C.4 The DC inductance of a wire with rectangular cross section

The internal inductance expressions above apply to *any* cross sectional shape,

$$L_i = \mu_i \int_{in} dS\ (H_i/I)^2 \qquad (C.3.7)$$

$$L_e = \mu_e \int_{out} dS\ (H_e/I)^2 \qquad (C.3.8)$$

so the only problem is how to compute H for a non-round wire. That problem is solved in Appendix B where it is shown that

$$\mathbf{H}(x,y) = -\frac{1}{4\pi}\int d^2\mathbf{x}'\ \ln(R^2)\ \mathrm{curl}'\ \mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \ . \qquad (B.2.6) \qquad (C.4.1)$$

We can do a wire of rectangular cross section 2a x 2b (uniform J_z) as an example.

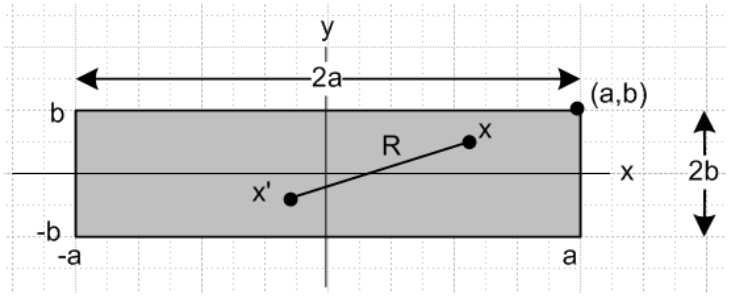


Fig C.1

The current density is given by

$$J_z(x) = (I/4ab)\theta(-a \le x \le a)\ \theta(-b \le y \le b)$$

$$= (I/4ab)\ \theta(x \le a)\theta(x \ge -a)\ \theta(y \le b)\theta(y \ge -b) \qquad // \ \theta(s \ge r) \text{ means } \theta(s-r) \text{ Heaviside}$$

$$= (I/4ab)\ \theta(a-x)\theta(x+a)\ \theta(b-y)\theta(y+b) \ . \qquad (C.4.2)$$

Calculate curl **J** :

$$\mathrm{curl}\ \mathbf{J} = \hat{\mathbf{x}}\ (\partial_y J_z - \partial_z J_y) + \hat{\mathbf{y}}\ (\partial_z J_x - \partial_x J_z) + \hat{\mathbf{z}}\ (\partial_x J_y - \partial_y J_x)$$

$$= \hat{\mathbf{x}}\ (\partial_y J_z) + \hat{\mathbf{y}}\ (-\partial_x J_z) \qquad (C.4.3)$$

$$\partial_x J_z = (I/4ab)\ \partial_x[\theta(a-x)\theta(x+a)]\ \theta(b-y)\theta(y+b)$$
$$= (I/4ab)\ [\theta(a-x)\delta(x+a) - \delta(x-a)\ \theta(x+a)]\ \theta(b-y)\theta(y+b) \qquad // \ \partial_x\theta(a-x) = -\delta(x-a)$$

$$\partial_y J_z = (I/4ab)\,\theta(a-x)\theta(x+a)\,\partial_y[\theta(b-y)\theta(y+b)]$$
$$= (I/4ab)\,\theta(a-x)\theta(x+a)\,[\theta(b-y)\delta(y+b) - \delta(y-b)\,\theta(y+b)]$$

so then

$$[\text{curl } \mathbf{J}]_x = (I/4ab)\,\theta(a-x)\theta(x+a)\,[\theta(b-y)\delta(y+b) - \delta(y-b)\,\theta(y+b)]$$
$$[\text{cur l}\mathbf{J}]_y = -\,(I/4ab)\,[\theta(a-x)\delta(x+a) - \delta(x-a)\,\theta(x+a)]\,\theta(b-y)\theta(y+b)\ . \qquad \text{(C.4.4)}$$

Notice that we can obtain $[\text{cur l}\mathbf{J}]_y$ from $[\text{curl } \mathbf{J}]_x$ by doing a↔b, x↔y and adding a minus sign.

Finally calculate H_x from (C.4.1).

$$H_x(x,y) = -\frac{1}{4\pi}\int d^2\mathbf{x}'\,\ln(R^2)\,[\text{curl}'\,\mathbf{J}(\mathbf{x}')]_x \qquad R = |\mathbf{x}-\mathbf{x}'|$$
$$= -\frac{1}{16\pi ab}\int_{-a}^{a} dx' \int_{-\infty}^{\infty} dy'\ \ln[(x-x')^2 + (y-y')^2]\,[\theta(b-y')\delta(y'+b) - \theta(y'+b)\delta(y'-b)]\ \}$$

$$= -\frac{1}{16\pi ab}\int_{-a}^{a} dx' \int_{-\infty}^{b} dy'\ \ln[(x-x')^2 + (y-y')^2]\,\delta(y'+b)$$
$$+\frac{1}{16\pi ab}\int_{-a}^{a} dx' \int_{-b}^{\infty} dy'\ \ln[(x-x')^2 + (y-y')^2]\,\delta(y'-b)\ ]\ \}$$

$$= -\frac{1}{16\pi ab}\int_{-a}^{a} dx'\,\ln[(x'-x)^2 + (y+b)^2] \quad + \frac{1}{16\pi ab}\int_{-a}^{a} dx'\,\ln[(x'-x)^2 + (y-b)^2]\ \}$$

$$= \frac{1}{16\pi ab}(-I_1+I_2) \qquad I_1(b) = \int_{-a}^{a} dx'\,\ln[(x'-x)^2 + (y+b)^2]$$
$$I_2(b) \equiv \int_{-a}^{a} dx'\,\ln[(x'-x)^2 + (y-b)^2] = I_1(-b)\ . \qquad \text{(C.4.5)}$$

Since we will be plotting things below, we assist Maple in computing these integrals in simple form:

```
> int(ln(xp^2 +c^2),xp=-a-x..a-x):t1 := collect(%,ln);
```

$$t1 := (a+x)\ln(a^2+2ax+x^2+c^2) + (a-x)\ln(a^2-2ax+x^2+c^2) + 2c\arctan\left(\frac{a+x}{c}\right) + 2c\arctan\left(\frac{a-x}{c}\right) - 4a$$

```
> c := y+b:
> t2 := unapply(t1,b):
> I1 := t2(b);
```

$$I1 := (a+x)\ln(a^2+2ax+x^2+(y+b)^2) + (a-x)\ln(a^2-2ax+x^2+(y+b)^2) + 2(y+b)\arctan\left(\frac{a+x}{y+b}\right) + 2(y+b)\arctan\left(\frac{a-x}{y+b}\right) - 4a$$

```
> I2 := t2(-b);
```

$$I2 := (a+x)\ln(a^2+2ax+x^2+(y-b)^2) + (a-x)\ln(a^2-2ax+x^2+(y-b)^2) + 2(y-b)\arctan\left(\frac{a+x}{y-b}\right) + 2(y-b)\arctan\left(\frac{a-x}{y-b}\right) - 4a$$

Although the font is small, one can see that each integral contains two ln terms, two arctan terms, and a constant term. Next, write

$$H_x(x,y) = \frac{1}{16\pi ab} F(x,y,a,b) \qquad \text{where} \quad F(x,y,a,b) = I_2 - I_1$$

The Maple unapply command causes an expression to be a function of the specified arguments, so

```
> F := unapply(I2-I1,x,y,a,b):
> F(x,y,a,b);
```

$$(a-x)\ln(a^2-2ax+x^2+(y-b)^2)+(a+x)\ln(a^2+2ax+x^2+(y-b)^2)+2(y-b)\arctan\left(\frac{a+x}{y-b}\right)+2(y-b)\arctan\left(\frac{a-x}{y-b}\right)$$
$$-(a-x)\ln(a^2-2ax+x^2+(y+b)^2)-(a+x)\ln(a^2+2ax+x^2+(y+b)^2)-2(y+b)\arctan\left(\frac{a+x}{y+b}\right)-2(y+b)\arctan\left(\frac{a-x}{y+b}\right)$$

The function F(x,y,a,b) has four ln terms and four arctan terms (the constant cancelled). Now based on the comment below (C.4.4) above, we may conclude that

$$H_y(x,y) = -\frac{1}{4\pi}\int d^2\mathbf{x}'\,\ln(R^2)\,[\text{curl}'\,\mathbf{J}(\mathbf{x}')]_y = H_x(x,y) \quad \text{if we swap } a\leftrightarrow b,\ x\leftrightarrow y \text{ and add a minus}$$

$$= -\frac{1}{16\pi ab} F(y,x,b,a)\ . \qquad \text{(C.4.6)}$$

Just for the record,

```
> F(y,x,b,a);
```

$$(b-y)\ln(b^2-2by+y^2+(-a+x)^2)+(y+b)\ln(b^2+2by+y^2+(-a+x)^2)+2(-a+x)\arctan\left(\frac{y+b}{-a+x}\right)+2(-a+x)\arctan\left(\frac{b-y}{-a+x}\right)$$
$$-(b-y)\ln(b^2-2by+y^2+(a+x)^2)-(y+b)\ln(b^2+2by+y^2+(a+x)^2)-2(a+x)\arctan\left(\frac{y+b}{a+x}\right)-2(a+x)\arctan\left(\frac{b-y}{a+x}\right)$$

We can now make a "field plot" showing the H field (direction and magnitude) in the cross section plane of our rectangular conductor, where we stick with the 4:1 ratio of edges,

```
a := 4: b := 1: I := 1:
Hx := F(x,y,a,b)/(16*Pi*a*b):
Hy := -F(y,x,b,a)/(16*Pi*a*b):
p1 := fieldplot([Hx,Hy],x= -2*a..2*a,y= -6*b..6*b,arrows=THICK,grid = [30,30],scaling=constrained):
p2 := PLOT(CURVES([[a,b],[a,-b],[-a,-b],[-a,b],[a,b]],COLOR(RGB,1,0,0))):
display(p1,p2);
```

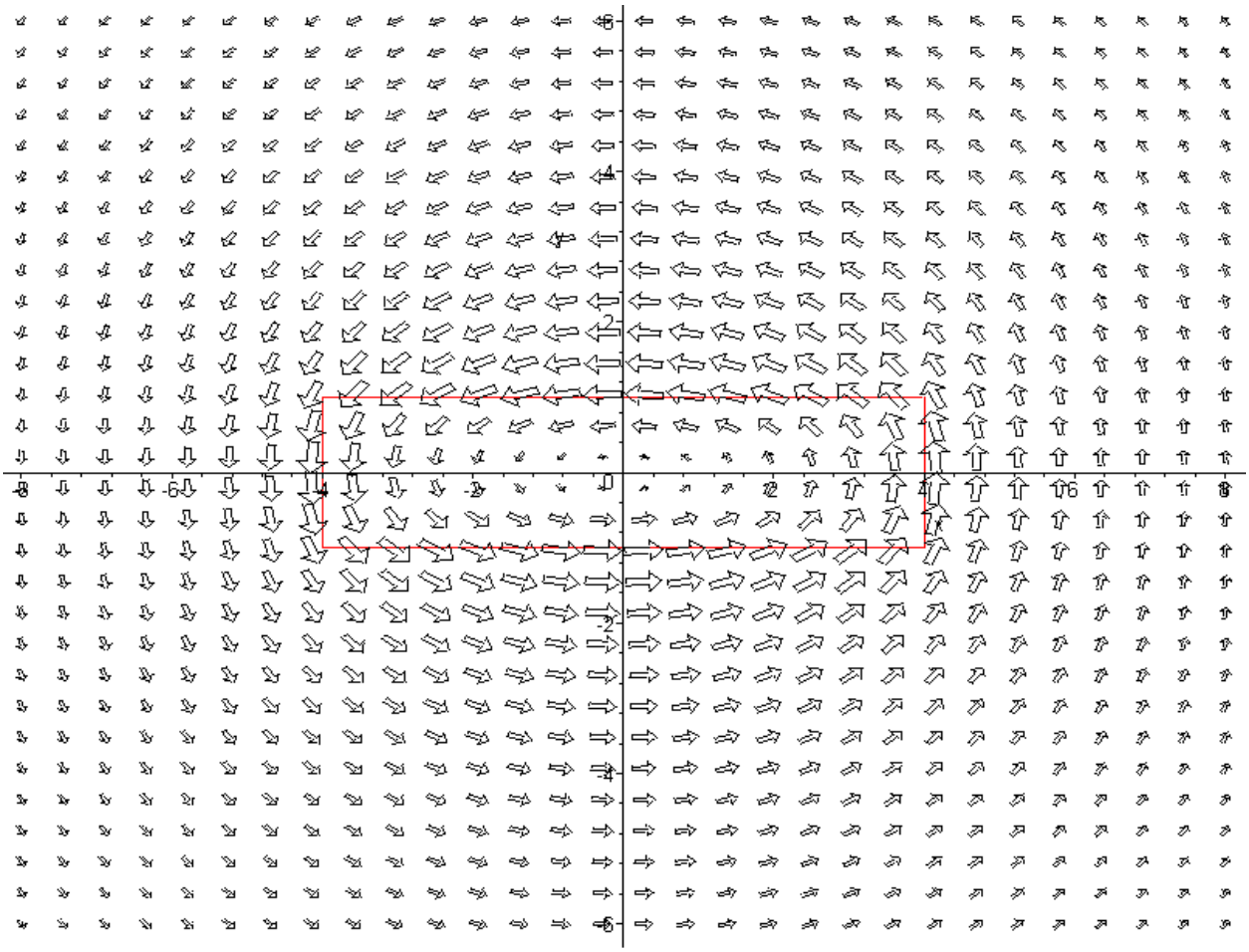
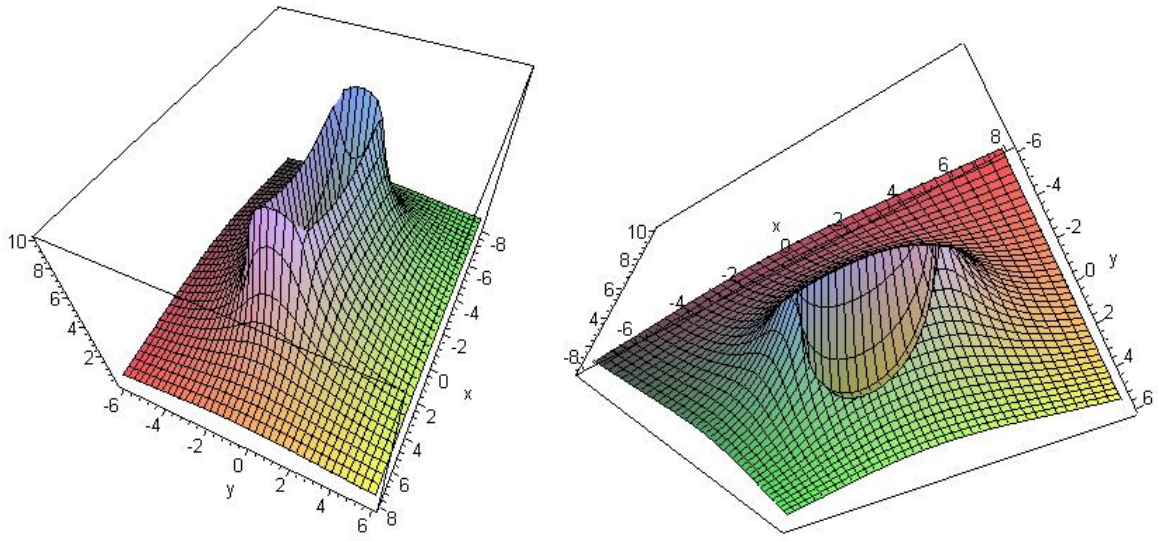


Fig C.2

In this plot the H field appears to be maximal at the conductor boundary (shown in red) and as one moves away it becomes the field of a thin round wire. Current J_z is flowing in the z direction toward the viewer.

The second plot is of $|\mathbf{H}|^2$ as a surface over the x,y plane. Recall that $|\mathbf{H}|^2$ is proportional to the energy density in the magnetic field.

```
plot3d(3000*(Hx^2+Hy^2),x= -2*a..2*a,y= -6*b..6*b, grid = [40,40],projection =
normal,axes=boxed,scaling=constrained);
```



view from above view from below

Fig C.3

The $|\mathbf{H}|^2$ surface is very steep at the conductor boundaries, somewhat resembling a rectangular volcano which dips all the way down to 0 in the center, as shown on the right. The L_i integration discussed below is over this central "cone" of the volcano.

The red plot below is a slice through the volcano at x = 0 :

```
x := 0: plot([Hx^2+Hy^2,(0.03)*sqrt(Hx^2+Hy^2)],y=-6..6, color =
[red,black],thickness=2,numpoints = 500);
```

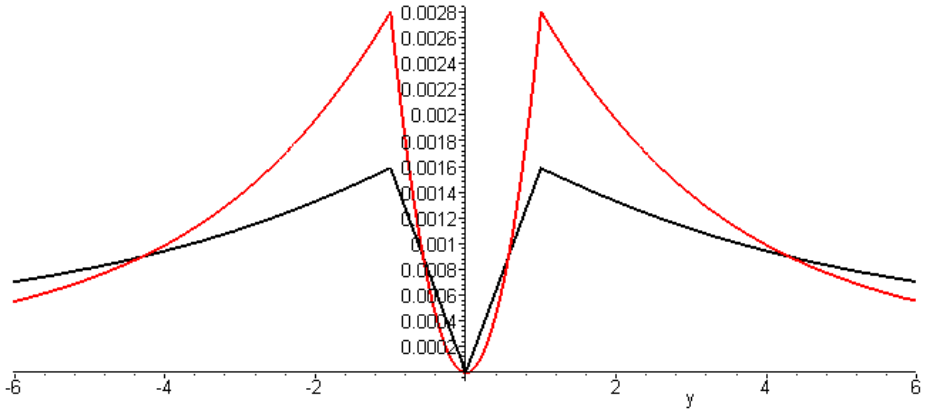


Fig C.4

The black plot is of | **H** | (but scaled down) and resembles the H_θ plot for the round wire shown in Fig B.8.

We return now to a computation of the **internal inductance** L_i of the rectangular wire. From (C.3.7),

$$L_i = \mu_i \int_i dS\ (H/I)^2 = \mu_i \left(\frac{1}{16\pi ab}\right)^2 \int_{-a}^{a} dx \int_{-b}^{b} dy\ [F(x,y,a,b)^2 + F(y,x,b,a)^2] \ . \qquad (C.4.7)$$

Since $F(x,y,\alpha a,\alpha b) = \alpha F(x/\alpha, y/\alpha, a, b)$ one can show that L_i must have this form

$$L_i = (\mu_i/8\pi)\, \ell(b/a) \qquad \text{(C.4.8)}$$

though this conclusion is obvious based on dimensions alone. The problem is to find function ℓ. We can set a = 1 with no loss of generality, and we can use the obvious four-fold symmetry of the energy density so that

$$L_i = 4\,\mu_i \left(\frac{1}{16\pi b}\right)^2 \int_0^1 dx \int_0^b dy\, [F(x,y,1,b)^2 + F(y,x,b,1)^2]$$

$$= \left(\frac{\mu_i}{8\pi}\right) \left\{ \frac{1}{8\pi^2 b^2} \int_0^1 dx \int_0^b dy\, [F(x,y,1,b)^2 + F(y,x,b,1)^2] \right\} . \qquad \text{(C.4.9)}$$

Thus our function of interest is

$$\ell(b) = \frac{1}{8\pi^2 b^2} \int_0^1 dx \int_0^b dy\, [F(x,y,1,b)^2 + F(y,x,b,1)^2] \qquad \text{(C.4.10)}$$

where

```
> F1 := F(x,y,1,b)^2 + F(y,x,b,a)^2;
```

$$F1 := \left((1+x)\ln(1+2x+x^2+(y-b)^2) + (1-x)\ln(1-2x+x^2+(y-b)^2) + 2(y-b)\arctan\left(\frac{1+x}{y-b}\right) - 2(y-b)\arctan\left(\frac{-1+x}{y-b}\right)\right.$$
$$\left. - (1+x)\ln(1+2x+x^2+(y+b)^2) - (1-x)\ln(1-2x+x^2+(y+b)^2) - 2(y+b)\arctan\left(\frac{1+x}{y+b}\right) + 2(y+b)\arctan\left(\frac{-1+x}{y+b}\right)\right)^2 +$$
$$\left((y+b)\ln(b^2+2by+y^2+(-4+x)^2) + (b-y)\ln(b^2-2by+y^2+(-4+x)^2) + 2(-4+x)\arctan\left(\frac{y+b}{-4+x}\right)\right.$$
$$\left. - 2(-4+x)\arctan\left(\frac{y-b}{-4+x}\right) - (y+b)\ln(b^2+2by+y^2+(4+x)^2) - (b-y)\ln(b^2-2by+y^2+(4+x)^2)\right.$$
$$\left. - 2(4+x)\arctan\left(\frac{y+b}{4+x}\right) + 2(4+x)\arctan\left(\frac{y-b}{4+x}\right)\right)^2$$

If one were to expand this expression, there would be $16^2 + 16^2 = 256 + 256 = 512$ terms if no terms combined. In fact there are 232 terms:

```
F2 := expand(F1): nops(F2);
```

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Here are four sample terms in the integrand of (C.4.10),

```
for n from 13 to 16 do
  print(op(n,F2));
od;
```

$$\ln(1-2x+x^2+(y-b)^2)^2\,x^2$$

$$-16\ln(b^2-2by+y^2+(4+x)^2)\,b\arctan\left(\frac{y-b}{4+x}\right)$$

$$-2\ln(b^2+2by+y^2+(-4+x)^2)\,y^2\ln(b^2-2by+y^2+(-4+x)^2)$$

$$-8\arctan\left(\frac{1+x}{y-b}\right)y^2\arctan\left(\frac{-1+x}{y-b}\right)$$

It seems rather unlikely that all 232 terms can be double-integrated analytically! For example, if we ask Maple to analytically integrate the last term shown above just over the x range, it gives up,

```
int(op(16,F2),x=0..1);
```

$$\int_0^1 -8\arctan\left(\frac{1+x}{y-b}\right)y^2\arctan\left(\frac{-1+x}{y-b}\right)dx$$

Thus, in order to compute $\ell(b)$ we must turn to numerical integration which, for each value of b, requires doing 232 numerical double integrals and adding up the results. As is visible in Fig C.3 and Fig C.4, the overall integrand is singular at the conductor edge, so we might expect some difficulties with the numeric integrations near the upper endpoints.

We were not successful trying for a hour to get Maple to compute the integral (C.4.10) analytically or numerically, but certainly the numerical integration can be done. De Smedt quotes the following numerical approximate formulas for two special cases:

$L_i = (\mu_i/8\pi)$ [0.96639] a = b (square wire) // very close to the round conductor

$L_i = (\mu_i/8\pi)\,[(4\pi/3)\,a/b\,]$ $a/b \ll 1$ (flat wire) $// \approx (\mu_i/8\pi)\,[\,4.2\,a/b\,]$ (C.4.11)

Reader Exercise: Do the numerical integration outlined above to determine function $\ell(b)$ for several b values and plot for b = 1 to 10. Is $\ell(1) = 0.96639$?