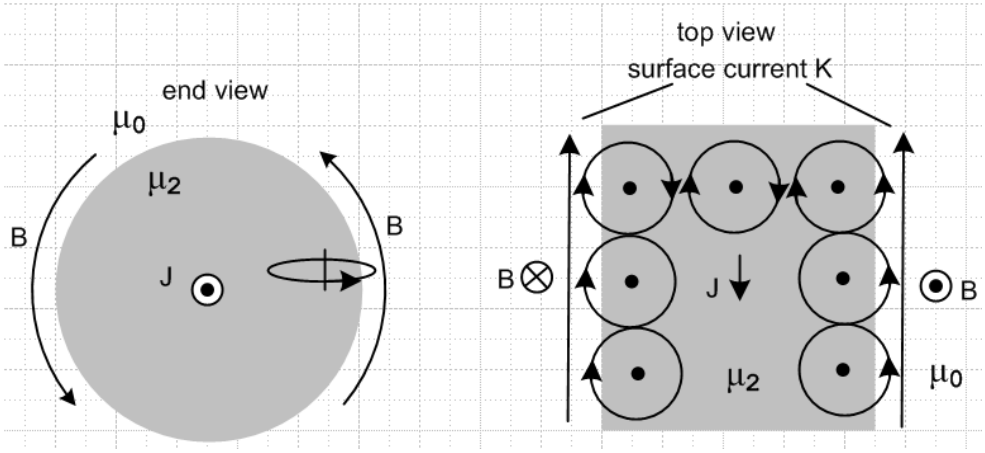

Fig B.3

On the left we look at a round wire end on, while the right shows a top view where the wire has been tilted down. Here $\mu_1 = \mu_0$ so there is only vacuum outside the wire. The **B** field lines up the little magnetic dipoles (or creates them) which we represent schematically as little atoms with orbiting electrons. The atoms in the interior always have cancelling current arrows, but there is an imbalance on the outer surface which is the surface magnetization current. The picture shows why it is that the surface current is directed opposite to the current **J** which creates it, a sort of magnetic Lenz's Law. The surface current is not seen by **H**, but it is seen by **B**.

In the case that the outer medium has some $\mu_1 > \mu_0$, both media have surface currents at the boundary, and then when $\mu_1 \neq \mu_2$ there is a surface current imbalance resulting in a net surface current. If it happens that $\mu_1 < \mu_2$, then the directions shown above are correct, but if $\mu_1 > \mu_2$, the surface current runs in the opposite direction to that shown.