## C.3 The DC inductance of a round wire

We assume here that $\mu_i$ is the permeability of the wire, and $\mu_e$ of the region outside the wire.

### (a) Internal DC inductance of a round wire

Inductance is a slightly harder problem. The energy density (joules/$m^3$) stored in an electromagnetic field within a medium of negligible loss is given by $u_{em} = (\mathbf{E}\bullet\mathbf{D} + \mathbf{B}\bullet\mathbf{H})/2$ [ Jackson p 259 Eq. (6.106) ] . We are interested only in the portion of this energy density stored in the magnetic field, so $u = (1/2)\ \mathbf{B}\bullet\mathbf{H}$. Since $\mu$ and $\varepsilon$ are assumed to be non-tensor in nature, $u = (1/2)\ \mu\ H^2$. Denoting by dU the field energy stored in a volume $d\mathcal{V}$, so that $u = dU/d\mathcal{V}$, one finds

$$dU = (1/2)\ \mu H^2\ d\mathcal{V}\ . \qquad (C.3.1)$$

For any inductor of inductance L carrying current I and having potential difference V, we know that $V = L\ dI/dt$ and power $P = IV = I\ (L\ dI/dt) = d/dt\ [\ (1/2)L\ I^2\ ]$. The total energy U stored in the full field of the inductor must then be $U = (1/2)L\ I^2$ in order to obtain $P = dU/dt$.

If we consider a piece of uniform round wire of length dz, we can compute the total energy U stored in the field portion that lies inside the wire, and this piece of wire then has "internal inductance" $L_i dz$ where then $L_i$ is the internal inductance of the wire per unit length. So,

$$U_i = (1/2)\ (L_i\ dz)\ I^2\ . \qquad (C.3.2)$$

Looking at (C.3.1), we see that $U_i = \int_{inside} (1/2)\ \mu H^2 d\mathcal{V}$, so our next task is to determine H inside the wire. Due to the symmetry of the round wire, we know that $\mathbf{H} = H(r)\hat{\boldsymbol{\theta}}$ **.** We may use the static Maxwell H curl equation (1.1.1) and the static integral form (1.1.37),

$$\text{curl}\ \mathbf{H} = \mathbf{J} \quad \Leftrightarrow \quad \oint \mathbf{H} \bullet d\mathbf{s}\ = \int_S\ [\mathbf{J}] \bullet d\mathbf{A} \quad \text{"Ampere's Law"}\ . \qquad (1.1.23)$$

Consider a transverse circular loop of radius r centered on the wire's center line. The enclosed current is $\int \mathbf{J}\bullet d\mathbf{A}\ = J \int_0^r 2\pi r dr\ = (I/\pi a^2)\ (\pi r^2) = I\ (r/a)^2$ while $\oint \mathbf{H}\bullet d\mathbf{s} = 2\pi r H(r)$. Thus one finds that $I\ (r/a)^2\ = 2\pi r\ H(r)$ so

$$H(r) = (I/2\pi a^2)\ r \qquad r \le a\ . \qquad (C.3.3)$$

From (C.3.1) the energy stored in a ring of volume $d\mathcal{V} = 2\pi r dr dz$ is

$$dU(r) = (1/2)\ \mu_i H^2\ d\mathcal{V}\ = (1/2)\ \mu_i\ (Ir/2\pi a^2)^2 2\pi r dr dz\ = \mu_i\ (I^2 r^2/4\pi^2 a^4)\ \pi r dr dz\ = \mu_i\ (I^2 r^3/4\pi a^4)\ dr dz$$

$= (\mu_i dz I^2/4\pi a^4)\ r^{-3} dr$ .

Integrate this from r=0 to r=a to get the total magnetic energy stored inside the wire,

$$U_i = (\mu_i dz\, I^2/16\pi) = (1/2)\,[\mu_i/8\pi]\, dz\, I^2 . \qquad (C.3.4)$$

Set this equal to $(1/2)L_i dz\, I^2$ in (C.3.2) to find that

$$L_i = \mu_i/8\pi . \qquad (C.3.5)$$

If this medium is non-magnetic, then $\mu_i = \mu_0$. Since $\mu_0 = 4\pi \times 10^{-7}$ henry/m we find that $(\mu_0/8\pi) = (1/2)\times 10^{-7} = 50 \times 10^{-9}$ so the internal inductance per unit length of a round wire is given by

$$L_i = (\mu_i/\mu_0) * 50\ \text{nH/m} \qquad \text{// nH = nanohenries} \qquad (C.3.5a)$$

which agrees with Matick p 97 (4-6) .

Interestingly, this result is independent of the wire radius a. For a given current I, the total field energy stored in the wire is independent of a. For small a, the field is stronger but in a smaller volume.

**(b) External DC inductance of a round wire**

What about the *external* inductance of a round wire? We can compute it by the same method. If we use a Stokesian loop of radius r ≥ a, the enclosed current is I. We set this enclosed current equal to $2\pi r H(r)$ to get the magnetic field at r,

$$H(r) = (I/2\pi)\, r^{-1} \quad . \qquad (C.3.6)$$

From (C.3.1) the energy stored in a ring of volume $d\mathcal{V} = 2\pi r dr dz$ is

$$dU(r) = (1/2)\,\mu_e H^2 d\mathcal{V} = (1/2)\mu_e (I/2\pi r)^2 [2\pi r dr dz] = \mu_e (I^2/4\pi^2 r^2)\pi r dr dz = \mu_e (I^2/4\pi r) dr dz$$

$$= (\mu_e dz I^2/4\pi)\, r^{-1}\, dr .$$

Integrate this from r=a to some large radius r=R to get

$$U_e = (\mu_e dz\, I^2/4\pi)\, \ln(R/a) = (1/2)\,[\,\frac{\mu_e}{2\pi}\,\ln\frac{R}{a}\,]\, dz\, I^2 \quad . \qquad (C.3.7)$$

Set this equal to $(1/2)L_e dz\, I^2$ in (C.3.2) to find that,

$$L_e = \frac{\mu_e}{2\pi}\,\ln\frac{R}{a} . \qquad (C.3.8)$$

If we set R = ∞ to get the total external inductance per unit length of a round wire, the result is logarithmically divergent. The total magnetic energy stored *per unit length* of an infinitely long round

wire in isolation is infinite. It takes an infinite amount of work to build up such a field even in 1 cm worth of the wire.

A "practical wire" is more like a loop of wire than an infinitely long wire. It is difficult to conjure up an experiment to test (C.3.8) even for a very long straight piece of wire without having some return path for the current to return to the driving "battery". For wire and dielectric both having $\mu_0$, Jackson shows (p 216-218) that the inductance per unit length of a loop of projected area A of radius-a wire is given by $L_e + L_i = (\mu_0/4\pi)\,[\ln(\xi A/a^2) + 1/2]$, where $\xi$ is a near-unity factor which accounts for messy details of the calculation. The 1/2 term accounts for the internal inductance $L_i = \mu_0/8\pi$ as in (C.3.5).

For a *circular* loop of radius R, one has $A = \pi R^2$ and, if $R >> a$, $\xi = 64/(\pi e^4) \approx .373$. So,

$$\ln(\xi A/a^2) = \ln(64\,\pi R^2/\pi e^4 a^2) = 2\ln(8R/ae^2) = 2\ln(8R/a) + 2\ln(e^{-2}) = 2\ln(8R/a) - 4$$

and then the total inductance per unit length is

$$L_e + L_i = (\mu_0/4\pi)\,[2\ln(8R/a) - 4 + 1/2] = (\mu_0/2\pi)\,[\ln(8R/a) - 2 + 1/4] = (\mu_0/2\pi)\,[\ln(8R/a) - 7/4]\ .$$

The *total* inductance of such a loop is then

$$L = 2\pi R(\mu_0/2\pi)\,[\ln(8R/a) - 7/4] = \mu_0 R\,[\ln(8R/a) - 7/4]$$

in agreement with Jackson Problem 5.32 p 234. If one omits the internal inductance, the last factor is -2 instead of -7/4, and this result is seen in some sources. The point is that this is a finite result, even though the (dipole) magnetic field of such a loop does extend to infinity. A loop of N turns gets an extra factor $N^2$ because in effect current $I \rightarrow NI$ in (C.3.4) and (C.3.7), so the total field energy increases by factor $N^2$.

Suppose there were two parallel wires with currents flowing in opposite directions. In this case, we could compute the magnetic field H at any point in space as the vector sum of the fields of the two wires, then we could integrate $H^2$ over all space to get the total energy U and from that the external inductance $L_e$. In this case, the ln(R) divergence does not appear. In effect, the divergence cancels between the two wires, similar to the way opposite short segements of the circular wire cancel to give the finite result quoted above. Since we will be doing this computation by another means in the main text, we do not bother with the parallel wire calculation here.

We really only care about the internal inductance $L_i$ of a wire in our transmission line analysis because the external inductance $L_e$ is already accounted for by the techniques of Chapter 4. That is, $L_e$ is computed by considering the magnetic potential $A_z$ ( or W ) between the wires.

**C.4 The DC inductance of a wire of arbitrary cross section**

The general method described above was this:

$$U_i = (1/2)\,(L_i\,dz)\,I^2 = \int_{inside} (1/2)\,\mu_i H^2 d\mathcal{V}$$

$$U_e = (1/2)\,(L_i\,dz)\,I^2 = \int_{inside} (1/2)\,\mu_e H^2 d\mathcal{V}$$

so that

$$L_i = \mu_i \int_{inside} H^2\ d\mathcal{S}$$

The main purposes of this section are to show how divergences can be handled, and to remind ourselves that we really have to attack differential equations directly to get real solutions, except in the simplest cases.

**(a) Statement of a Plan of Attack**

1. Compute **H** everywhere using (B.2.6),

$$\mathbf{H}(x,y) = -\frac{1}{4\pi}\int d^2\mathbf{x}'\,\ln(R^2)\ \mathrm{curl}'\,\mathbf{J}(\mathbf{x}') \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad (B.2.6)$$

This integral determines the H field both inside and outside the wire. Section B.5 evaluates (B.2.6) for the round wire as an example.

We make use of "the $J_m$ theorem" described in Appendix B.6 (b) where it is shown that the correct total solution for A(x) can be obtained by adding an artifical surface current source $J_m$ to the true conduction current source $J_c$.

1. Compute the surface current component of $J_m$ as described

Here is a possible program for computing the DC inductance of a wire of arbitrary cross section. It is very similar to the above except for the first step. The wire is assumed aligned with the z axis.

1. Compute **A**(x,y) for an arbitrary wire at DC using formula (1.5.9) with $\beta(\omega=0) = 0$ :

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int dV'\,[\mu_2\mathbf{J}_c(\mathbf{x}') + \mu_0\,\mathbf{J}_m(\mathbf{x}')]\,\frac{1}{R} \qquad R = |\mathbf{x}-\mathbf{x}'| \qquad \omega = 0 \qquad (1.5.9)$$

where **A**(**x**) means A(**x**,ω=0) and the same for the **J**'s. Here $\mu_2$ is magnetic permeability of the wire, while $\mu_1$ is for the dielectric outside the wire. $\mathbf{J}_m$ represents the surface magnetization current (expressed as a volume density) that appears on the conductor surface in the case $\mu_1 \neq \mu_2$. We presume that there is some

reasonable prescribed conduction current density $\mathbf{J_c}$ inside the wire, and from this one can (with some effort) compute $\mathbf{J_m}$ and then one can compute $\mathbf{A}(\mathbf{x},\omega)$ according to (1.5.9). Since (at DC) $\mathbf{J_c}$ is longitudinal along the wire (z direction), $\mathbf{J_m}$ is also longitudinal and $\mathbf{A} = A_z\hat{\mathbf{z}}$ . Thus, all equations below should really be written in terms of the z component only, but we continue to use the simpler vector notation.

2. Compute $\mathbf{B} = \text{curl } \mathbf{A}$.

3. The magnetic energy density is then $dU/dV = (1/2)B^2/\mu$. Integrate this over the interior or exterior of a slice of the wire to get $U_i$ or $U_e$. For $U_e$ we know we have to use a cutoff because the result is going to be log divergent as was the case for the round wire.

4. Set $U = (1/2)\, L\, I^2$ to extract the appropriate inductance. We shall be mainly interested in the internal inductance.

**(b) The divergence problem and its resolution**

Let us look more closely at the integral for $\mathbf{A}(\mathbf{x},\omega)$ stated above. At DC, both $\mathbf{J_c}$ and $\mathbf{J_m}$ will be constant in z, so we may write

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int dx'dy' \, [\, \mu_1 \mathbf{J_m}(x',y') + \mu_2 \mathbf{J_c}(x',y')] \int_{-\infty}^{\infty} dz' \, \frac{dz'}{\sqrt{s^2 + z'^2}} \qquad \text{(C.4.1)}$$

where

$$s = \sqrt{(x-x')^2 + (y-y')^2} \quad \text{and} \quad R = \sqrt{s^2 + z'^2}$$

At DC $\mathbf{J_c}$ will be uniform across the wire so we can write $\mathbf{J_c}(x',y') = \mathbf{J_c}$.

An attempt to do the dz' integration reveals the first sign of trouble because this integral diverges. This is a reflection of the same problem noted earlier, that there is something unphysical about an infinite wire. So we install a very large cutoff $\pm \Lambda/2$ on the z integration, as if the wire were of length $\Lambda$ instead of $\infty$. Then,

$$\int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2 + z'^2}} \rightarrow \int_{-\Lambda/2}^{\Lambda/2} \frac{dz'}{\sqrt{s^2 + z'^2}} = 2 \int_{0}^{\Lambda/2} \frac{dz'}{\sqrt{s^2 + z'^2}} = 2 \ln[\, z' + \sqrt{z'^2 + s^2}\, ] \,|^{\Lambda/2}_0$$

$$= 2\ln[\Lambda/2 + \sqrt{(\Lambda/2)^2 + s^2}\,] \; - \; 2 \ln s$$

$$\approx 2\ln(\Lambda) - 2\ln s = -2\ln(s/\Lambda)\,,$$

where in the last step we assume that the transverse dimensions of the wire are much smaller than the cutoff $\Lambda$ so that $s << \Lambda$ for all s in the transverse integration. We are then left with

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int dx'dy' \, [\, \mu_1 \mathbf{J_m}(x',y') + \mu_2 \mathbf{J_c}] \, \{ \; 2\ln(\Lambda) - 2\ln s \, \}$$

$$= -\frac{1}{2\pi} \int dx'dy' \, [\, \mu_1 \mathbf{J_m}(x',y') + \mu_2 \mathbf{J_c}] \;\; \{ \ln s - \ln \Lambda \} \qquad \text{(C.4.2)}$$

$$= -\frac{1}{2\pi} \int dx'dy' \, [\, \mu_1 \mathbf{J_m}(x',y') + \mu_2 \mathbf{J_c}] \ln[\sqrt{(x-x')^2 + (y-y')^2}\,] \;\; + \text{ a constant}$$

where the constant is proportional to ln(Λ). When we compute **B** = curl **A** , this constant has no effect on **B** so we just ignore the constant, setting it to 0 for the purpose of computing **B**. However, in order to keep track of dimensions, we shall keep the constant Λ around, perhaps later setting it to 1 since its value does not matter. So:

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2\pi} \int dx'dy' \, [\, \mu_1 \mathbf{J_m}(x',y') + \mu_2 \mathbf{J_c}] \;\; \ln(\Lambda/s) \qquad \text{(C.4.3)}$$

where the transverse integral is over the shape of the wire. Whatever shape wire we choose, we can in principle do the above integration and carry out the program. At worst, we have to resort to numerical techniques.

Comment: We could have arrived at (C.4.3) by starting with (1.5.4) treated as a 2D problem since none of the fields depends on z (recall $\beta^2 = 0$ since we are doing DC treatment ),

$$-(\nabla^2_{2D})\mathbf{A} = [\, \mu_1 \mathbf{J_m}^{(i)} + \mu_2 \mathbf{J_2}] \quad . \qquad \textit{(1.5.4)}$$

The solution to this equation is (C.4.3) "by inspection" since the 2D free-space propagator is $\frac{1}{2\pi} \ln(1/R)$ where $R = s = \sqrt{(x-x')^2 + (y-y')^2}$ . This concept is discussed in Appendix I. See in particular (I.1.8).

In the case that $\mu_1 = \mu_2$ where $\mathbf{J_m} = 0$, the equation (C.4.3) simplifies to

$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2\pi} \int dx'dy' \, [\mu_2 \mathbf{J_c}] \;\; \ln(s/\Lambda) \;\; = -\frac{1}{2\pi} \mu_1 \, \mathbf{J_c} \int dx'dy' \, \ln(s/\Lambda) \qquad // \; \mu_1 = \mu_2$$

which involves a purely geometric transverse integral over the wire cross section.

For a round wire still with $\mu_1 = \mu_2$ the above integration becomes

$$A_z(r) = -\frac{\mu_1 J_z}{2\pi} \int_0^a r'\, dr' \int_0^{2\pi} d\theta' \, \ln\left(\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}\,/\Lambda\right) \qquad // \; \mu_1 = \mu_2 \qquad \text{(C.4.4)}$$

Although it looks like a mess, these are standard integrals, and the **H** field results (C.3.3) and (C.3.6) of the previous section are easily duplicated using **H** = $(1/\mu_1)$**B** = $(1/\mu_1)$ curl **A**. The following exercise shows how this works. Need reference to somewhere else on this!!

Reader Exercise: In the expression (C.4.4) the -ln(Λ) term creates a constant in $A_z(r)$ which we ignore as noted above since we only care about **B** = curl **A**, so set Λ = 1. Then use GR7 4.224.9 to show that

$$\int_0^{2\pi} d\theta' \ln\left(\sqrt{r'^2+r^2-2rr'\cos\theta'}\right) = \pi \ln\left[\frac{(r'^2+r^2)+|r'^2-r^2|}{2}\right].$$

The denominator 2 can be ignored since it too just creates a constant. Then show that for r > a,

$$A_z(r) = -\frac{\mu_1 J_z}{2\pi} \int_0^a r'\,dr'\,\pi \ln\left[\frac{(r'^2+r^2)+|r'^2-r^2|}{2}\right] = -(\mu J_z/2)\, a^2 \ln(r).$$

Then compute **B** using the curl in cylindrical coordinates to get

$$\mathbf{B} = -\partial_r A_z(r)\,\hat{\boldsymbol{\theta}} = (\mu J_z a^2/2) r^{-1}\,\hat{\boldsymbol{\theta}} = B(r)\,\hat{\boldsymbol{\theta}} \qquad \text{where} \quad B(r) = (\mu J_z a^2/2) r^{-1}.$$

Finally, since $I = J_z(\pi a^2)$, show that one obtains the known result for field outside a round wire,

$$H(r) = (I/2\pi)\, r^{-1} \quad . \qquad (C.3.6)$$

For r < a, break up the integral into two parts to obtain the internal results

$$A_z(r) = -(\mu J_z/4)\, r^2 \quad \Rightarrow \quad H(r) = (I/2\pi a^2)\, r \qquad r \le a \qquad (C.3.3)$$

The reader may notice a similarity between (C.4.3) and results of Chapter 4 such as 4.4 (3) which contain factors ln(s1/s2). This is no coincidence of course since we took the small β limit in Chapter 4, and here we are dealing with the β = 0 limit (DC). // Remove this paragraph because reader probably has not read Chapter 4 when looking at this appendix.

**(c) Avoiding the divergence problem**

A way to avoid this divergence business is to apply transverse derivatives to both sides of (C.4.1) right at the start, before doing any integrations. It is really these derivatives of **A** that we need in order to compute **B** in Step (b) of the program outlined above. For example, in Cartesian coordinates,

$$\partial_x R^{-1} = -R^{-2}\partial_x R = -R^{-2}\partial_x = -R^{-2}(1/2)R^{-1}\,2(x-x') = -(x-x')\,R^{-3}$$

so the x derivative of (C.4.1) becomes

$$\partial_x A_z(\mathbf{x}) = -\frac{1}{4\pi} \int\int dx'dy'[\,\mu_1 J_{mz}(x',y') + \mu_2 J_{cz}]\,(x-x') \int_{-\infty}^{\infty} dz' \frac{1}{R^3} \quad . \qquad (C.4.5)$$

The z' integration is now a convergent integral equal to [ GR7 3.252.7 ],

$$\int_{-\infty}^{\infty} dz' \frac{1}{[z'^2+s^2]^{3/2}} = 2/s^2 \qquad s = \sqrt{(x-x')^2+(y-y')^2} \quad . \qquad (C.4.6)$$

so that

$$-B_y = \partial_x A_z(\mathbf{x}) = -\frac{1}{2\pi}\int\int dx'dy'[\,\mu_1 J_{mz}(x',y') + \mu_2 J_{cz}]\frac{(x-x')}{(x-x')^2+(y-y')^2}$$

$$B_x = \partial_y A_z(\mathbf{x}) = -\frac{1}{2\pi}\int\int dx'dy'[\,\mu_1 J_{mz}(x',y') + \mu_2 J_{cz}]\frac{(y-y')}{(x-x')^2+(y-y')^2} \qquad \text{(C.4.7)}$$

Now the need for a cutoff $\Lambda$ is completely avoided.

In the special case $\mu_1 = \mu_2$ so $J_{mz} = 0$ we have

$$-B_y = \partial_x A_z(\mathbf{x}) = -\frac{\mu_1 J_{cz}}{2\pi}\int\int dx'dy'\frac{(x-x')}{(x-x')^2+(y-y')^2}$$

$$B_x = \partial_y A_z(\mathbf{x}) = -\frac{\mu_1 J_{cz}}{2\pi}\int\int dx'dy'\frac{(y-y')}{(x-x')^2+(y-y')^2} \qquad //\ \mu_1 = \mu_2 \qquad \text{(C.4.8)}$$

and the fields are then determined by these purely geometric integrals (see example below)

In any event, we have outlined a method by which both the internal and external DC inductance of any wire can be calculated. For a round wire we know from (C.3.5) above that $L_i = \mu/8\pi$ . For a rectangular conductor of dimensions a and b, the result can be obtained using (C.4.7). The result must have the form $L_i = (\mu/8\pi)\, f(a/b)$ where f(x) is the function one obtains by doing the computation. A square wire would then have f(1). As a square wire is gradually deformed into a round wire, f(1) gradually deforms into 1.

**Example: Magnetic Field of a Rectangular Wire**

As one might expect, the **B** field for a rectangular conductor is not particularly simple. Here is a Maple calculation of the double integral appearing in the $\partial_x A_z$ expression of (C.4.7) where we have made a clumsy attempt to minimize the number of terms. We assume $\mu_1$

```
> f := (x-xp)/((x-xp)^2 + (y-yp)^2):  # integrand
> dxAz := Int(Int(f, xp=-a/2..a/2),yp=-b/2..b/2); # just to display the integral
```

$$dxAz := \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \frac{x - xp}{(x-xp)^2 + (y-yp)^2} \, dxp \, dyp$$

```
> dxAz := int(int(f, xp=-a/2..a/2),yp=-b/2..b/2):  # makes a long expr of 16 terms
> t := collect(dxAz,arctan): # factor 8 arctan terms into 4 terms
> t1 := sum(op(n,t),n=1..4): # pick off just these 4 factored arctan terms
> t2 := collect(t - t1,ln):  # t2 is then a set of 8 ln terms factored into 4 terms
> dxAz_ := t1+t2;            # put the pieces back together
```

$$dxAz_ := \left(-x + \frac{1}{2}a\right) \arctan\left(\frac{2y-b}{-2x+a}\right) + \left(-x - \frac{1}{2}a\right) \arctan\left(\frac{2y-b}{2x+a}\right) + \left(-\frac{1}{2}a + x\right) \arctan\left(\frac{2y+b}{-2x+a}\right) + \left(x + \frac{1}{2}a\right) \arctan\left(\frac{2y+b}{2x+a}\right)$$
$$+ \left(-\frac{1}{4}b - \frac{1}{2}y\right) \ln(4x^2 - 4xa + 4yb + a^2 + 4y^2 + b^2) + \left(\frac{1}{2}y - \frac{1}{4}b\right) \ln(4x^2 - 4xa - 4yb + a^2 + 4y^2 + b^2)$$
$$+ \left(\frac{1}{2}y + \frac{1}{4}b\right) \ln(4x^2 + 4xa + 4yb + a^2 + 4y^2 + b^2) + \left(\frac{1}{4}b - \frac{1}{2}y\right) \ln(4x^2 + 4xa - 4yb + a^2 + 4y^2 + b^2)$$

At least the result is expressible in closed form. Using $B_y = -\partial_x A_z$ as above and $B_x = \partial_y A_z$ similarly, we can plot the magnetic field lines for a = 1, b = 4 both inside and outside the conductor shown in red:

```
a := 1: b := 4:
p1 := fieldplot([-dyAz_,dxAz_],x=-5..5,y=-5..5,arrows=THICK,grid = [30,30]):
p2 := PLOT(CURVES([[a,b],[a,-b],[-a,-b],[-a,b],[a,b]],COLOR(RGB,1,0,0))):
display(p1,p2);
```

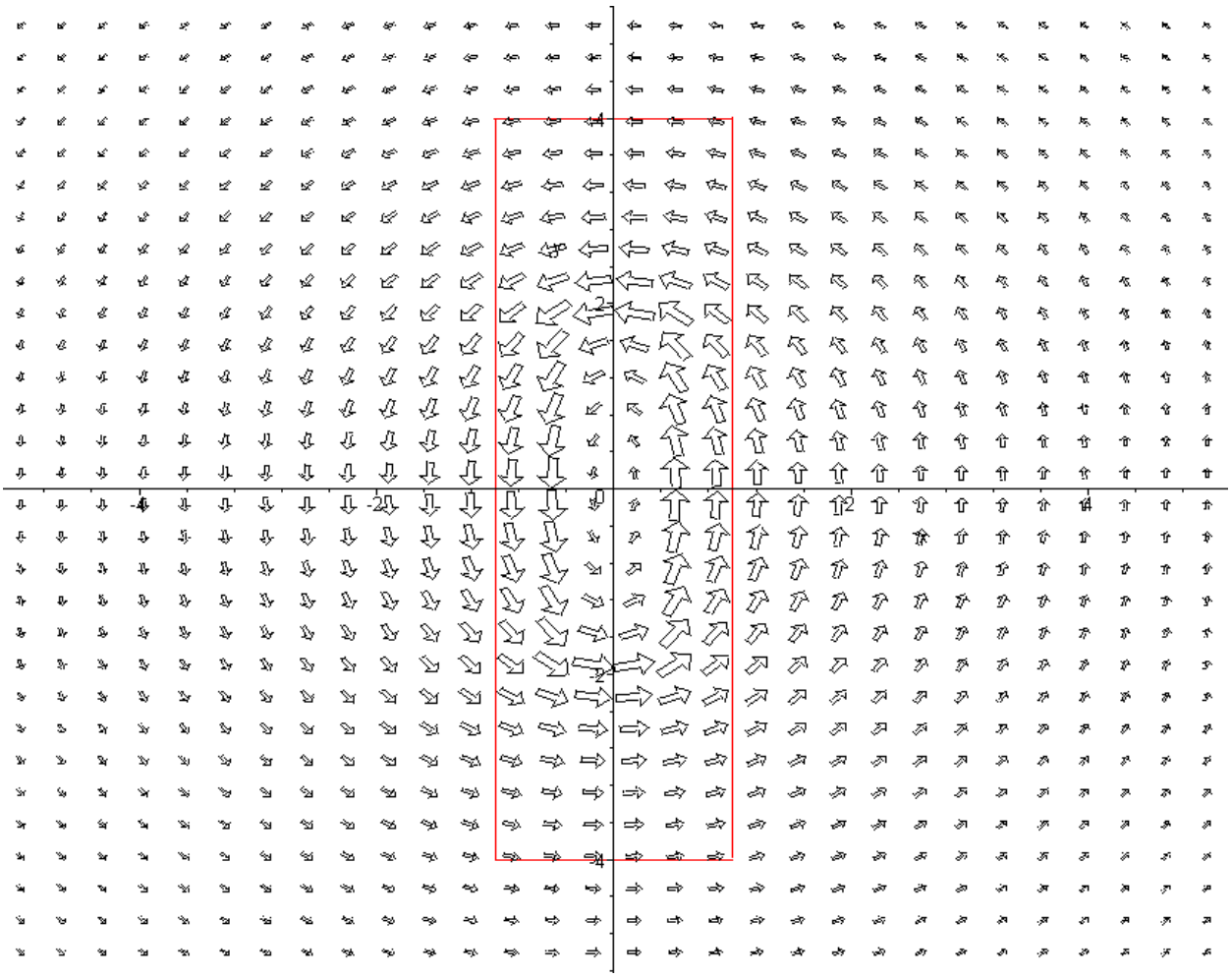

Fig C.1: The DC magnetic field of a rectangular conductor

Far away, the field is the same as that of a round wire carrying the same total current. It would take some work to integrate the energy in this field to obtain $L_i$ and $L_e$ for such a conductor, but it could be done.

The method presented in this section is *not* very useful for an AC calculation, because then we can no longer assume that the current density is uniform across the wire. This will become clear later when we compute the exact current density for a round wire at arbitrary frequency. Again, this is similar to Chapter 4 results such as (4.4.3). We need some other method to compute the distribution of charge and current on a conductor.