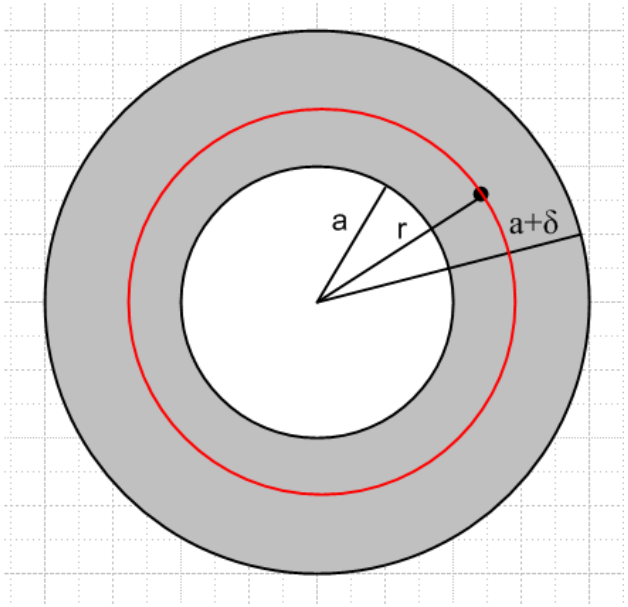
This is the Title PhL 3.26.05

This is now installed in Appendix C, and I found external verification for the result in a 1922 book!

A new problem:

Consider a cylindrical shell conductor of inner radius a and thickness δ. What is the DC internal inductance of such a shell?



First, compute the H field. The total area of the shell is this

$$A = \pi(a+\delta)^2 - \pi a^2 \quad = \pi\left[(a+\delta)^2 - a^2\right]$$

If we assume a constant current density in this shell, we find that

$$J_z(r) = I/A \; = \frac{I}{\pi\left[(a+\delta)^2-a^2\right]}$$

The current enclosed by a loop of radius r is given by

$$I_{enc}(r) \;=\; \int dS\, J_z = 2\pi \int_a^r r\,dr\, J_z(r)dS \;=\; 2\pi \frac{I}{\pi\left[(a+\delta)^2-a^2\right]} \int_a^r r'\,dr'$$

$$= 2\pi \frac{I}{\pi\left[(a+\delta)^2-a^2\right]} \left[r^2-a^2\right](1/2) \;=\; \frac{r^2-a^2}{\left[(a+\delta)^2-a^2\right]}\, I$$

The current enclosed at radii ranging from a to a+δ is this

$$I_{enc}(r) = \frac{r^2 - a^2}{[(a+\delta)^2 - a^2]}\ I \qquad\qquad I_{enc}(r) = \frac{r^2}{\delta^2}\ I$$

which I could have started off with, it is a completely obvious fact.

Now compute $H_\theta(r)$

$$2\pi r H(r) = \text{current enclosed} = I_{enc}(r) = \frac{r^2 - a^2}{[(a+\delta)^2 - a^2]}\ I \qquad \text{dim OK}$$

$$H(r) = \frac{r^2 - a^2}{[(a+\delta)^2 - a^2]}\ (I/2\pi r)$$

$$u = (1/2)\mu_1 H^2 = (1/2)\ \mu_1 \left(\frac{1}{[(a+\delta)^2 - a^2]}\right)^2\ \ (1/2\pi)^2\ I^2\ \frac{(r^2-a^2)^2}{r^2}$$

$$U = \int dS\ u\ = 2\pi\ \int_a^{a+\delta} dr\ r\ u\ =\ \ (1/2)\ \mu_1 \left(\frac{1}{[(a+\delta)^2 - a^2]}\right)^2\ \ (1/2\pi)\ I^2\ \int_a^{a+\delta} dr\ r\ \frac{(r^2-a^2)^2}{r^2}$$

Maple does the integral

```
Int( (r^2-a^2)^2/r,r=a..a+delta);
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$$\int_a^{a+\delta} \frac{(r^2 - a^2)^2}{r}\, dr$$

```
J := int( (r^2-a^2)^2/r,r=a..a+delta);
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$$J := -a^3\delta + \frac{1}{2}a^2\delta^2 + a\,\delta^3 + \frac{1}{4}\delta^4 + a^4\ln(a+\delta) - a^4\ln(a)$$

```
delta := b*a: J := int( (r^2-a^2)^2/r,r=a..a+delta);
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$$J := -b\,a^4 + \frac{1}{2}b^2a^4 + b^3a^4 + \frac{1}{4}b^4a^4 + a^4\ln(a\,(1+b)) - a^4\ln(a)$$

which result we re-express in terms of the ratio $b \equiv \delta/a$,

```
delta := b*a: J := int( (r^2-a^2)^2/r,r=a..a+delta);
```

$$J := -b\,a^4 + \frac{1}{2}b^2a^4 + b^3a^4 + \frac{1}{4}b^4a^4 + a^4\ln(a\,(1+b)) - a^4\ln(a)$$

Given this expression for J, we have

$$U = (1/2)\ \mu_1 \left(\frac{1}{[(a+\delta)^2 - a^2]}\right)^2\ \ (1/2\pi)\ I^2\ \ J$$

$$= (1/2)\ \mu_1 a^{-4} \left(\frac{1}{[(1+b)^2 - 1]}\right)^2\ \ (1/2\pi)\ I^2\ \ J\ \ = (1/2)\ \mu_1 a^{-4} \left(\frac{1}{[b^2(2+b)^2]}\right)\ \ (1/2\pi)\ I^2\ \ J$$

and then

$$U = (1/2) L I^2$$

so

$$L = \mu_1 a^{-4} \left(\frac{1}{[b^2(2+b)^2]} \right) (1/2\pi)\, J(b)$$

In the limit $a \to 0$ we have $b \to \infty$ and then $J(b) = (b^4 a^4)/4$ so

$$L = \mu_1 a^{-4}\, b^{-4}\ (1/2\pi)\ (b^4 a^4)/4 = \mu_1\, (1/8\pi)$$

which is the correct result for a solid round wire.

For a thin cylindrical shell, we instead take $b \to 0$. Maple tells us that

```
K := J/(b^2*(2+b)^2):
series(K,b=0,4);
```

$$\frac{1}{3} a^4 b + O(b^2)$$

so that in this limit we get

$$L_i = \mu_1 a^{-4}\ (1/2\pi)\ \{ J(b) \frac{1}{[b^2(2+b)^2]} \} = \mu_1 a^{-4}\, (1/2\pi)\, (a^4 b/3) = \mu_1\, (1/2\pi)\, (b/3) = (\mu_1/6\pi) b$$

$$= (\mu_1/6\pi)(\delta/a) = (\mu_1/8\pi)(4/3)(\delta/a)$$

Now if I crudely think of this as a rolled up microstrip of width $w = 2\pi a$ then I am getting

$$L_i = (\mu_1/8\pi)\, (4/3)\, (2\pi\delta/w) = (\mu_1/8\pi)\, [8\pi/3]\, (\delta/w)$$

This is not too different from the quoted result for an unrolled microstrip,

$$L_i = (\mu_i/8\pi)\, [(4\pi/3)\, a/b\,] \quad a/b << 1 \text{ (flat wire)} \qquad // \approx (\mu_i/8\pi)\, [\, 4.2\, a/b\ ] \qquad \text{(C.4.11)}$$

I think my result is the right one!

It is true that the flat wire and the annulus have different H field distributions so that might account for the difference.

Amazingly I found as PDF which has these quotes

Following up, the above lightning interaction applies only to the system level ground-test with a nearby return current creating an external inductance between the differential mode currents. In-flight, there is no return current and the inductance of the common mode cable currents is the internal inductance. For a conducting cylinder of length, *l*, radius, *R*, thickness, *t*, permeability, μ, and conductivity, σ, the low frequency ($\omega \cdot \tau_d = \omega \cdot \mu \cdot \sigma \cdot t^2 < 1$) internal inductance is as follows:[1]

(1) $L_{int} = \frac{\mu_0 \cdot l}{4 \cdot \pi} * \frac{t}{R}$.

The high frequency ($\omega \cdot \tau_d = \omega \cdot \mu \cdot \sigma \cdot t^2 \geq 1$) internal inductance is as follows:

(2) $L_{int} = \frac{\mu_0 \cdot l}{4 \cdot \pi} \cdot \frac{\delta}{R}$, where $\delta = \sqrt{1/\pi \cdot f \cdot \mu \cdot \sigma}$ is the skin depth.

My result was $\mu_1(1/4\pi)[\ 3\text{-}4\ln 2]\ (\delta/a)$, but their result is $\mu_1(1/4\pi)\ (\delta/a)$. They give this reference for their results

1. Schelkunoff, S. A., "The Electromagnetic Theory of Coaxial Transmission Lines and Cylindrical Shields", The Bell System Technical Journal, Volume XIII, 1934, 532-579

and I think I have access to that stuff now. I download the thing and save it. I then do OCR with Acrobat so I can search the thing. None of them come searchable for some reason. It is 48 pages long. But I don't see this formula in there. The result quoted does match my Appendix K high frequency result

$$L_i(\omega) = (1/\omega)\ \mathrm{Im}(Z_s) = (1/\omega)\ \frac{1}{\sigma_1(2\pi a_1)\delta_1} = \mu_1\sigma_1\delta^2/2\ \frac{1}{\sigma_1(2\pi a_1)\delta_1} = \mu_1\ (1/4\pi)\ (\delta_1/a_1)$$

except the quote shows some R' thing. I cannot find anything like this in that PDF! But I know exactly what they are saying. So no need for The BSTJ thing.

There is no H field inside my cylinder by the way. I think my result is correct and that airline PDF paper is a red herring result that is wrong.

So I will redo each step in the a = 0 limit and see where it disagrees with the round wire step.