Surface currents on non-round conductor PhL 11.11.13

This doc became the new Appendix B and was installed in lines doc on 11.15.13.

Appendix B: Magnetization Surface Currents on a Conductor 1
B.1 Relationship between surface current K and the field H at a conductor boundary 1
B.2 Calculation of H from the current in a conductor 4
B.3 General Method for computing the surface current J_m on a wire 5
B.4 Surface current on a round wire with uniform J 6
B.5 Surface current on a round wire using the General Method 6
B.6 An Acrobatic Exercise: Compute H from J and J_m via the vector potential A. 10

Appendix B: Magnetization Surface Currents on a Conductor

Overview

When a conductor of magnetic susceptibility μ_2 is embedded in a medium of μ_1 with $\mu_1 \neq \mu_2$, a "bound current" appears on the conductor surface. This so-called magnetization current must be accounted for when computing quantities like the **B** field or vector potential **A**.

Section B.1 shows how this surface current **K** is related to the **H** field at the surface.
Section B.2 shows how to compute **H** from the volume current density **J**.
Section B.3 then outlines a general plan for computing surface current **K** for an arbitrary conductor.

The above sections apply in both AC and DC conditions ($\omega > 0$ and $\omega = 0$). The remaining three sections consider the simple case of a round wire where the volume current J_z is uniform in the conductor, a situation that arises at DC and low frequency where there is no skin effect.

Section B.4 computes **H** and the surface current **K** for a round wire using symmetry.
Section B.5 repeats the calculation using the general method outlined in Section B.3.
Section B.6 assumes the **K** computed above, along with uniform J_z, and computes from these the vector potential **A**. Then **B** = curl **A** and **H** = **B**/μ and the simple results of Section B.4 are replicated.

The conductors considered here are those of a transmission line in the "transmission line limit" in which it is assumed that the wavelength along the line is much longer than the transverse dimensions of the line. In this case, it is reasonable to use 2D wave equations whose solutions then involve use of the 2D Poisson free-space propagator $\ln(R/2\pi)$ as discussed in Appendix I.

B.1 Relationship between surface current K and the field H at a conductor boundary

First, consider this blow up of a piece of the boundary between a conductor (medium 2) and a dielectric (medium 1). Both media extend uniformly in the z direction, so we are looking at a piece of the cross section of a transmission line at a particular point on the surface of one of the conductors.

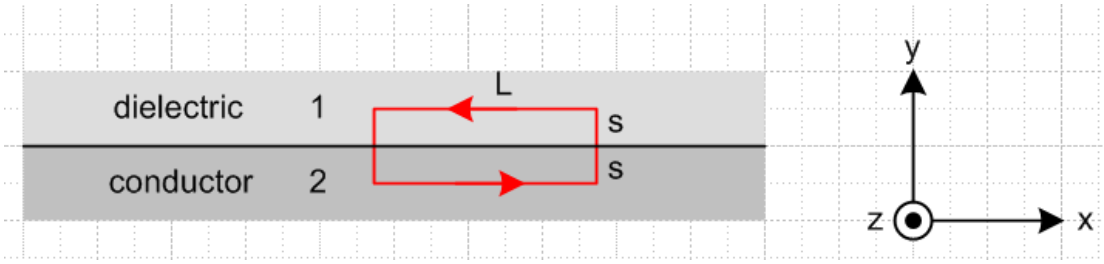

Fig B.1

We shall assume that the conduction current is positive in the z direction, so $\mathbf{J} = J_z\hat{\mathbf{z}}$ with $J_z > 0$. Since the lower medium is the conductor in the drawing, the **B** and **H** field at the boundary are in the $-\hat{\mathbf{x}}$ direction, that is to say, they point to the left due to the right hand rule relating **J** and **B** or **H** .

At the boundary, we know from (1.1.26) that

$$H_{x2} = H_{x1} \qquad (1/\mu_1)B_{x1} = (1/\mu_2)B_{x2} \quad . \tag{B.1.1}$$

Assuming $\mu_2 \geq \mu_1$ (which would be the case if $\mu_1 = \mu_0$), the right equation implies $|B_{x2}| \geq |B_{x1}|$ so the B field is larger inside the conductor. But in our picture, both B_{x2} and B_{x1} are negative, so $-B_{x2} \geq -B_{x1}$ which then says $B_{x2} \leq B_{x1}$ and finally $(B_{x2} - B_{x1}) \leq 0$. Also, $H_{x2} = H_{x1} \leq 0$. For the red loop shown in the figure one then has,

$$\oint \mathbf{B} \bullet d\mathbf{s} \approx B_{x2}L - B_{x1}L = (B_{x2} - B_{x1})L \leq 0 \, . \tag{B.1.2}$$

Now consider (1.1.24) which says (in the ω domain),

$$\oint \mathbf{B} \bullet d\mathbf{s} = \int_S [\, j\omega\varepsilon\mathbf{E} + \mu\mathbf{J}_c + \mu\mathbf{J}_m] \bullet d\mathbf{A} \tag{B.1.3}$$

where $d\mathbf{A} = dA\,\hat{\mathbf{z}}$. Since $\mathbf{E} \bullet d\mathbf{A}$ involves only E_z (parallel to surface), and since by (1.1.25) such E_z is continuous at the boundary, and since $E_z \approx 0$ inside the conductor, the $\varepsilon j\omega\mathbf{E}$ term makes no contribution, giving then

$$\oint \mathbf{B} \bullet d\mathbf{s} = \int_S [\mu\mathbf{J}_c + \mu\mathbf{J}_m] \bullet d\mathbf{A} \quad . \tag{B.1.4}$$

Since $\mathbf{J}_c$ flows in the $+\hat{\mathbf{z}}$ direction, the surface current $\mathbf{J}_m$ flows in the $-\hat{\mathbf{z}}$ direction, so write

$$\mathbf{J}_m = K_z\,\delta(y)\,\hat{\mathbf{z}} \tag{B.1.5}$$

where $K_z \leq 0$ is the magnitude of the surface current. The area integral on the right side of (B.1.4) is written as two terms, one for the area above the boundary, and one below:

$$\int_S \mu\,[\mathbf{J}_c + \mathbf{J}_m] \bullet d\mathbf{A} = \int \mu_1\,[\mathbf{J}_{c1} + \mathbf{J}_{m1}] \bullet d\mathbf{A} + \int \mu_2\,[\mathbf{J}_{c2} + \mathbf{J}_{m2}] \bullet d\mathbf{A} \quad . \tag{B.1.6}$$

The surface current lies on the outside of the conductor surface, so it is region 1. Then

$$\int_S \mu\,[\mathbf{J}_c + \mathbf{J}_m] \bullet d\mathbf{A} \approx \mu_1 J_{c1z}\ (s/2)L + \mu_2 J_{c2z}\ (s/2)L + \mu_1 \int dxdy\ [K_z \delta(y)]\ \hat{\mathbf{z}} \bullet d\mathbf{A}$$

$$= [\mu_1 J_{c1z} + \mu_2 J_{c2z}]\ (s/2)L + \mu_1 K_z L\ .$$

Then (B.1.4) with (B.1.2) says

$$(B_{x2} - B_{x1})L = [\mu_1 J_{c1z} + \mu_2 J_{c2z}]\ (s/2)L + \mu_1 K_z L \qquad \text{(B.1.7)}$$

Since the volume currents are not infinite at the boundary, as s→ 0 we find

$$\mu_1 K_z = (B_{x2} - B_{x1}) \qquad \text{(B.1.8)}$$

The signs are consistent with $K_z \le 0$ and $(B_{x2} - B_{x1}) \le 0$ as noted above. Then from (B.1.1) we find

$$\mu_1 K_z = (\mu_2 H_{x2} - \mu_1 H_{x1}) = (\mu_2 - \mu_1)\, H_{x2}$$

and finally

$$K_z = (\mu_2/\mu_1 - 1)\, H_{x2} \qquad \text{(B.1.9)}$$

As noted earlier, $H_{x2} < 0$ so $K_z \le 0$ is consistent with $\mu_2 \ge \mu_1$.

We now rewrite this result in terms of a different picture:

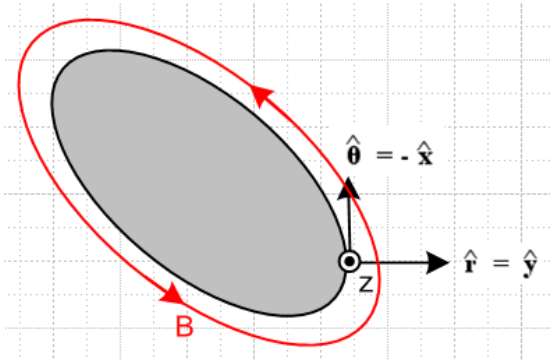

Fig B.2

This shows the cross section of the entire conductor in gray, and $\mathbf{J}_c$ is still directed toward the viewer. In this picture a point on the surface is associated with a local coordinate system for which $\hat{\mathbf{r}} = \hat{\mathbf{y}}$ is normal to the surface and $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}$ is tangent to the surface (so $H_\theta = -H_x$). We are thinking of (r,θ,z) as local cylindrical coordinates at the point shown on the conductor surface, where $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{z}}$, and the x,y,z directions of the figure match those of the previous figure where as usual $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$. Then (B.1.9) says

$$K_z = -(\mu_2/\mu_1 - 1)\, H_\theta \qquad \text{(B.1.10)}$$

where $H_\theta > 0$ and $K_z \leq 0$.

Conclusion: At the surface of a conductor, if the conduction current is in the + z direction, the surface current is in the - z direction and has a magnitude given by $K_z = -(\mu_2/\mu_1 - 1)\, H_\theta$ where H_θ is the tangential H field at the surface of the conductor (either inside or outside since both are the same). If $\mu_1 = \mu_2$ then there is no surface current since there is no imbalance of magnetization current at the boundary.

Example: For a round wire of radius a carrying an axially symmetric current distribution, we know that $2\pi a H_\theta = I$ so $H_\theta = I/(2\pi a)$ at the surface. Then

$$K_z = -(\mu_2/\mu_1 - 1)[\, I/(2\pi a)] \qquad \text{round wire of radius a and } \mu_2\text{, dielectric } \mu_1 \qquad \text{(B.1.11)}$$

Physical mechanism of the surface current. As a reminder, a surface magnetization current arises at a boundary between media with different μ values just the way surface polarization charge arises at a boundary between media with different ε. In the μ case, here is a suggestive picture :

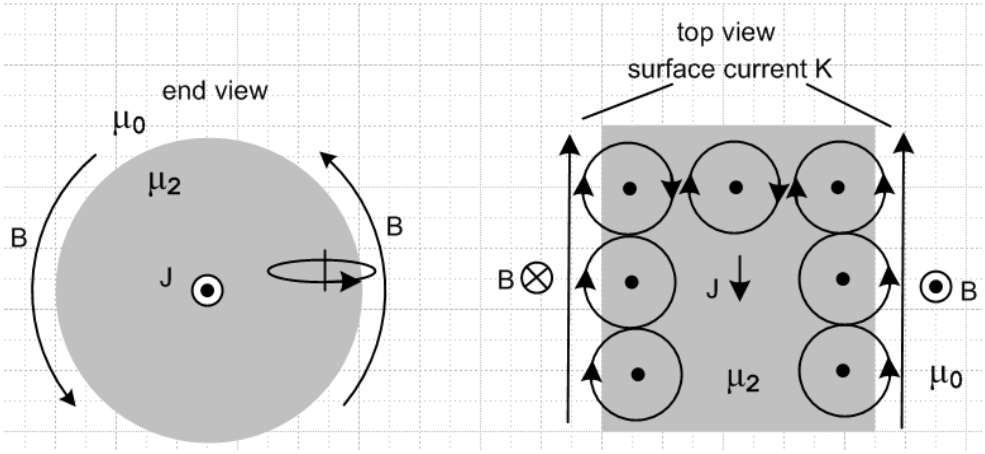

Fig B.3

On the left we look at a round wire end on, while the right shows a top view where the wire has been tilted down. Here $\mu_1 = \mu_0$ so there is only vacuum outside the wire. The **B** field lines up the little magnetic dipoles (or creates them) which we represent schematically as little atoms with orbiting electrons. The atoms in the interior always have cancelling current arrows, but there is an imbalance on the outer surface which is the surface magnetization current. The picture shows why it is that the surface current is directed opposite to the current **J** which creates it, a sort of magnetic Lenz's Law. The surface current is not seen by **H**, but it is seen by **B**.

In the case that the outer medium has some $\mu_1 > \mu_0$, both media have surface currents at the boundary, and then when $\mu_1 \neq \mu_2$ there is a surface current imbalance resulting in a net surface current. If it happens that $\mu_1 < \mu_2$, then the directions shown above are correct, but if $\mu_1 > \mu_2$, the surface current runs in the opposite direction to that shown.

B.2 Calculation of H from the current in a conductor

Start with Maxwell's equation (1.1.1), and we are now working inside a conductor so $\mathbf{E} = 0$ and then

$$\text{curl } \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} = \mathbf{J} \tag{B.2.1}$$

where $\mathbf{J}$ is the conduction current. Apply curl to both sides and use curl curl = grad div $-\nabla^2$ to get

$$\text{grad div } \mathbf{H} - \nabla^2\mathbf{H} = \text{curl } \mathbf{J} . \tag{B.2.2}$$

But in a uniform medium div $\mathbf{H} = 0$ since div $\mathbf{B} = 0$ so

$$\nabla^2\mathbf{H} = -\text{ curl } \mathbf{J} . \tag{B.2.3}$$

Now let's assume that we have $\mathbf{J} = J_z(x,y)\hat{\mathbf{z}}$ and assume that the solution $\mathbf{H}$ does not depend on z. In that case we have

$$\nabla^2_{2D}\,\mathbf{H}(x,y) = -\text{ curl } \mathbf{J} \qquad \mathbf{J} = J_z(x,y)\hat{\mathbf{z}} \tag{B.2.4}$$

The particular solution to this PDE is shown in (I.1.8) to be

$$\mathbf{H}(x,y) = \int d^2\mathbf{x}' \left[\frac{1}{2\pi} \ln(1/R) \right] \text{curl}' \, \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \tag{B.2.5}$$

or

$$\mathbf{H}(x,y) = -\frac{1}{4\pi} \int d^2\mathbf{x}' \ln(R^2) \text{ curl}' \, \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \tag{B.2.6}$$

where $\frac{1}{2\pi} \ln(1/R)$ is the Poisson 2D free-space propagator.

B.3 General Method for computing the surface current J_m on a wire

Here are the steps for a wire of arbitrary cross sectional shape:

1. From the prescribed conduction current $\mathbf{J}$ in the wire, compute curl $\mathbf{J}$. As will be shown in the example below, some or all of curl $\mathbf{J}$ may lie on the outer surface of the wire.

2. Compute the $\mathbf{H}$ field at all points in the wire cross-sectional plane section using (B.2.5)

$$\mathbf{H}(x,y) = -\frac{1}{4\pi} \int d^2\mathbf{x}' \ln(R^2) \text{ curl}' \, \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| . \tag{B.2.5}$$

3. Evaluate this $\mathbf{H}$ field at $\mathbf{x}_b = (x_b,y_b)$ for all points $\mathbf{x}_b$ on the cross section boundary.

4. Compute the component of $\mathbf{H}$ which is tangential to the boundary in the cross sectional plane. Call this component H_θ.

5. The surface current density is then given by (B.1.10),

$$K_z = -(\mu_2/\mu_1 - 1)\, H_\theta \quad . \qquad \textit{(B.1.10)}$$

B.4 Surface current on a round wire with uniform J

For a round wire of radius a with uniform J_z (as would be the DC case $\omega = 0$) , geometric symmetry makes the calculation of **H** very easy. One need only apply Ampere's Law separately for a point r outside the wire, and for another point r inside the wire. For the outside case one finds

$$2\pi r\, H_\theta(r) = I \qquad \Rightarrow \qquad H_\theta(r) = I/(2\pi r) \qquad r \geq a \quad . \qquad (B.4.1)$$

And then for the inside case the "current enclosed" is determined by a simple area fraction.

$$2\pi r\, H_\theta(r) = I\,(\pi r^2/\pi a^2) \qquad \Rightarrow \qquad H_\theta(r) = I\, r/(2\pi a^2) \qquad r \leq a \quad . \qquad (B.4.2)$$

At the boundary the two expressions agree and we have

$$H_\theta = I/(2\pi a) \quad . \qquad (B.4.3)$$

If this wire has magnetic susceptibility μ_2 and is embedded in an infinite medium of μ_1, then the surface magnetization current induced on the wire is

$$K_z = -(\mu_2/\mu_1 - 1)\, H_\theta = -(\mu_2/\mu_1 - 1)\, I/(2\pi a) \qquad \text{amp/m} \qquad (B.4.4)$$

$$\mathbf{K_z} = K_z \hat{\mathbf{z}} \qquad (B.4.5)$$

and this surface current is in the direction opposite **J** if $\mu_2 > \mu_1$. If $\mu_1 = \mu_2$, the surface current vanishes. This result could be expressed in volume density form as

$$\mathbf{J_m} = K_z \delta(r-a)\hat{\mathbf{z}} \qquad \text{amp/m}^2 \quad . \qquad (B.4.6)$$

B.5 Surface current on a round wire using the General Method

For a wire of some general cross section, symmetry is not available to allow the simple solution outlined in the previous section. We then have to use the more general method outlined in section 3 above. As a check on the viability of this general method, we shall apply it here to the round wire and attempt to replicate the results found in Section 4 above.

The conduction current density in a round wire with uniform J_z is given by

$$J_z(r) = J_0 \theta(a-r) \qquad (B.5.1)$$

where θ is the Heaviside step function. Our first step is to compute curl **J**, and we do this in cylindrical coordinates by just staring at the cylindrical-coordinates curl formula,

$$\text{curl } \mathbf{J} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta J_z - \partial_z J_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z J_r - \partial_r J_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rJ_\theta) - r^{-1}\partial_\theta J_r\,] \qquad \text{(B.5.2)}$$

and finding the only non-zero piece which is this (uniform J_z)

$$\text{curl } \mathbf{J} \;=\; [-\partial_r J_z(r)]\;\hat{\boldsymbol{\theta}}\;. \qquad \text{(B.5.3)}$$

Inserting $J_z(r)$ from above we find

$$\partial_r\, J_z(r) = J_0\;\partial_r\theta(a-r) \;= -\,J_0\;\delta(r-a) \qquad \text{(B.5.4)}$$

$$\Rightarrow\;\; \text{curl } \mathbf{J}(r) = \hat{\boldsymbol{\theta}}\;J_0\;\delta(r-a)\;. \qquad \text{(B.5.5)}$$

so we have a "ring source of curl **J**". For use in our integral for **H** we then have

$$\text{curl}'\; \mathbf{J}(r') = \hat{\boldsymbol{\theta}}'\;J_0\;\delta(r'-a) \qquad \text{(B.5.6)}$$

For a current distribution which tapers off smoothly to 0 at the wire edge one would not have this singular contribution, but for a wire with prescribed uniform current, it is present, and curl **J** vanishes everywhere but on the boundary. The relevant picture is this:

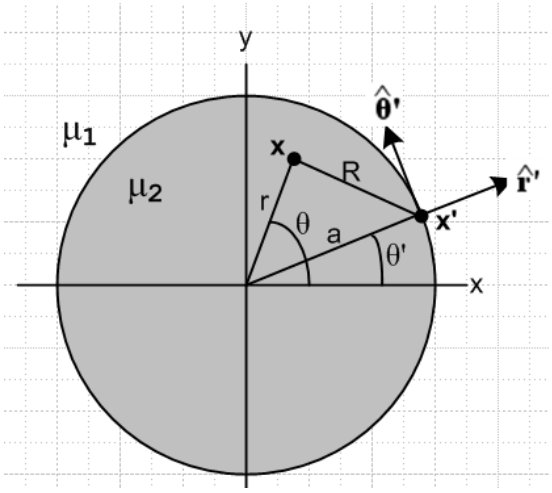

Fig B.4

From (B.2.5) the **H** field at any point **x** = (x,y) is then given by

$$\mathbf{H}(x,y) \;=\; \frac{1}{2\pi}\int d^2\mathbf{x}'\,\ln(1/R)\;\hat{\boldsymbol{\theta}}'\;J_0\;\delta(r'-a)$$

$$= \frac{J_0}{2\pi}\;\int_0^{\infty} r'dr'\;\;\int_{-\pi}^{\pi} d\theta'\,(1/2)\ln(1/R^2)\;\hat{\boldsymbol{\theta}}'\;\delta(r'-a) \;= \frac{J_0 a}{4\pi}\;\;\int_{-\pi}^{\pi} d\theta'\,\ln(1/R^2)|^{r'=a}\;\hat{\boldsymbol{\theta}}'$$

$$= -\;\frac{J_0 a}{4\pi}\;\;\int_{-\pi}^{\pi} d\theta'\,\ln(R^2)|^{r'=a}\;\hat{\boldsymbol{\theta}}'$$

or

$$\mathbf{H}(r,\theta) = -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} d\theta' \ln\left[r^2 + a^2 - 2ar\cos(\theta'-\theta) \right] \hat{\boldsymbol{\theta}}' \,. \tag{B.5.7}$$

The figure shows that

$$\hat{\boldsymbol{\theta}}' = \cos\theta' \,\hat{\mathbf{y}} - \sin\theta' \,\hat{\mathbf{x}} \tag{B.5.8}$$

so then

$$\mathbf{H}(r,\theta) = -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} d\theta' \ln\left[r^2 + a^2 - 2ar\cos(\theta'-\theta) \right] \left[\cos\theta' \,\hat{\mathbf{y}} - \sin\theta' \,\hat{\mathbf{x}}\right] \,. \tag{B.5.9}$$

Next, let $x \equiv \theta'-\theta$. Since the $\int d\theta'$ has full range 2π, one can replace $\int_{-\pi}^{\pi} d\theta' = \int_{-\pi}^{\pi} dx$. Then

$$\mathbf{H}(r,\theta) = -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \left[\cos(x+\theta) \,\hat{\mathbf{y}} - \sin(x+\theta) \,\hat{\mathbf{x}}\right] . \tag{B.5.10}$$

Now writing $\mathbf{H} = H_x \,\hat{\mathbf{x}} + H_y \,\hat{\mathbf{y}}$, decompose the above into two equations

$$\begin{aligned} H_x(r,\theta) &= +\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \sin(x+\theta) \\ H_y(r,\theta) &= -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos(x+\theta) \end{aligned} \tag{B.5.11}$$

or

$$\begin{aligned} H_x(r,\theta) &= +\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \left[\sin x\cos\theta + \cos x\sin\theta \right] \\ H_y(r,\theta) &= -\frac{J_0 a}{4\pi} \int_{-\pi}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \left[\cos x\cos\theta - \sin x\sin\theta \right] . \end{aligned} \tag{B.5.12}$$

Since $\int_{-\pi}^{\pi} dx$ is over an even range, throw out odd integrand terms, and *then* fold the negative range into the positive adding a factor of 2 to get

$$\begin{aligned} H_x(r,\theta) &= +\sin\theta \,\frac{J_0 a}{2\pi} \int_{0}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos x \quad \equiv \quad \sin\theta \,\frac{J_0 a}{2\pi} K \\ H_y(r,\theta) &= -\cos\theta \,\frac{J_0 a}{2\pi} \int_{0}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos x \quad = -\cos\theta \,\frac{J_0 a}{2\pi} K \end{aligned} \tag{B.5.13}$$

where

$$K \equiv \int_{0}^{\pi} dx \ln\left[r^2 + a^2 - 2ar\cos(x) \right] \cos x \,. \tag{B.5.14}$$

Before evaluating this integral, we see that

$$\mathbf{H} = H_x\,\hat{\mathbf{x}} + H_y\,\hat{\mathbf{y}} = -\frac{J_0 a}{2\pi} K\,[\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{x}}] = -\frac{J_0 a}{2\pi} K\,\hat{\boldsymbol{\theta}} = H_\theta\,\hat{\boldsymbol{\theta}} \quad . \tag{B.5.15}$$

Thus we find that the resulting **H** is entirely in the $\hat{\boldsymbol{\theta}}$ direction and

$$H_\theta = -\frac{J_0 a}{2\pi} K \quad . \tag{B.5.16}$$

We seek now to evaluate this integral K

$$K \equiv \int_0^\pi dx \ln[\, r^2 + a^2 - 2ar\cos(x)\,]\cos x$$

$$= \int_0^\pi dx \ln\,[\,\{a^2\}\{(r/a)^2 + 1 - 2(r/a)\cos(x)\}\,]\cos x$$

$$= \int_0^\pi dx\,\{\ln(a^2) + \ln[(r/a)^2 + 1 - 2(r/a)\cos(x)]\}\cos x$$

$$= \ln(a^2)[\int_0^\pi dx \cos x\,] + \int_0^\pi dx \ln[(r/a)^2 + 1 - 2(r/a)\cos(x)]\cos x$$

$$= \ln(a^2)[0] + \int_0^\pi dx \ln[\alpha^2 + 1 - 2\alpha\cos(x)]\cos x \qquad \text{where} \quad \alpha \equiv r/a$$

$$= \int_0^\pi dx \ln[\alpha^2 + 1 - 2\alpha\cos(x)]\cos x \quad . \tag{B.5.17}$$

This integral is the n=1 special case of the following integral from GR7 page 589,

6. $$\int_0^\pi \ln\left(1 - 2a\cos x + a^2\right)\cos nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \ln\left(1 - 2a\cos x + a^2\right)\cos nx\,dx$$

$$= -\frac{\pi}{n}a^n \qquad [a^2 < 1] \quad \text{BI (330)(11), BI (332)(5)}$$

$$= -\frac{\pi}{na^n} \qquad [a^2 > 1] \quad \text{GW (338)(13a)}$$

Therefore we find

$$K = \begin{cases} -\pi(r/a) & r<a \\ -\pi(a/r) & r>a \end{cases} \tag{B.5.18}$$

so

$$H_\theta = -\frac{J_0 a}{2\pi} K = \begin{cases} (aJ_0/2)\,(r/a) & r<a \\ (aJ_0/2)\,(a/r) & r>a \end{cases} .$$

(B.5.19)

Now the total current in the wire is $I = J_0 \pi a^2$ so $(aJ_0/2) = (I/2\pi a)$ and then

$$H_\theta = \begin{cases} (I/2\pi a)\,(r/a) & r<a \\ (I/2\pi a)\,(a/r) & r>a \end{cases} = \begin{cases} I\,(r/2\pi a^2) & r<a \\ I\,(1/2\pi r) & r>a \end{cases} .$$

(B.5.20)

Thus, we finally arrive at the same results for H as obtained in (B.4.2) and (B.4.1). We then evaluate H_θ on the boundary $H_\theta = I/2\pi a$ and conclude as in (B.1.11) that the surface current is $K_z = -(\mu_2/\mu_1 - 1)\, I/(2\pi a)$.

B.6 An Acrobatic Exercise: Compute H from J and J_m via the vector potential A.

For the round wire with a uniform J_z we are aware of both the volume current J_z and the surface current K_z. In this exercise, we insert *both* these currents into our formula for the vector potential **A**, we do the required integrals, and at the end we compute **B** = curl **A** and finally use **H** = **B**/μ. Hopefully in the end we shall recover once again the results for H_θ shown above. Here again is the picture of interest:

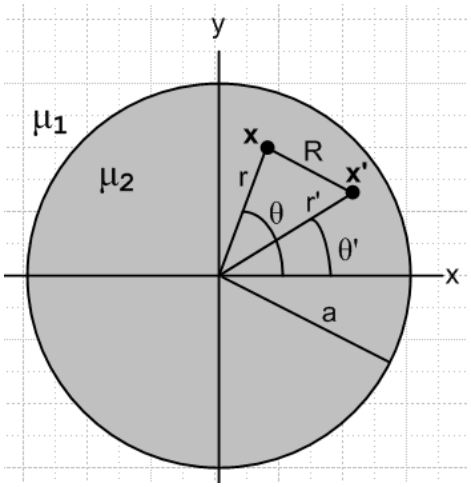

Fig B.5

Start with a 2D magnetostatics equation which is (1.5.4) with $\beta^2 = 0$,

$$\nabla^2_{2D}\mathbf{A}(\mathbf{x}) = -\mu_1 \mathbf{J_m}(\mathbf{x}) - \mu_2 \mathbf{J_c}(\mathbf{x}) . \qquad \text{(B.6.1)}$$

The particular solution from (I.1.8) is

$$\mathbf{A} = (1/2\pi) \int dV' \,[\mu_1 \mathbf{J_m}(\mathbf{x}') + \mu_2 \mathbf{J_c}(\mathbf{x}')] \ln(1/R) \qquad R = |\mathbf{x}-\mathbf{x}'|$$

$$= (1/2\pi) \{ \int dS' [\mu_1 \mathbf{K_m(x')}] \ln(1/R) + \int dV'[\mu_2 \mathbf{J_c(x')}] \ln(1/R) \}$$

so

$$A_z = (1/2\pi) \{ \int dS' [\mu_1 K_z\mathbf{(x')}] \ln(1/R) + \int dV'[\mu_2 J_{cz}\mathbf{(x')}] \ln(1/R) \}$$

$$= -(1/2\pi) \{ \int dS' [\mu_1 K_z\mathbf{(x')}] \ln(R) + \int dV'[\mu_2 J_{cz}\mathbf{(x')}] \ln(R) \}$$

$$= -(1/4\pi) \{ \mu_1 \int dS' [K_z\mathbf{(x')}] \ln(R^2) + \mu_2 \int dV'[J_{cz}\mathbf{(x')}] \ln(R^2) \} . \quad (B.6.1)$$

Here $\int dV'$ represents an integral over the "volume" of the wire slice shown in the figure (which is really a disk) and $\int dS'$ represents an integral over the "surface" of the wire slice (which is really a circle).

We now insert the uniform prescribed current J_{cz} *and* our previously computed surface current K_z,

$$J_{cz}\mathbf{(x')} = J_{cz} = I/(\pi a^2) \quad (B.6.2)$$

$$K_z\mathbf{(x')} = K_z = (1-\mu_2/\mu_1) (1/2\pi a) I \quad (B.6.3)$$

with the result that

$$A_z = - (1/4\pi) \{ \mu_1(1-\mu_2/\mu_1) (1/2\pi a) I \int dS' \ln(R^2) + \mu_2 I/(\pi a^2) \int dV' \ln(R^2) \}$$

$$= - [I/(4\pi^2 a^2)] \{ (\mu_1-\mu_2) (a/2) \int dS' \ln(R^2) + \mu_2 \int dV' \ln(R^2) \} . \quad (B.6.4)$$

The volume integral (per unit length in the z direction, so really an area integral as just noted) is,

$$\int dV' \ln(R^2) = \int_0^a r' dr' \int_{-\pi}^{\pi} d\theta' \ln [r'^2 + r^2 - 2rr' \cos(\theta-\theta')]$$

$$= 2 \int_0^a r' dr' \int_0^{\pi} d\theta' \ln [r'^2 + r^2 - 2rr' \cos(\theta-\theta')]$$

$$= 2 \int_0^a r' dr' \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x] . \quad (B.6.5)$$

Define the x integral as

$$Q(r',r) \equiv \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x] \quad (B.6.6)$$

so then

$$\int dV' \ln(R^2) = 2 \int_0^a r' dr' \; Q(r',r) . \qquad \text{(B.6.7)}$$

Meanwhile, our surface integral of interest (really a line integral) is

$$\int dS' \ln(R^2) = \int_{-\pi}^{\pi} [a \, d\theta'] \ln [r'^2 + r^2 - 2rr' \cos(\theta-\theta')] \, |^{r'=a}$$

$$= 2a \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x] \, |^{r'=a} = 2a \, Q(a,r) \qquad \text{(B.6.8)}$$

so that both integrals of interest require computation of Q(r',r), which we replicate here,

$$Q(r',r) \equiv \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x] . \qquad \textit{(B.6.6)}$$

This integral may be evaluated using GR7 p 531 4.224,

$$9.^8 \qquad \int_0^{\pi} \ln (a + b\cos x) \, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

with $a = r'^2 + r^2$ and $b = -2rr'$ and $a^2 - b^2 = (r'^2 - r^2)^2$ so that $\sqrt{a^2-b^2} = | r'^2 - r^2 |$ .The condition $a > |b| > 0$ is met since $(r \pm r')^2 > 0 \Rightarrow r^2 + r'^2 > \pm 2rr'$ which says $a > \pm b$ so $a > |b|$. Thus,

$$Q(r',r) = \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x)] = \pi \ln \left[\frac{(r'^2 + r^2) + | r'^2 - r^2 |}{2} \right] = \begin{cases} \pi \ln [r'^2] & r' > r \\ \pi \ln [r^2] & r' < r \end{cases}$$

$$= 2\pi \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases} . \qquad \text{(B.6.9)}$$

This our two integrals of interest are

$$\int dS' \ln(R^2) = 2a \, Q(a,r) = 4\pi a \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases} \qquad \text{(B.6.10)}$$

$$\int dV' \ln(R^2) = 2 \int_0^a r' dr' \; Q(r',r) = 4\pi \int_0^a r' dr' \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases} , \qquad \text{(B.6.11)}$$

where we still have a dr' integral to carry out for the $\int dV'$ case. For $r > a$ we know that $r > r'$ since r' integrates over the inside of the wire. Then

$$\int dV' \ln(R^2) = 4\pi \int_0^a r' \, dr' \ln(r) = 4\pi \ln(r) [\, a^2/2 \,] = 2\pi a^2 \ln(r) \qquad r > a \qquad (B.6.12)$$

On the other hand, for r < a we have to break the integral into two parts :

$$\int dV' \ln(R^2) = 4\pi \int_0^r r' \, dr' \ln(r) + 4\pi \int_r^a r' \, dr' \ln(r')$$

$$= 4\pi \ln(r) (r^2/2) + 4\pi [(1/2)x^2(\ln x - 1/2)]|_r^a$$

$$= 2\pi r^2 \ln(r) + 4\pi [(1/2)a^2(\ln a - 1/2) - (1/2)r^2(\ln r - 1/2)]$$

$$= 2\pi r^2 \ln(r) + 2\pi [a^2(\ln a - 1/2) - r^2(\ln r - 1/2)]$$

$$= 2\pi r^2 \ln(r) + 2\pi a^2 \ln a - \pi a^2 - 2\pi r^2 \ln r + \pi r^2$$

$$= 2\pi a^2 \ln a + \pi(r^2 - a^2) \ . \qquad r < a \qquad (B.6.13)$$

The result then is the following, where below it we repeat the earlier surface integral result,

$$\int dV' \ln(R^2) = \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases} \qquad (B.6.14)$$

$$\int dS' \ln(R^2) = 4\pi a \begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases} \ . \qquad (B.6.15)$$

Now install these integrals into the A_z expression (B.6.4) to get

$$A_z = - [I/(4\pi^2 a^2)] \{(\mu_1 - \mu_2) (a/2) \int dS' \ln(R^2) + \mu_2 \int dV' \ln(R^2) \} \qquad \textit{(B.6.4)}$$

$$= - [I/(4\pi^2 a^2)] \{(\mu_1 - \mu_2) (a/2) 4\pi a \begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases} + \mu_2 \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases} \} \qquad (B.6.16)$$

Then write these out for the two separate regions :

$$A_z(r>a) = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) (a/2) 4\pi a \ln(r) + \mu_2 2\pi a^2 \ln(r) \}$$

$$= -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) 2\pi a^2 \ln(r) + \mu_2 2\pi a^2 \ln(r) \}$$

$$= -[I/(2\pi)] \{ \mu_1 - \mu_2) \ln(r) + \mu_2 \ln(r) \}$$

$$= -[I/(2\pi)] \{ \mu_1 \ln(r) \} \qquad (B.6.17)$$

$$A_z(r<a) = -[I/(4\pi^2 a^2)] \{ \mu_1 - \mu_2) (a/2) 4\pi a \ln(a) + \mu_2(2\pi a^2 \ln(a) + \pi(r^2 - a^2)) \}$$

$$= -[I/(4\pi a^2)]\ \{\ \mu_1-\mu_2)\ 2a^2\ \ln(a) + \mu_2 2a^2 \ln(a) + \mu_2(r^2-a^2)\ )\ \ \}$$

$$= -[I/(4\pi a^2)]\ \{\ \mu_1\ 2a^2\ \ln(a) + \mu_2(r^2-a^2)\ )\ \ \}$$

$$= -[I/(2\pi)]\ \{\ \mu_1\ \ln(a) + \mu_2(r^2-a^2)/(2a^2)\ )\ \ \} \qquad \text{(B.6.18)}$$

At the boundary r = 0 both expressions give

$$A_z(r=a) = -[I/(2\pi)]\ \{\ \mu_1 \ln(a)\ \}\ . \qquad \text{(B.6.19)}$$

The next step is to compute the magnetic field **B** = curl **A** given our function $A_z(r)$. We find from the usual cylindrical coordinates curl formula (B.5.2),

$$\mathbf{B} = \text{curl}\ \mathbf{A} = -\ \partial_r A_z(r)\ \hat{\boldsymbol{\theta}}\ . \qquad \text{(B.6.20)}$$

First, for r > a one finds,

$$-\partial_r A_z(r) = \partial_r[I/(2\pi)\ \{\ \mu_1 \ln(r)\ \}] = (I\ \mu_1/2\pi)\ (1/r) \qquad r > a \qquad \text{(B.6.21)}$$

and then for r < a,

$$-\partial_r A_z(r) = I/(2\pi)\ \partial_r\ \{\ \mu_1\ \ln(a) + \mu_2(r^2-a^2)/(2a^2)\ )\ \ \} = (I\ \mu_2/2\pi)\ \mu_2\ r/a^2 \qquad \text{(B.6.22)}$$

Thus

$$\mathbf{B} = (I\ \mu_1/2\pi)\ (1/r)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r > a \text{ which is in the dielectric medium with } \mu_1$$
$$\mathbf{B} = (I\ \mu_2/2\pi)\ (r/a^2)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r < a \text{ which is in the conductor medium with } \mu_2\ . \qquad \text{(B.6.23)}$$

Using **B** = μ**H** in each region we then get

$$\mathbf{H} = (I/2\pi)\ (1/r)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r > a \qquad \Rightarrow \qquad H_\theta = (I/2\pi r)$$
$$\mathbf{H} = (I/2\pi)\ (r/a^2)\ \hat{\boldsymbol{\theta}} \qquad \text{for } r < a \qquad \Rightarrow \qquad H_\theta = (Ir/2\pi a^2) \qquad \text{(B.6.24)}$$

Amazingly, these results agree with the elementary calculation results given in (B.4.1) and (B.4.2). The important point here is that these results only come out right when both the volume and surface currents on the round wire are included in the calculation. The elementary Ampere's Law calculation of H_θ is simple because in the Maxwell equation curl **H** = **J** (Ampere's Law), the field **H** does not "see" the surface magnetization current, so one can forget about it.

Maple provides plots of A_z from (B.6.17,18), B_θ from (B.6.23) and H_θ from (B.6.24) for this round wire situation. Parameters are set to I = 1, a = 2, μ_1 = 2, μ_2 = 3.

```
restart;alias(I=I);
Az := -(I/2*Pi)*piecewise(r>a,mu1*log(r),r<=a,mu1*log(a) + mu2*(r^2-a^2)/(2*a^2));
```

$$Az := -\frac{1}{2} I \pi \left(\begin{cases} \mu 1 \ln(r) & a < r \\ \mu 1 \ln(a) + \frac{1}{2} \frac{\mu 2 \left(r^2 - a^2\right)}{a^2} & r \le a \end{cases} \right)$$

```
B := (I/2*Pi)*piecewise(r>a,mu1/r,r<=a,mu2*r/(a^2));
```

$$B := \frac{1}{2} I \pi \left(\begin{cases} \frac{\mu 1}{r} & a < r \\ \frac{\mu 2 \, r}{a^2} & r \le a \end{cases} \right)$$

```
H := (I/2*Pi)*piecewise(r>a,1/r,r<=a,r/(a^2));
```

$$H := \frac{1}{2} I \pi \left(\begin{cases} \frac{1}{r} & a < r \\ \frac{r}{a^2} & r \le a \end{cases} \right)$$

```
I := 1: a := 2: mu1 := 2: mu2 := 3:
plot([Az,B,H],r=0..8, color=[black,red,blue]);
```

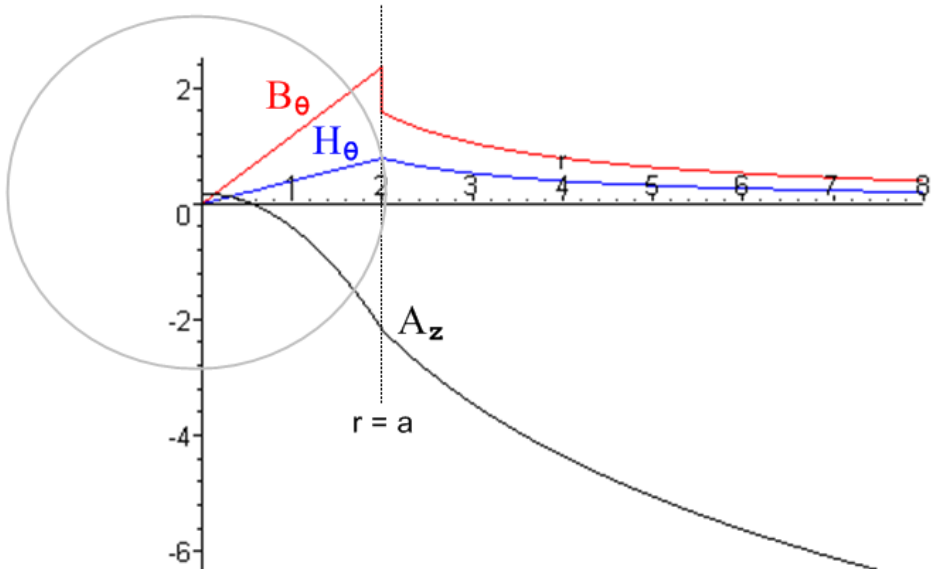

Fig B.6

A_z wanders down as ~ -ln(r) for large r, B_θ jumps at r = a while H_θ is continuous there