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C.3 The DC inductance of a round wire

We assume here that μ_i is the permeability of the wire, and μ_e of the region outside the wire.

The energy density (joules/m^3) stored in an electromagnetic field within a medium of negligible loss is given by $u_{em} = (\mathbf{E}\bullet\mathbf{D} + \mathbf{B}\bullet\mathbf{H})/2$ [Jackson p 259 Eq. (6.106)] . We are interested only in the portion of this energy density stored in the magnetic field, so $u = (1/2)\ \mathbf{B}\bullet\mathbf{H}$. Since μ and ε are assumed to be non-tensor in nature, we find that

$$u = (1/2)\ \mu\ H^2. \qquad \text{(C.3.1)}$$

For any inductor of inductance L carrying current I and having potential difference V, we know that

$$V = L\ dI/dt \qquad \text{(C.3.2)}$$

$$P = IV = I\ (L\ dI/dt) = d/dt\ [\ (1/2)L\ I^2\ ]. \qquad \text{(C.3.3)}$$

But since the power fed into an ideal inductor goes into the magnetic field energy, $P = dU/dt$ and so

$$U = (1/2)L\ I^2\ . \qquad \text{(C.3.4)}$$

For our straight wire, we now redefine symbol L to mean inductance per unit length, so then

$$U = (1/2)[Ldz]\ I^2 \qquad \text{(C.3.5)}$$

For either the internal or external region we have $U = \int d\mathcal{V}\ u = dz \int dS\ u$ where $d\mathcal{V}$ is a volume element and dS is a cross sectional area element. Thus,

$$U = dz \int dS\ (1/2)\ \mu\ H^2 = (1/2)[Ldz]\ I^2 \quad \Rightarrow \quad L = \mu \int dS\ (H/I)^2 \qquad \text{(C.3.6)}$$

In particular

$$L_i = \mu_i \int_{in} dS\ (H_i/I)^2 \qquad \text{(C.3.7)}$$

$$L_e = \mu_e \int_{out} dS\ (H_e/I)^2 \qquad \text{(C.3.8)}$$

For the round wire it is a simple matter to compute H_i and H_e using Ampere's Law (1.1.37),

$$\oint \mathbf{H} \bullet d\mathbf{s} = \int_S \mathbf{J} \bullet d\mathbf{S}$$

$$2\pi r\, H_i = I(\pi r^2/\pi a^2) \Rightarrow \quad H_i/I = \frac{1}{2\pi}(r/a^2) \ .$$

$$2\pi r\, H_e = I \quad \Rightarrow \quad H_e/I = \frac{1}{2\pi}(1/r) \qquad \text{(C.3.9)}$$

We then compute the two inductances as follows:

$$L_i = \mu_i \int dS\, (H_i/I)^2 = \mu_i \int_0^a r dr \int_{-\pi}^{\pi} d\theta \left[\frac{1}{2\pi}(r/a^2)\right]^2 = \frac{\mu_i}{2\pi a^4} \int_0^a r^3 dr = \frac{\mu_i}{8\pi}$$

$$L_e = \mu_i \int dS\, (H_e/I)^2 = \mu_e \int_a^\infty r dr \int_{-\pi}^{\pi} d\theta \left[\frac{1}{2\pi}(1/r)\right]^2 = \frac{\mu_e}{2\pi} \int_a^\infty (1/r)\, dr = \infty$$

Both results are interesting. L_i is interesting because it is independent of the wire radius a. For the same current, a smaller a results in a larger H and B field, which is then offset by the smaller volume (area).

L_e is interesting because it is infinite ! Even a tiny 1 cm piece of round wire stores an infinite amount of energy in its magnetic field. We therefore limit the external region by some large radius R and then we have

$$L_i = \frac{\mu_i}{8\pi} = \frac{\mu_i}{\mu_0}\frac{\mu_0}{8\pi} = \frac{\mu_i}{\mu_0}\frac{4\pi \times 10^{-7}}{8\pi} = \frac{\mu_i}{\mu_0} * 50 \text{ nH/m} \qquad \text{(C.3.10)}$$

$$L_e = \frac{\mu_e}{2\pi}\ln(R/a) \qquad \text{(C.3.11)}$$

Thus for a round wire in air the internal inductance is exactly 50 nH/m.

A "practical wire" is more like a loop of wire than an infinitely long wire. It is difficult to conjure up an experiment to test (C.3.11) even for a very long straight piece of wire without having some return path for the current to return to the driving "battery". For wire and dielectric both having μ_0, Jackson shows (p 216-218) that the inductance per unit length of a loop of projected area A of radius-a wire is given by $L_e + L_i = (\mu_0/4\pi)\,[\ln(\xi A/a^2) + 1/2]$, where ξ is a near-unity factor which accounts for messy details of the calculation. The 1/2 term accounts for the internal inductance $L_i = \mu_0/8\pi$ as in (C.3.10).

For a *circular* loop of radius R, one has $A = \pi R^2$ and, if $R >> a$, $\xi = 64/(\pi e^4) \approx .373$. So,

$$\ln(\xi A/a^2) = \ln(64\, \pi R^2/\pi e^4 a^2) = 2\ln(8R/ae^2) = 2\ln(8R/a) + 2\ln(e^{-2}) = 2\ln(8R/a) - 4$$

and then the total inductance per unit length is

$$L_e + L_i = (\mu_0/4\pi)\,[2\ln(8R/a) - 4 + 1/2] = (\mu_0/2\pi)\,[\ln(8R/a) - 2 + 1/4] = (\mu_0/2\pi)\,[\ln(8R/a) - 7/4] \ .$$

The *total* inductance of such a loop is then

$$L = 2\pi R(\mu_0/2\pi)\,[\,\ln(8R/a) - 7/4\,] \;= \mu_0 R\,[\,\ln(8R/a) - 7/4\,] \qquad \text{(C.3.12)}$$

in agreement with Jackson Problem 5.32 p 234. If one omits the internal inductance, the last factor is -2 instead of -7/4, and this result is seen in some sources. The point is that this is a finite result, even though the (dipole) magnetic field of such a loop does extend to infinity. A loop of N turns gets an extra factor N^2 because in effect current $I \rightarrow NI$ in (C.3.5), so the total field energy increases by factor N^2.

Suppose there were two parallel wires with currents flowing in opposite directions. In this case, we could compute the magnetic field H at any point in space as the vector sum of the fields of the two wires, then we could integrate H^2 over all space to get the total energy U and from that the external inductance L_e. In this case, the ln(R) divergence does not appear. In effect, the divergence cancels between the two wires, similar to the way opposite short segments of the circular wire cancel to give the finite result quoted above. Since we will be doing this computation by another means in the main text, we do not bother with the parallel wire calculation here.

We really only care about the internal inductance L_i of a wire in our transmission line analysis because the external inductance L_e is already accounted for by the techniques of Chapter 4. That is, L_e is computed by considering the magnetic potential A_z (or W) between the wires.

C.4 The DC inductance of a wire of arbitrary cross section

The expressions above apply to any cross sectional shape,

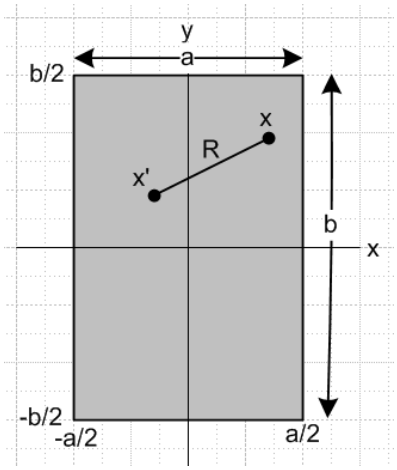
$$L_i = \mu_i \int_{in} dS\,(H_i/I)^2 \qquad \textit{(C.3.7)}$$

$$L_e = \mu_e \int_{out} dS\,(H_e/I)^2 \qquad \textit{(C.3.8)}$$

so the only problem is how to compute H for a non-round wire. That problem is solved in Appendix B where it is shown that

$$\mathbf{H}(x,y) = -\frac{1}{4\pi}\int d^2\mathbf{x}'\,\ln(R^2)\;\text{curl}'\,\mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \;. \qquad \textit{(B.2.6)} \qquad \text{(C.4.1)}$$

We can do a wire of rectangular cross section a x b (uniform J_z) as an example.



The current density is given by

$$J_z(x) = (I/ab)\theta(-a/2 \le x \le a/2)\ \theta(-b/2 \le y \le b/2)$$

$$= (I/ab)\ \theta(x \le a/2)\theta(x \ge -a/2)\ \theta(y \le b/2)\theta(y \ge -b/2) \qquad // \ \theta(s \ge r) \text{ means } \theta(s-r) \text{ Heaviside}$$

$$= (I/ab)\ \theta(a/2 - x)\theta(x + a/2)\ \theta(b/2 - y)\theta(y + b/2) \qquad \text{(C.4.2)}$$

Then calculate curl **J** :

$$\text{curl } \mathbf{J} = \hat{\mathbf{x}}\ (\partial_y J_z - \partial_z J_y) + \hat{\mathbf{y}}\ (\partial_z J_x - \partial_x J_z) + \hat{\mathbf{z}}\ (\partial_x J_y - \partial_y J_x)$$

$$= \hat{\mathbf{x}}\ (\partial_y J_z) + \hat{\mathbf{y}}\ (-\partial_x J_z) \qquad \text{(C.4.3)}$$

$$\partial_x J_z = (I/ab)\ \theta(b/2 - y)\theta(y + b/2)\ \partial_x\ [\theta(a/2 - x)\theta(x + a/2)] \qquad // \ \partial_x \theta(a-x) = -\ \delta(x-a)$$

$$= (I/ab)\ \theta(b/2 - y)\theta(y + b/2)\ [\theta(a/2 - x)\delta(x + a/2) - \theta(x + a/2)\delta(x - a/2)\ ]$$

Swap x↔y and a↔b to get

$$\partial_y J_z = (I/ab)\ \theta(a/2 - x)\theta(x + a/2)\ [\theta(b/2 - y)\delta(y + b/2) - \theta(y + b/2)\delta(y - b/2)\ ]$$

so then

$$[\text{curl}J]_x = (I/ab)\ \theta(a/2 - x)\theta(x + a/2)\ [\theta(b/2 - y)\delta(y + b/2) - \theta(y + b/2)\delta(y - b/2)\ ]$$
$$[\text{curl}J]_y = -\ (I/ab)\ \theta(b/2 - y)\theta(y + b/2)\ [\theta(a/2 - x)\delta(x + a/2) - \theta(x + a/2)\delta(x - a/2)\ ] \qquad \text{(C.4.4)}$$

Finally calculate H from (C.4.1):

$$H_x(x,y) = -\frac{1}{4\pi} \int d^2\mathbf{x}'\ \ln(R^2)\ \ [\text{curl}'\ \mathbf{J}(\mathbf{x}')]_x \qquad R = |\mathbf{x} - \mathbf{x}'|$$

$$= -\frac{1}{4\pi} \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy' \ \ln\left[(x-x')^2 + (y-y')^2 \right] \ (I/ab) \left[\delta(y'+b/2) - \delta(y'-b/2) \right]$$

$$= -\frac{1}{4\pi} \ (I/ab) \int_{-a/2}^{a/2} dx' \left\{ \ln\left[(x-x')^2 + (y+b/2)^2 \right] - \ln\left[(x-x')^2 + (y-b/2)^2 \right] \right\}$$

$$= -\frac{1}{4\pi} \ (I/ab) \int_{-a/2}^{a/2} ds \ln\left[\frac{(x-s)^2 + (y+b/2)^2}{(x-s)^2 + (y-b/2)^2} \right]$$

$$\equiv -\frac{1}{4\pi} \ (I/ab) \ F(x,y,a,b)$$

We shall evaluate this F integral below. Meanwhile, the expression for $H_{\mathbf{x}}(x,y)$ is

$$H_{\mathbf{y}}(x,y) = -\frac{1}{4\pi} \int d^2\mathbf{x}' \ln(R^2) \ [\text{curl}' \ \mathbf{J}(\mathbf{x}')]_{\mathbf{y}} \qquad R = |\mathbf{x}-\mathbf{x}'|$$

$$= \frac{1}{4\pi} \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy' \ \ln\left[(x-x')^2 + (y-y')^2 \right] \ (I/ab) \left[\delta(x'+a/2) - \delta(x'-a/2) \right]$$

$$= \frac{1}{4\pi} \ (I/ab) \int_{-b/2}^{b/2} dy' \left\{ \ln\left[(x+a/2)^2 + (y-y')^2 \right] - \ln\left[(x-a/2)^2 + (y-y')^2 \right] \right\}$$

$$= \frac{1}{4\pi} \ (I/ab) \int_{-a/2}^{a/2} ds \ln\left[\frac{(y-s)^2 + (x+a/2)^2}{(y-s)^2 + (x-a/2)^2} \right]$$

$$= \frac{1}{4\pi} \ (I/ab) \ F(y,x,b,a)$$

To summarize

$$H_{\mathbf{x}}(x,y) = -\frac{1}{4\pi} \ (I/ab) \ F(x,y,a,b)$$

$$H_{\mathbf{y}}(x,y) = +\frac{1}{4\pi} \ (I/ab) \ F(y,x,b,a) \qquad \text{(C.4.5)}$$

The function F(x,y,a,b) is rather unpleasant, although expressible in closed form. Maple computes F as follows:

```
restart; alias(I=I, j= sqrt(-1)): with(plots): with(plottools):
FF := Int(ln( ((x-s)^2+(y+b/2)^2)/((x-s)^2+(y-b/2)^2)),s=-a/2..a/2);
```

$$FF := \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \ln\left(\frac{(x-s)^2+\left(y+\frac{1}{2}b\right)^2}{(x-s)^2+\left(y-\frac{1}{2}b\right)^2}\right) ds$$

```
t := value(FF): f := collect(%,arctan):
F := unapply(f,x,y,a,b);  # create the function F(x,y,a,b)
```

$$:= (x,y,a,b) \to (2y+b)\arctan\left(\frac{-2x+a}{2y+b}\right)+(-2y+b)\arctan\left(\frac{-2x+a}{2y-b}\right)+(2y+b)\arctan\left(\frac{2x+a}{2y+b}\right)+(-2y+b)\arctan\left(\frac{2x+a}{2y-b}\right)$$
$$+x\ln(4x^2-4xa-4yb+a^2+4y^2+b^2)-x\ln(4x^2-4xa+4yb+a^2+4y^2+b^2)+\frac{1}{2}a\ln\left(\frac{4x^2+4xa+4yb+a^2+4y^2+b^2}{4x^2+4xa-4yb+a^2+4y^2+b^2}\right)$$
$$+\frac{1}{2}a\ln\left(\frac{4x^2-4xa+4yb+a^2+4y^2+b^2}{4x^2-4xa-4yb+a^2+4y^2+b^2}\right)+x\ln(4x^2+4xa+4yb+a^2+4y^2+b^2)-x\ln(4x^2+4xa-4yb+a^2+4y^2+b^2)$$

```
Hx := -(I/(4*Pi*a*b))*F(x,y,a,b):
Hy := +(I/(4*Pi*a*b))*F(y,x,b,a):
```

The 2nd line displays the integral of interest as FF. The t = value(FF) causes Maple to do the integration analytically, then the collect command puts the four arctan terms first (cosmetic). The unapply command causes the expression f to be interpreted as F(x,y,a,b). The last two lines then compute H_x and H_y.

The internal inductance of the rectangular wire is then given by

$$L_i = \mu_i \int dS\, (H_i/I)^2 = \mu_i\left(\frac{1}{4\pi ab}\right)^2 \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, [F(x,y,a,b)^2 + F(y,x,b,a)^2] \quad . \tag{C.4.6}$$

Since $F(x,y,\alpha a,\alpha b) = \alpha F(x/\alpha, y/\alpha, a, b)$ one can show that L_i must have this form

$$L_i = (\mu_i/8\pi)\, \ell(b/a) \tag{C.4.7}$$

though that conclusion is obvious based on dimensions alone. The problem is to find function ℓ

$$\ell(b/a) = (8\pi/\mu_i)\, \mu_i\left(\frac{1}{4\pi ab}\right)^2 \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, [F(x,y,a,b)^2 + F(y,x,b,a)^2]$$

$$= \frac{1}{2\pi a^2 b^2} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, [F(x,y,a,b)^2 + F(y,x,b,a)^2]$$

$$\ell(b) = (8\pi/\mu_i)\, \mu_i\left(\frac{1}{4\pi b}\right)^2 \int_{-1/2}^{1/2} dx \int_{-b/2}^{b/2} dy\, [F(x,y,1,b)^2 + F(y,x,b,1)^2] \,. \tag{C.4.8}$$

Notice that $\ell(b/a) = \ell(a/b)$, which seems also clear on physical grounds, so $\ell(b) = \ell(1/b)$.

We were not successful (in a few minutes) trying to get Maple to compute this integral analytically or numerically, but certainly the numerical integration can be done. De Smedt quotes the following numerical approximate formulas for two special cases:

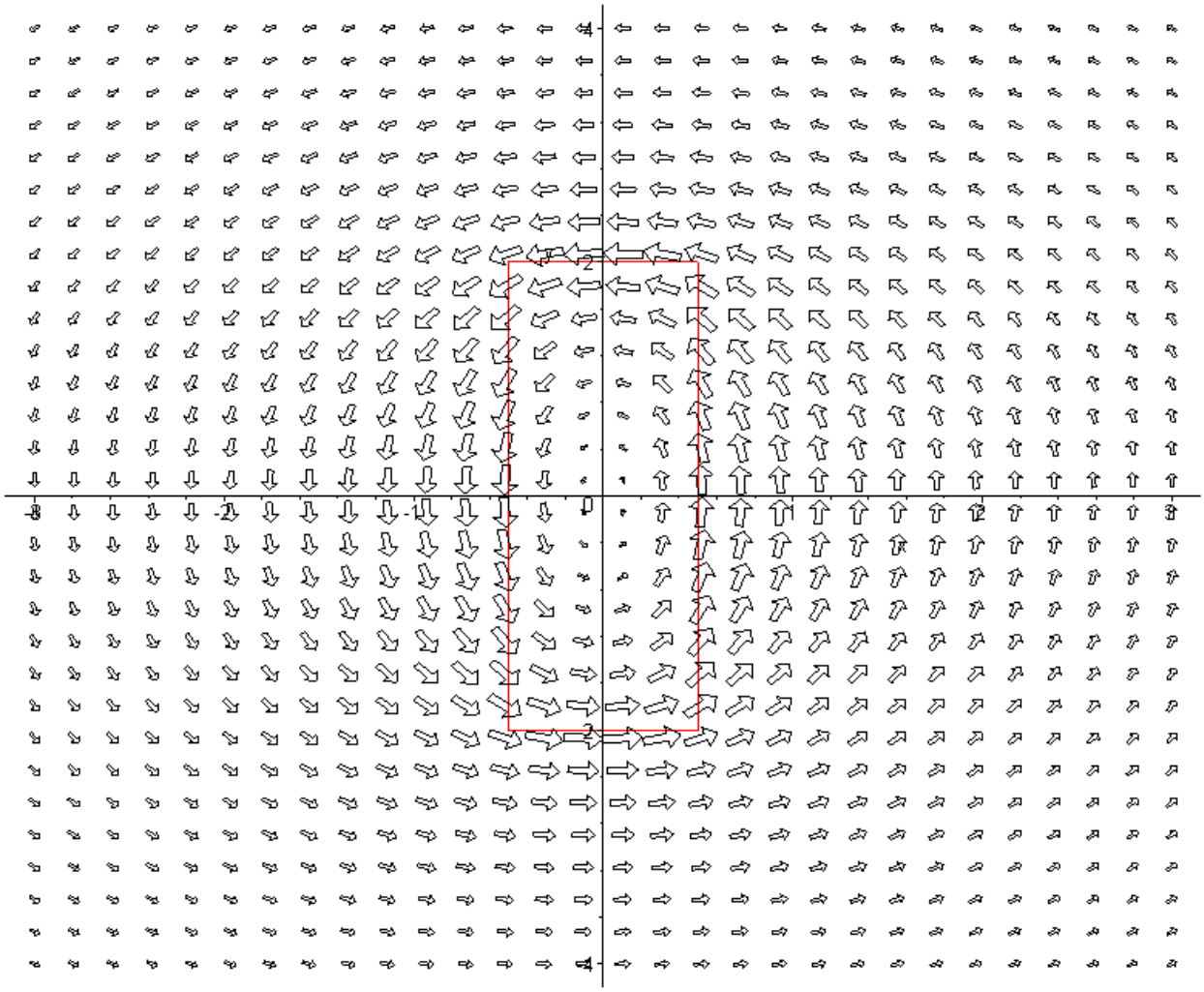
$L_i = (\mu_i/8\pi)$ [0.96639] a = b (square wire) // very close to the round conductor

$L_i = (\mu_i/8\pi)$ [$(4\pi/3)$ a/b] a/b << 1 (flat wire) // ≈ $(\mu_i/8\pi)$ [4.2 a/b] (C.4.9)

Reader Exercise: Do the numerical integration (C.4.8) to determine function $\ell(b)$ for several b values and plot for b = 0 to 1. Is $\ell(1)$ = 0.96639 ?

We now plot the **H** field for the rectangular conductor in two ways. First, setting a = 1, b = 2, I = 1:

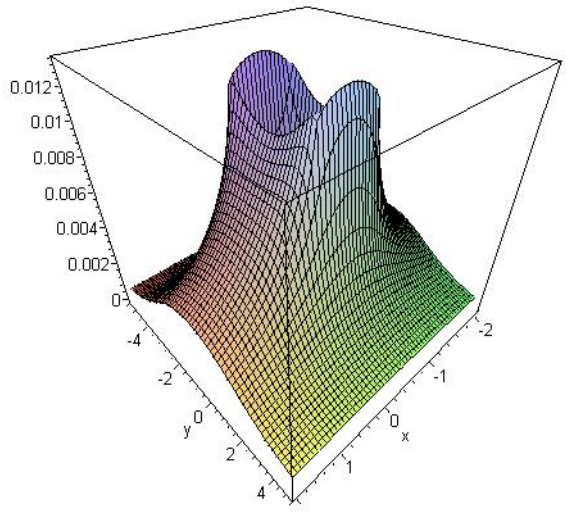
```
a := 1: b := 4: I := 1:
p1 := fieldplot([Hx,Hy],x=-3..3,y=-4..4,arrows=THICK,grid = [30,30]):
p2 := PLOT(CURVES([[a/2,b/2],[a/2,-b/2],[-a/2,-b/2],[-a/2,b/2],[a/2,b/2]],COLOR(RGB,1,0,0))):
display(p1,p2);
```



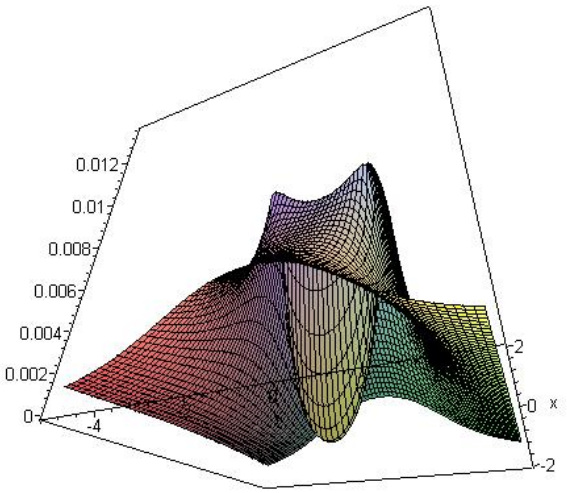
In this plot the H field appears to be maximal at the conductor boundary (shown in red) and as one moves away it becomes the field of a thin round wire. Current J_z is flowing in the z direction toward the viewer.

The second plot is of $|\mathbf{H}|^2$ as a surface over the x,y plane:

```
plot3d(Hx^2+Hy^2,x=-2..2,y=-5..5, grid = [60,60],projection = normal,axes=boxed);
```



view from above

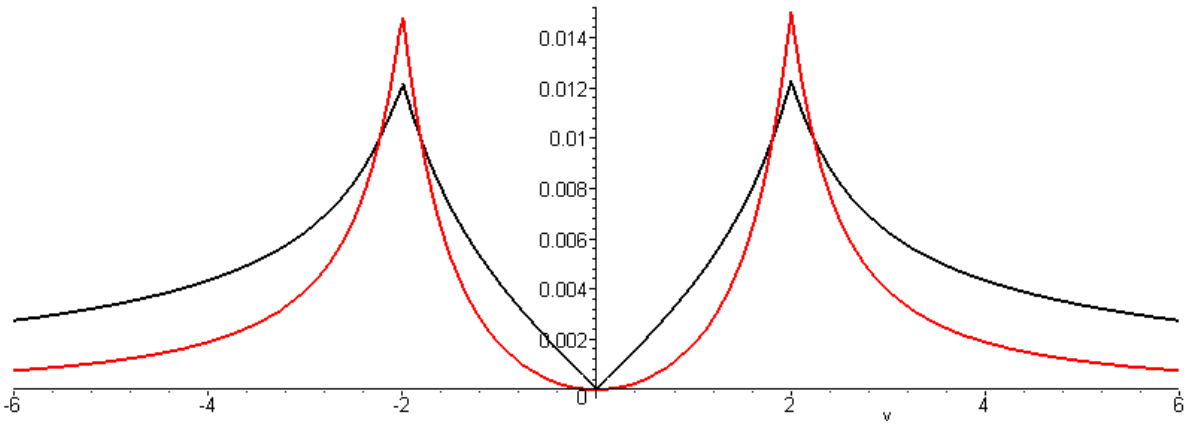


view from below

The H field surface is very steep at the conductor boundaries, somewhat resembling a rectangular volcano which dips all the way down to 0 in the center, as shown on the right. The L_i integration is over this central part of the volcano.

The red plot below is a slice through the volcano at x = 0 :

```
x := 0: plot([Hx^2+Hy^2,(0.1)*sqrt(Hx^2+Hy^2)],y=-6..6, color = [red,black],thickness=2);
```



The black plot is of | **H** | (but scaled down) and resembles the H_θ plot for the round wire shown in Fig B.8.