Perhaps this is my original version of this section where I still have the "unrolling the shell" section included. I later had to drop this section. Only later did I add my discovery of external verification from an old book I found on line.

C.6 The DC internal inductance of a hollow pipe

The pipe geometry is as follows, where we assume J_z is uniform and total current is I :

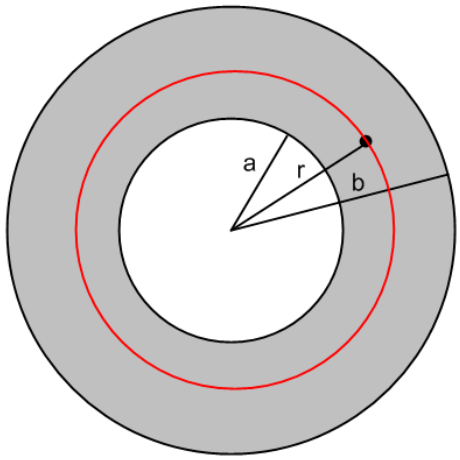


Fig C.7

As with the round wire case, we can avoid using (C.4.1) to compute H due to symmetry. The total current enclosed within the red circle is this (area = πr^2 for any disk),

$$I_{enc}(r) = \frac{\text{area inner annulus}}{\text{area full annulus}} = \frac{r^2 - a^2}{b^2 - a^2} I \qquad \text{valid for } a \le r \le b \ . \qquad \text{(C.6.1)}$$

For $r < a$ $I_{enc}(r) = 0$, and for $r > b$ $I_{enc}(r) = I$. Ampere's Law says

$$2\pi r H_\theta(r) = I_{enc}(r) \ . \qquad \text{(C.6.2)}$$

In passing, note that $H_\theta(r) = 0$ inside the pipe, so this region makes no contribution to L_e or L_i. Inside the annulus,

$$H_\theta(r) = I_{enc}(r)/(2\pi r) = \frac{1}{2\pi}\frac{1}{r}\frac{r^2 - a^2}{b^2 - a^2} I \ . \qquad a \le r \le b \qquad \text{(C.6.3)}$$

Recalling that

$$L_i = \mu_i \int_{in} dS\, (H/I)^2 \qquad \textit{(C.3.7)}$$

we conclude that

$$L_i = \mu_i \frac{1}{(2\pi)^2} \frac{1}{(b^2-a^2)^2} \int_i dS \frac{1}{r^2} (r^2 - a^2)^2$$

and then using $dS = 2\pi r dr$ we find

$$L_i = \mu_i \frac{1}{2\pi} \frac{1}{(b^2-a^2)^2} \int_a^b dr \frac{1}{r} (r^2 - a^2)^2$$

$$= \mu_i \frac{1}{2\pi} \frac{1}{(b^2-a^2)^2} J \ . \qquad \text{(C.6.4)}$$

Maple computes the integral J as follows

```
J := Int((r^2-a^2)^2/r,r=a..b);
```

$$J := \int_a^b \frac{(r^2 - a^2)^2}{r} dr$$

```
J := int((r^2-a^2)^2/r,r=a..b);
```

$$J := \frac{1}{4} b^4 - b^2 a^2 + a^4 \ln(b) + \frac{3}{4} a^4 - a^4 \ln(a)$$

which we restate as

$$J = (1/4)b^4 + (3/4)a^4 - a^2b^2 + a^4 \ln(b/a) \qquad \text{(C.6.5)}$$

and then the final result is

$$L_i = \mu_i \frac{1}{2\pi} \frac{1}{(b^2-a^2)^2} [(1/4)b^4 + (3/4)a^4 - a^2b^2 + a^4 \ln(b/a)] \ . \qquad \text{(C.6.6)}$$

(a) limit as a→ 0: should be round wire of radius b

We can read off this limit from (C.6.6)

$$L_i = \mu_i \frac{1}{2\pi} \frac{1}{(b^2-a^2)^2} [(1/4)b^4 + (3/4)a^4 - a^2b^2 + a^4 \ln(b/a)]$$

$$= \mu_i \frac{1}{2\pi} \frac{1}{b^4} [(1/4)b^4 + 0 - 0 + 0 \ln(b/a)]$$

$$= \mu_i \frac{1}{8\pi} \qquad \text{(C.6.7)}$$

and we recover the L_i of a round wire as found in (C.3.10).

(b) limit as b-a → 0: thin shell radius a and thickness d

In this limit, the hollow pipe is a thin cylindrical shell of inner radius a and thickness d. Continuing the Maple code, we find

```
x := mu*(1/(2*Pi*(b^2-a^2)^2));
```

$$x := \frac{1}{2}\frac{\mu}{\pi\left(b^2-a^2\right)^2}$$

```
LI := x*J;
```

$$LI := \frac{1}{2}\frac{\mu\left(\frac{1}{4}b^4 - b^2a^2 + a^4\ln(b) + \frac{3}{4}a^4 - a^4\ln(a)\right)}{\pi\left(b^2-a^2\right)^2}$$

```
b := a+d;
```

$$b := a + d$$

```
LI;
```

$$\frac{1}{2}\frac{\mu\left(\frac{1}{4}(a+d)^4 - (a+d)^2a^2 + a^4\ln(a+d) + \frac{3}{4}a^4 - a^4\ln(a)\right)}{\pi\left((a+d)^2-a^2\right)^2}$$

Maple then expands this L_i function about d = 0,

```
series(LI,d=0,4);
```

$$\frac{1}{6}\frac{\mu}{\pi a}d - \frac{1}{6}\frac{\mu}{\pi a^2}d^2 + \frac{3}{20}\frac{\mu}{\pi a^3}d^3 + O(d^4)$$

Of course our only interest is in the first term, so in this limit we have found that

$$L_i = \mu_i \frac{1}{6\pi}(d/a) = \frac{\mu_i}{8\pi}[(4/3)(d/a)] \qquad \text{thin shell, valid for } d << a \qquad (C.6.8)$$

where again $(\mu_i/8\pi)$ is L_i for a round conductor of any radius.

(c) Unrolling the thin shell?

Suppose we were to unroll the thin shell of the previous section to arrive at a wire whose cross section was a strip of with w = 2πa and thickness d,

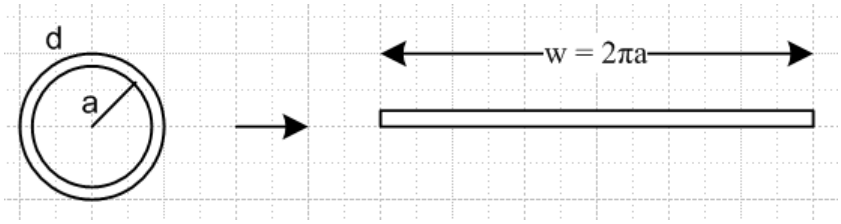
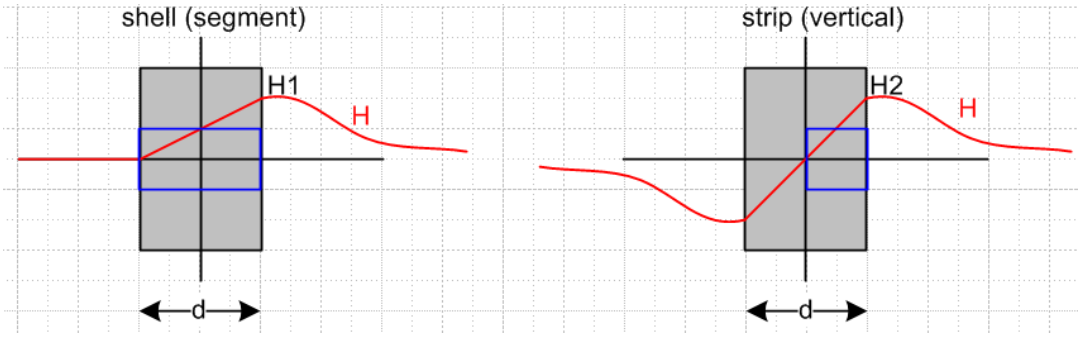


Fig C.8

If this unrolling could be done without affecting L_i, we would then have an expression for L_i for the strip, which would be

$$L_i = \frac{\mu_i}{8\pi}\,[\,(4/3)(d/a)\,] \;=\; = \frac{\mu_i}{8\pi}\,[\,(4/3)(2\pi d/w)\,] \;=\; \frac{\mu_i}{8\pi}\,[\,(8\pi/3)(d/w)\,] \qquad //\ \text{"if ..."}$$

Although the current density J_z is the same for rolled and unrolled, the H field pattern is in fact not the same in the two situations:



$J_z = I/(wd)$ $\qquad w = 2\pi a$ $\qquad\qquad$ $J_z = I/(wd)$ $\qquad\qquad$ Fig C.9

In the above figures we plot the H field in red, giving it linear form inside the conductor. This linear form is justified for the thin shell from (C.3.6),

$$H_\theta(r) = I_{enc}(r)/\,(2\pi r) \;= \frac{1}{2\pi}\,\frac{1}{r}\,\frac{r^2-a^2}{b^2-a^2}\;I \;\approx\; \frac{1}{2\pi}\,\frac{1}{a}\,\frac{(r+a)(r-a)}{(b+a)(b-a)}\;I \approx \frac{1}{2\pi}\,\frac{1}{a}\,\frac{r-a}{b-a}\;I \;=\; \frac{1}{2\pi}\,\frac{1}{a}\,\frac{r-a}{d}\;I \qquad (C.6.9)$$

For the wide strip, we found the linear form H(x) = Ix/(wd) in (C.5.2). Ignoring these specific expressions and just using the fact that H is linear inside the conductor in both cases, and using the J_x expressions above, we will now show that the thin shell has four times more L_i than the equivalent strip.

For each of the blue loops in Fig * apply Ampere's Law:

$$\text{shell:} \qquad H_1 s = J_z\ sd \quad \Rightarrow \quad H_1 = J_z d = I/2\pi a = I/w \quad \Rightarrow \quad H_{shell}(x) = H_1\,(\frac{x}{d} + \frac{1}{2})$$

$$\text{strip:} \qquad H_2 s = J_z\ s(d/2) \;\Rightarrow \quad H_2 = J_z d/2 = I/(2w) \quad \Rightarrow \quad H_{strip}(x) = H_2\,(2\,\frac{x}{d})$$

(C.6.10)

Then,

$$f_1 = H_1^2 \int_{-d/2}^{d/2} dx \left(\frac{x}{d} + \frac{1}{2}\right)^2 = H_1^2 \ (d/3) = (I/w)^2 (d/3) = (1/3)\, I\, (d/w^2) \qquad \text{// shell}$$

(C.6.11)

$$f_2 = H_2^2 \int_{-d/2}^{d/2} dx \ \left(2\frac{x}{d}\right)^2 = H_2^2 \ (d/3) = [I/(2w)]^2 (d/3) = (1/12)\, I\, (d/w^2) \qquad \text{// strip}$$

Thus $f_1 = 4f_2$. Since $L_i = \mu_i\, w\, (f_i/I^2)$ we conclude that $L_1 = 4L_2$. This is consistent with our results obtained above,

$$L_i = \mu_i \frac{1}{6\pi} (d/a) = \frac{\mu_i}{8\pi} [\, (4/3)(d/a)\,] \qquad \text{thin shell, valid for } d << a \qquad \textit{(C.6.8)}$$

$$L_i = \frac{\mu_i}{8\pi} [\, 2\pi/3\ (d/w)\,] = \frac{\mu_i}{8\pi} [\, (1/3)\ (d/a)\,] \qquad \text{// strip } w = 2\pi a >> d \qquad \textit{(C.5.3)}$$

As noted earlier, this strip result is half the value quoted in (C.4.11).