scratch save PhL 11.13.13

Reviewed 11.16.13, keep for first part only.

Verify GR7: Let's make sure the GR7 integral is correct. Use **Stak and (6.22)** which claims that

$$\ln(1 + a^2 - 2a\cos x) = -2\sum_{n=1}^{\infty}(a^n/n)\cos(nx) \qquad // \text{ where } |A| < 1$$

Then my integral is

$$\int_0^{\pi} dx\, \ln[\alpha^2 + 1 - 2\alpha\cos(x)]\cos x$$

$$= \int_0^{\pi} dx\ \{-2\sum_{n=1}^{\infty}(\alpha^n/n)\cos(nx)\}\cos x$$

$$= -2\sum_{n=1}^{\infty}(\alpha^n/n)\int_0^{\pi} dx\ \{\cos(nx)\}\cos x$$

Now according to Spiegel p 96 15.27 this integral vanishes unless n =1 in which case it is $\pi/2$, so

$$= -2(\alpha)\pi/2 = -\pi\alpha$$

which agrees with GR7.

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This is a big chunk from my surface currents doc where I think I had the wrong propagator:

Now write

$$\text{curl } \mathbf{J} = (\partial_x J_y - \partial_y J_x)\hat{\mathbf{z}} + (\partial_y J_z - \partial_z J_y)\hat{\mathbf{x}} + (\partial_z J_x - \partial_x J_z)\hat{\mathbf{y}}$$

Coordinates are given by

$$\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$
$$\mathbf{x}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$$
$$\mathbf{J}(\mathbf{x}) = J_x(\mathbf{x})\hat{\mathbf{x}} + J_y(\mathbf{x})\hat{\mathbf{y}} + J_z(\mathbf{x})\hat{\mathbf{z}}$$
$$\mathbf{J}(\mathbf{x}') = J_x(\mathbf{x}')\hat{\mathbf{x}} + J_y(\mathbf{x}')\hat{\mathbf{y}} + J_z(\mathbf{x}')\hat{\mathbf{z}}$$

If **J** has only a z component, then

curl $\mathbf{J} = (\partial_y J_z)\,\hat{\mathbf{x}} + (-\partial_x J_z)\,\hat{\mathbf{y}}$

curl' $\mathbf{J}(\mathbf{x}') = (\partial_{y'} J_z(\mathbf{x}'))\,\hat{\mathbf{x}} + (-\partial_{x'} J_z(\mathbf{x}'))\,\hat{\mathbf{y}}$

Then ** becomes these two equations, where J has no z dependence,

$H_x(x,y) = (1/4\pi) \int d^3x'\ \partial_{y'} J_z(x',y')/R \qquad R = |\mathbf{x} - \mathbf{x}'|$ wrong propagator.

$H_y(x,y) = -(1/4\pi) \int d^3x'\ \partial_{x'} J_z(x',y')/R$

Doing parts integration (the parts vanish since J_z has finite x,y extent) we find

$$H_x(x,y) = (1/4\pi) \int dx'dz' \{ \ [J_z(x',y')R^{-1}]_{-\infty}^{\infty} - \int dy'\ J_z(x',y')\ \partial_{y'} R^{-1} \ \}$$

Assuming a finite J_z transverse distribution, the "parts" contribution vanishes. Meanwhile,

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$2RdR = 2(y'-y)dy' \qquad \Rightarrow \qquad \partial_{y'} R^{-1} = -R^{-2}\partial_{y'} R = -R^{-2}\,(y'-y)/R = -R^{-3}(y'-y)$$

Thus,

$$H_x(x,y) = (1/4\pi) \int dx'dz' \{\ 0\ - \int dy'\ J_z(x',y')\ [-R^{-3}(y'-y)]\ \}$$

$$= (1/4\pi) \int dx'dy'dz' \{\ J_z(x',y')\ [R^{-3}(y'-y)]\ \}$$

$$= (1/4\pi) \int dx'dy'\ (y'-y)dz'\ J_z(x',y') \int dz'\ [R^{-3}]$$

The dz integration over the conductor gives

$$\int dz'\ [R^{-3}] = \int_{-\infty}^{\infty} dz'\ (s^2+z'^2)^{-3/2} = 1/s^2 \qquad \text{where } s \equiv \sqrt{(x-x')^2 + (y-y')^2}$$

Therefore

$$H_x(x,y) = (1/4\pi) \int dx'dy'\ J_z(x',y') \frac{y'-y}{(x-x')^2 + (y-y')^2}$$

Doing similar processing for H_y gives this pair of results:

$$H_x(x,y) = (1/4\pi) \int dx'dy' J_z(x',y') \frac{y'-y}{(x-x')^2 + (y-y')^2}$$

$$H_y(x,y) = (1/4\pi) \int dx'dy' J_z(x',y') \frac{x'-x}{(x-x')^2 + (y-y')^2}$$

Let's see how this does for a round wire with origin at center of cross section and radius a. Then

$x = a\cos\theta$
$y = a\sin\theta$

$x' = r'\cos\theta'$ $\quad$ $x-x' = a\cos\theta - r'\cos\theta'$
$y' = r'\sin\theta'$ $\quad$ $y-y' = a\sin\theta - r'\sin\theta'$

$$R^2 = (a\cos\theta - r'\cos\theta')^2 + (a\sin\theta - r'\sin\theta')^2 = a^2 + r'^2 - 2ar'\cos(\theta-\theta')$$

$dx'dy' = r'dr'd\theta'$

$H_\theta = H_y \cos\theta - H_x \sin\theta$ $\quad$ // scratch drawing

So start with

$$H_x(x,y) = (1/4\pi) \int dx'dy' J_z(x',y') \frac{y'-y}{(x-x')^2 + (y-y')^2}$$

$$= -(1/4\pi) \int r'dr'd\theta' J_z(x',y') \frac{a\sin\theta - r'\sin\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

So we then have these two equations

$$H_x(x,y) = -(1/4\pi) \int r'dr'd\theta' J_z(x',y') \frac{a\sin\theta - r'\sin\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

$$H_y(x,y) = -(1/4\pi) \int r'dr'd\theta' J_z(x',y') \frac{a\cos\theta - r'\cos\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

Now for the round wire example, assume $J_z = I/(\pi a^2)$ so then

$$H_x(x,y) = -(1/4\pi)\, I/(\pi a^2) \int_0^a r'dr' \int_{-\pi}^{\pi} d\theta' \frac{a\sin\theta - r'\sin\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

$$H_y(x,y) = -(1/4\pi)\, I/(\pi a^2) \int_0^a r'dr' \int_{-\pi}^{\pi} d\theta' \frac{a\cos\theta - r'\cos\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

There are three angle integrals to evaluate.

$$I_1 \equiv \int_{-\pi}^{\pi} d\theta' \frac{1}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

$$I_2 = \int_{-\pi}^{\pi} d\theta' \frac{\sin\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

$$I_3 = \int_{-\pi}^{\pi} d\theta' \frac{\cos\theta'}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

in terms of which we then have

$$H_x(x,y) = -(1/4\pi)\, I/(\pi a^2) \int_0^a r'dr' \{ a\sin\theta\, I_1 - r'\, I_2 \}$$

$$H_y(x,y) = -(1/4\pi)\, I/(\pi a^2) \int_0^a r'dr' \{ a\cos\theta\, I_1 - r'\, I_3 \}$$

Let $x = \theta-\theta'$ so

$$\sin\theta' = \sin(\theta-x) = \sin\theta\cos x - \cos\theta\sin x$$

$$\cos\theta' = \cos(\theta-x) = \cos\theta\cos x + \sin\theta\sin x$$

Then

$$I_1 = \int dx \frac{1}{a^2 + r'^2 - 2ar'\cos x}$$

$$I_2 = \int dx \frac{\sin\theta\cos x - \cos\theta\sin x}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

$$= \sin\theta \int dx \frac{\cos x}{a^2 + r'^2 - 2ar'\cos x} - \cos\theta \int dx \frac{\sin x}{a^2 + r'^2 - 2ar'\cos x}$$

$$= \sin\theta\ I_4 - \cos\theta\, I_5$$

$$I_3 = \int dx \frac{\cos\theta\cos x + \sin\theta\sin x}{a^2 + r'^2 - 2ar'\cos(\theta-\theta')}$$

$$= \cos\theta \int dx \frac{\cos x}{a^2 + r'^2 - 2ar'\cos x} + \sin\theta \int dx \frac{\sin x}{a^2 + r'^2 - 2ar'\cos x}$$

$$= \cos\theta\, I_4 + \sin\theta\, I_5$$

We now have to evaluate

$$I_1 = \int dx \frac{1}{a^2 + r'^2 - 2ar'\cos x}$$

$$I_4 = \int dx \frac{\cos x}{a^2 + r'^2 - 2ar'\cos x}$$

$$I_5 = \int dx \frac{\sin x}{a^2 + r'^2 - 2ar'\cos x}$$

But taking x from $-\pi$ to π, we see that I_5 has an odd integrand so we write $I_5 = 0$. We then have two to worry about

$$I_1 = \int dx \frac{1}{a^2 + r'^2 - 2ar'\cos x} = (1/a^2) \int dx \frac{1}{1 + (r'/a)^2 - 2(r'/a)\cos x}$$

$$= (1/a^2) \int dx \frac{1}{1 + \alpha^2 - 2\alpha\cos x} \qquad \alpha = r'/a$$

then we have two to worry about

$$I_1 = (1/a^2) \int dx \frac{1}{1 + \alpha^2 - 2\alpha\cos x}$$

$$I_4 = (1/a^2) \int dx \frac{\cos x}{1 + \alpha^2 - 2\alpha\cos x}$$

Now these appear on page 391 circa of GR7, and I quote

$$2.^6 \qquad \int_0^\pi \frac{\cos nx\, dx}{1 - 2a\cos x + a^2} = \frac{\pi a^n}{1 - a^2} \qquad \left[a^2 < 1, \quad n \geq 0\right]$$

$$= \frac{\pi}{(a^2 - 1)\, a^n} \qquad \left[a^2 > 1, \quad n \geq 0\right]$$

BI (65)(3)

Therefore

$$I_1 = (1/a^2)\ 2\pi\alpha^0/(1-\alpha^2) \qquad \alpha = r'/a < 1$$

$$= (2\pi/a^2)\ /(1-(r'/a)^2)$$

$$= 2\pi/(a^2-r'^2)$$

$$I_4 = (1/a^2)\ 2\ \pi\alpha^1/(1-\alpha^2) = (2\pi/a^2)(r'/a)/(1-(r'/a)^2) =$$

$$= 2\pi\,(r'/a) / (a^2-r'^2)$$

Now we go backwards up the stack:

$$I_2 = \sin\theta\ I_4 - \cos\theta\ I_5 = \sin\theta\ I_4 = \sin\theta\ 2\pi\ (r'/a) / (a^2-r'^2)$$

$$I_3 = \cos\theta\ I_4 + \sin\theta\ I_5 = \cos\theta\ I_4 = \cos\theta\ 2\pi\ (r'/a) / (a^2-r'^2)$$

which we summarize as

$$I_1 = 2\pi/(a^2-r'^2)$$
$$I_2 = \sin\theta\ 2\pi\ (r'/a) / (a^2-r'^2)$$
$$I_3 = \cos\theta\ 2\pi\ (r'/a) / (a^2-r'^2)$$

Then go back to

$$H_x(x,y) = -(1/4\pi)\ I/(\pi a^2) \int_0^a r'dr' \{ a\sin\theta\ I_1 - r'\ I_2 \}$$

$$H_y(x,y) = -(1/4\pi)\ I/(\pi a^2) \int_0^a r'dr' \{ a\cos\theta\ I_1 - r'\ I_3 \}$$

$$H_x(x,y) = -(1/4\pi)\ I/(\pi a^2) \int_0^a r'dr' \{ a\sin\theta\ 2\pi/(a^2-r'^2) - r'\ \sin\theta\ 2\pi\ (r'/a) / (a^2-r'^2) \}$$

$$H_y(x,y) = -(1/4\pi)\ I/(\pi a^2) \int_0^a r'dr' \{ a\cos\theta\ 2\pi/(a^2-r'^2) - r'\ \cos\theta\ 2\pi\ (r'/a) / (a^2-r'^2) \}$$

$$H_x(x,y) = -(1/2)\ I/(\pi a^2)\ \sin\theta \int_0^a r'dr' \{ a\ 1/(a^2-r'^2) - r'\ (r'/a) / (a^2-r'^2) \}$$

$$H_y(x,y) = -(1/2)\ I/(\pi a^2)\ \cos\theta \int_0^a r'dr' \{ a\ 1/(a^2-r'^2) - r'\ (r'/a) / (a^2-r'^2) \}$$

Now we have various r' integrals to worry about.

$$J_1 = \int_0^a r'dr'/ (a^2-r'^2)$$

$$J_2 = \int_0^a r'^3dr'/(a^2-r'^2)$$

in terms of which

$$H_x(x,y) = -(1/2)\ I/(\pi a^2)\ \sin\theta \{ a\ J_1 - (1/a)\ J_2 \}$$
$$H_y(x,y) = -(1/2)\ I/(\pi a^2)\cos\theta \{ a\ J_1 - (1/a)\ J_2 \}$$

But oops, both J_1 and J_2 diverge at the $r' \approx a$ endpoint. That was completely unexpected by me.

********** this was at the start of the surface currents deal **********

I am hopeful of generalizing the Chapter 4 development of transmission line parameters to allow for the possibility of magnetic conductors, if it is not too hard! I already know the surface currents on a round conductor with uniform DC current density, but I would like to find an expression for this current in a more general case.

For any boundary we know that

$$H_{t2} = H_{t1} \qquad (1/\mu_1)B_{t1} = (1/\mu_2)B_{t2} \qquad \text{at the boundary}$$

If we have $\mu_2 > \mu_1$ then $B_{t2} > B_{t1}$.