App D Actions 1 v 2 PhL 11.30.13

This document supersedes App D Actions 1 in which I made a math error. So no need ever to look at the older App D Actions 1 ever again. [I threw that doc out!]

This doc addresses a whole battery of ugly questions and I think as of 12/5/13 they have all been resolved so this doc is now "fully reviewed". See TOC below for subject matter.

1. This is done now for general m and for m = 0 in Appendix D Section D.4
2. I showed this explicitly and it is reported now in Appendix D section D.5.
3. $\beta_d^2 << |\beta|^2$ is now addressed in Appendix D just below equation (D.2.2).
4. I only assume $\beta_d^2 << |\beta|^2$ very late in Appendix D in (D.4.12) for the B fields only
5.6. The $E_\varphi(r=0) = 0$ idea is rejected as a real BC, it will just be true. I explain $E_\varphi = 0$ at the surface now at the very start of Appendix D below (D.1.2), so that BC is now understood.
7,8. Resolving the above BC then resolved these questions.

1. Compute the Appendix D Magnetic Field B for m = 0. 1
2. Do these E and B fields satisfy Maxwell's equations? 3
3. What exactly does the condition $\beta_d^2 << |\beta|^2$ imply? 7
4. Did I assume $\beta_d^2 << |\beta|^2$ somewhere in my Appendix D development? Answer is no! 9
5. My strange E_φ boundary condition. 10
6. Would it be better to say maybe that $E_\varphi(r=0,m) = 0$? 12
7. So now I am stuck with a new question: how to compute a_m and K_m ? 13
8. What should I do about boundary conditions then for m > 0 ? 15

1. Compute the Appendix D Magnetic Field B for m = 0.

I did not do this in App D and maybe I should, but I will try to do it right here for the first time. I comment on this at the end of App D and perhaps will adjust that comment. It seems the easiest method is this:

curl **E** = - jω**B** **B** = (-1/jω) curl **E**

Notes before starting:

- In this section, write β' where things should be β'.
- Assume $E_\varphi = E_\theta \equiv 0$ for this m=0 analysis, since can select $C_{\varphi 0} = 0$ as shown below
- Everything here is for inside the wire only
- Here are the Appendix D m = 0 results for the E field

$$E_z(r,0) = (1/2)\,(a\beta')\, I\, R_{dc}\, [\, J_0(\beta' r)/\, J_1(\beta' a)\,]$$
$$E_r(r,0) = (j/2)\,(a\beta_d)\, I\, R_{dc}\, [J_1(\beta' r)/\, J_1(\beta' a)]$$
$$E_\varphi(r,0) = C_{\varphi 0}\, J_1(\beta' r) = 0$$

- $\beta'^2 = \beta^2 - \beta_d^2 \Rightarrow \quad \beta'^2 + \beta_d^2 = \beta^2$

- Remember that β has a strange and large phase, but think of β_d as real.
- $R_{dc} = \frac{1}{\sigma\pi a^2}$
- $\beta^2 \approx -j\omega\mu\sigma$ and therefore $(-j\omega/\beta) \approx \beta/(\mu\sigma)$

So we use (quote from the web just to back up mine from my TK doc)

$$\nabla \times \boldsymbol{u} = \hat{\boldsymbol{r}} \left(\frac{1}{r} \frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + \hat{\boldsymbol{z}} \frac{1}{r} \left[\frac{\partial (r u_\phi)}{\partial r} - \frac{\partial u_r}{\partial \phi} \right].$$

curl $\mathbf{E} = \hat{\mathbf{r}} [r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}} [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}} [r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r]$

$= \hat{\mathbf{r}} [r^{-1}\partial_\theta E_z] + \hat{\boldsymbol{\theta}} [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}} [- r^{-1}\partial_\theta E_r]$ // $E_\theta \equiv 0$

$= \hat{\boldsymbol{\theta}} [\partial_z E_r - \partial_r E_z]$

$= \hat{\boldsymbol{\theta}} [-j\beta_d E_r - \partial_r E_z] = \hat{\boldsymbol{\theta}} [-j\beta_d \{E_r\} + \{- \partial_r E_z\}]$ // ok

with

$E_z = (1/2) (a\beta') I R_{dc} [J_0(\beta' r)/ J_1(\beta' a)]$ // sign agrees with Chap 2 and Matick

$E_r = (j/2) (a\beta_d) I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)]$. // I have no verification of this sign!

First, compute the second term:

$E_z = (1/2) (a\beta') I R_{dc} [J_0(\beta' r)/ J_1(\beta' a)]$

$\partial_r E_z = (1/2) (a\beta') I R_{dc} \beta' [J_0'(\beta' r)/ J_1(\beta' a)]$

$= - (1/2) (a\beta') I R_{dc} \beta' [J_1(\beta' r)/ J_1(\beta' a)]$

Then we have

$[\text{curl } \mathbf{E}]_\theta = -j\beta_d \{ (j/2) (a\beta_d) I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)]\} + \{(1/2) (a\beta') I R_{dc} \beta' [J_1(\beta' r)/ J_1(\beta' a)]\}$

$= + \beta_d (1/2) (a\beta_d) I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)] + (1/2) (a\beta'^2) I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)]$

$= (1/2) a\beta_d^2 I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)] + (1/2)(a\beta'^2 I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)]$

$= (1/2) a I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)] \{ \beta_d^2 + \beta'^2 \}$

$= (1/2) a \beta^2 I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)]$

Therefore,

$\mathbf{B} = (-1/j\omega)\ \text{curl}\ \mathbf{E} = \hat{\boldsymbol{\theta}}\ [(-1/j\omega)\ (1/2)\ a\ \beta^2\ I\ R_{dc}\ [J_1(\beta' r)/\ J_1(\beta' a)]]$ // exact

How does this compare to the Chapter 2 result?

$B_\theta = (-1/j\omega)\ (1/2)\ a\ \beta^2\ I\ R_{dc}\ [J_1(\beta' r)/\ J_1(\beta' a)]$

$= (-1/j\omega)\ (1/2)\ a\ \beta^2\ I \frac{1}{\sigma\pi a^2}\ [J_1(\beta' r)/\ J_1(\beta' a)]$

$\approx (-1/j\omega)\ (1/2)\ a\ (-j\omega\mu\sigma)\ I \frac{1}{\sigma\pi a^2}\ [J_1(\beta' r)/\ J_1(\beta' a)]$ // here assume $\beta^2 \approx -j\omega\mu\sigma$

$\approx (\ (\mu)\ I \frac{1}{2\pi a}\ [J_1(\beta' r)/\ J_1(\beta' a)]$

$\approx [\mu I/(2\pi a)]\ [J_1(\beta r)/\ J_1(\beta a)]$ // This is exactly the Chapter 2 result!

Recall that inside Chapter 2 we make the assumption that $\beta^2 \approx -j\omega\mu\sigma$, which says $\beta^2 \approx \beta'^2$ and all that stuff, so our exact App D result → Chapter 2 result when this approx is added.

Comments:

- The B field is entirely in the $\hat{\boldsymbol{\theta}}$ direction which is certainly a simplifying fact. This result is exact with the assumptions I have made about z dependence.

So here are the m = 0 field results a la Appendix D:

$E_z(r,0) = (1/2)\ (a\beta')\ I\ R_{dc}\ [J_0(\beta' r)/\ J_1(\beta' a)]$
$E_r(r,0) = (j/2)\ (a\beta_d)\ I\ R_{dc}\ [J_1(\beta' r)/\ J_1(\beta' a)]$
$B_\theta(r,0) = (-1/j\omega)(a/2)\ I\ R_{dc}\ [J_1(\beta' r)/\ J_1(\beta' a)]\ (\beta_d^2 + \beta'^2)$

The other three field components are each identically 0.

2. Do these E and B fields satisfy Maxwell's equations?

It would be good the check them all, one at time! Trust but verify.

(1) div $\mathbf{B} = 0$?

$\text{div}\ \mathbf{B} = r^{-1}\partial_r(rB_r) + r^{-1}\partial_\theta B_\theta + \partial_z B_z = 0$

because there is no B_r or B_z , and because B_θ does not depend on θ. One down!

(2) div $\mathbf{E} = 0$?

$\text{div}\ \mathbf{E} = r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z$

$= r^{-1}\partial_r(rE_r) + \partial_z E_z$

$= r^{-1}\partial_r(rE_r) - j\beta_d E_z$

Compute the first term

$rE_r = (j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a))\ [\ rJ_1(\beta' r)\ ]$

$\partial_r(rE_r) = (j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a))\ [\ r\beta' J_1'(\beta' r) + J_1(\beta' r)\ ]$

$r^{-1}\partial_r(rE_r) = (j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a))\ [\ \beta' J_1'(\beta' r) + r^{-1}J_1(\beta' r)\ ]$

Therefore

$$\text{div } \mathbf{E} = \{(j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a))\ [\ \beta' J_1'(\beta' r) + r^{-1}J_1(\beta' r)\ ]\} - j\beta_d \{ (1/2)(a\beta') I R_{dc}\ [J_0(\beta' r)/J_1(\beta' a)]\}$$

$$= (j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a))\ \{ J_1'(\beta' r) + r^{-1}J_1(\beta' r) - \beta' J_0(\beta' r)\ \}$$

$$= (j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a)\ \beta'\ \{\ J_1'(\beta' r) + (r\beta')^{-1}J_1(\beta' r) - J_0(\beta' r)\ \}$$

$$= (j/2)(a\beta_d) I R_{dc}(1/J_1(\beta' a)\ (\beta'/x)\ \{\ xJ_1'(x) + J_1(x) - xJ_0(x)\ \}$$

Now here is what Spiegel 24.19 says

$$xJ_1'(x) = xJ_0(x) - J_1(x) = 0 \quad \Rightarrow \quad xJ_1'(x) + J_1(x) - xJ_0(x) = 0$$

and therefore we have shown that div **E** = 0, hurray for once! Since this was built into App D, it is certainly an expected result.

(3) curl **E** = $-\partial_t \mathbf{B}$ or curl **E** = $-j\omega\mathbf{B}$

This is the equation we used to compute **B** in the first place, so it will be OK. But let's do it anyway just to make sure:

curl **E** = $\hat{\boldsymbol{\theta}}\ [(1/2)\ a\ I\ R_{dc}\ [J_1(\beta' r)/J_1(\beta' a)]\ (\beta_d^2 + \beta'^2)]$ from above

$-j\omega\mathbf{B} = (-j\omega)\ \hat{\boldsymbol{\theta}}\ [(-1/j\omega)\ (1/2)\ a\ I\ R_{dc}\ [J_1(\beta' r)/J_1(\beta' a)]\ (\beta_d^2 + \beta'^2)]$

and yes we are OK.

(4) curl **H** = $\partial_t \mathbf{D} + \mathbf{J}$
or
curl **B** = $j\omega\mu\varepsilon\ \mathbf{E} + \mu\sigma\mathbf{E} = (\mu\sigma + j\omega\mu\varepsilon)\ \mathbf{E}$

or

$$\text{curl } \mathbf{B} = \mu(\sigma+j\omega\varepsilon)\,\mathbf{E}$$

This is the most dangerous one (my experience of failures), I hope it works. Start on the left,

$$\begin{aligned}\text{curl } \mathbf{B} &= \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta B_z - \partial_z B_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z B_r - \partial_r B_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB_\theta) - r^{-1}\partial_\theta B_r\,] \\ &= \hat{\mathbf{r}}\,[-\partial_z B_\theta] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB_\theta)] \\ &= \hat{\mathbf{r}}\,[+j\beta_d B_\theta] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB_\theta)]\end{aligned}$$

Compute the second term:

$$B_\theta = (-1/j\omega)(a/2)\, I\, R_{dc}\, [J_1(\beta' r)/ J_1(\beta' a)]\, (\beta^2)$$

$$rB_\theta = (-1/j\omega)(a/2)\, I\, R_{dc}(1/ J_1(\beta' a))\, (\beta^2)\ [r\, J_1(\beta' r)\,]$$

$$\partial_r(rB_\theta) = (-1/j\omega)(a/2)\, I\, R_{dc}(1/ J_1(\beta' a))\, (\beta^2)\, [\, r\, \beta' J_1'(\beta' r) + J_1(\beta' r)\,]$$

$$r^{-1}\partial_r(rB_\theta) = (-1/j\omega)(a/2)\, I\, R_{dc}(1/ J_1(\beta' a))\, (\beta^2)\, [\ \beta' J_1'(\beta' r) + r^{-1}\, J_1(\beta' r)\,]$$

Therefore we have

$$[\text{curl } \mathbf{B}]_z = (-1/j\omega)(a/2)\, I\, R_{dc}(1/ J_1(\beta' a))\, (\beta^2)\, [\ \beta' J_1'(\beta' r) + r^{-1}\, J_1(\beta' r)\,]$$

Our Maxwell says this is supposed to be equal to

$$\mu(\sigma+j\omega\varepsilon)E_z = \mu(\sigma+j\omega\varepsilon)\ (1/2)\ (a\beta')\ I\, R_{dc}\, [\, J_0(\beta' r)/ J_1(\beta' a)\,]$$

It does not look promising. We now compare the two sides

$$(-1/j\omega)(a/2)\, I\, R_{dc}(1/ J_1(\beta' a))\, (\beta^2)\, [\ \beta' J_1'(\beta' r) + r^{-1}\, J_1(\beta' r)\,]$$
$$= \mu(\sigma+j\omega\varepsilon)\ (1/2)\ (a\beta')\ I\, R_{dc}\, [\, J_0(\beta' r)/ J_1(\beta' a)\,] \qquad ?$$

$$(-1/j\omega)\ (\beta^2)\, [\ \beta' J_1'(\beta' r) + r^{-1}\, J_1(\beta' r)\,]$$
$$= \mu(\sigma+j\omega\varepsilon)\ (\beta')\ J_0(\beta' r)\,] \qquad ?$$

$$(-1/j\omega)\ (\beta^2)\, [\ \beta' J_1'(\beta' r) + r^{-1}\, J_1(\beta' r)\,] \ = \ \mu(\sigma+j\omega\varepsilon)\ (\beta')\ J_0(\beta' r)\,] \qquad ?$$

Now we know that

$$\beta^2 = \omega^2\mu\xi = \omega^2\mu\,[\varepsilon + \sigma/(j\omega)] = \omega^2\mu/(j\omega) * [j\omega\varepsilon + \sigma] \ = \ -j\omega\,\mu(\sigma+j\omega\varepsilon)$$

which says

$$\mu(\sigma+j\omega\varepsilon) = (-1/j\omega)\ \beta^2$$

So now our test is this:

$$(-1/j\omega)\,(\beta^2)\,[\;\beta' J_1'(\beta' r) + r^{-1} J_1(\beta' r)\;] \;=\; (-1/j\omega)\,\beta^2\,(\beta')\,J_0(\beta' r)\;] \qquad ?$$

$$[\;\beta' J_1'(\beta' r) + r^{-1} J_1(\beta' r)\;] \;=\; (\beta')\,J_0(\beta' r)\;] \qquad ?$$

$$\beta'\,[J_1'(\beta' r) + (r\beta')^{-1} J_1(\beta' r)\;] \;=\; (\beta')\,J_0(\beta' r)\;] \qquad ?$$

$$[J_1'(\beta' r) + (r\beta')^{-1} J_1(\beta' r)\;] \;=\; (J_0(\beta' r)\;] \qquad ?$$

$$[J_1'(x) + x^{-1} J_1(x)\;] \;=\; J_0(x) \qquad ?$$

$$x J_1'(x) + J_1(x) \;=\; x J_0(x) \qquad ?$$

This is the same Spiegel 24.19 that I used above, so success for the $\hat{\mathbf{z}}$ component of curl **B** = $\mu(\sigma+j\omega\varepsilon)$ **E.**

Now let's examine the $\hat{\mathbf{r}}$ component of this Maxwell equation:

$$[\text{curl }\mathbf{B}]_r = +j\beta_d B_\theta = (j\beta_d)\,(-1/j\omega)(a/2)\,I\,R_{dc}\,[J_1(\beta' r)/\,J_1(\beta' a)]\,(\beta^2)$$

$$= -\,(\beta_d)\,(1/\omega)(a/2)\,I\,R_{dc}\,[J_1(\beta' r)/\,J_1(\beta' a)]\,(\beta^2)$$

Max ways this is supposed to equal

$$\mu(\sigma+j\omega\varepsilon)\,E_r = \mu(\sigma+j\omega\varepsilon)\,(j/2)\,(a\beta_d)\,I\,R_{dc}\,[J_1(\beta' r)/\,J_1(\beta' a)]$$

So we then have another equation to test out:

$$-\,(\beta_d)\,(1/\omega)(a/2)\,I\,R_{dc}\,[J_1(\beta' r)/\,J_1(\beta' a)]\,(\beta^2)$$
$$= \mu(\sigma+j\omega\varepsilon)\,(j/2)\,(a\beta_d)\,I\,R_{dc}\,[J_1(\beta' r)/\,J_1(\beta' a)] \qquad ?$$

$$-\,(\beta_d)\,(1/\omega)(a/2)\,(\beta^2)$$
$$= \mu(\sigma+j\omega\varepsilon)\,(j/2)\,(a\beta_d) \qquad ?$$

$$-\,(\beta_d)\,(1/\omega)(a/2)\,(\beta^2) \;=\; \mu(\sigma+j\omega\varepsilon)\,(j/2)\,(a\beta_d) \qquad ?$$

$$-\;(1/\omega)\,(\beta^2) \;=\; \mu(\sigma+j\omega\varepsilon)\,j \qquad ?$$

Recall from above that $\mu(\sigma+j\omega\varepsilon) = (-1/j\omega)\,\beta^2$ so we continue

$$-\;(1/\omega)\,(\beta^2) \;=\; (-1/j\omega)\,\beta^2 j \qquad ?$$

$$-\;(1/\omega)\,(\beta^2) \;=\; (-1/\omega)\,\beta^2 \qquad ?$$

and again agreement is exact so success for the $\hat{\mathbf{r}}$ component of curl **B** = μ(σ+jωε) **E.**

Conclusions on Verification of the four Maxwell equations.

(1) div **B** = 0 exactly with no assumptions.

(2) div **E** = 0 exactly with no assumptions.

(3) curl **E** = -jω**B** because that is how we compute **B**

(4) curl **B** = μ(σ+jωε) **E** with no assumptions

Conclusion: The ansatz $e^{j(\omega t-\beta_d z)}$ yields a solution of the round wire interior which satisfies all four of Maxwell's equations with no approximations. It works for all frequencies

3. What exactly does the condition $\beta_d^2 << |\beta|^2$ imply?

This assumption is not made in Appendix D nor above, but *is* made in Chapter 2.

$\beta_d^2 << |\beta|^2$

$\beta_d^2 = \omega^2\mu_d\varepsilon_d$ // I assume all is real to keep it simple

$\beta^2 = \omega^2\mu\xi = \omega^2\mu\,[\varepsilon + \sigma/(j\omega)]$

$|\beta^2|^2 = (\omega^2\mu)^2\,[\varepsilon - j\,\sigma/(\omega)]\,[\varepsilon + j\,\sigma/(\omega)] = (\omega^2\mu)^2\,[\,\varepsilon^2 + \sigma^2/\omega^2]$

$|\beta^2| = \omega^2\mu\,\sqrt{\varepsilon^2 + \sigma^2/\omega^2}$

So condition is

$\beta_d^2 << |\beta|^2$

$\omega^2\mu_d\varepsilon_d << \omega^2\mu\,\sqrt{\varepsilon^2 + \sigma^2/\omega^2}$

$\mu_d\varepsilon_d << \mu\,\sqrt{\varepsilon^2 + \sigma^2/\omega^2}$

$(\mu_d\varepsilon_d)^2 << \mu^2(\varepsilon^2+\sigma^2/\omega^2)$

$(\mu_d\varepsilon_d)^2 - (\mu\varepsilon)^2 << \mu^2\,\sigma^2/\omega^2$

$[(\mu_d\varepsilon_d)^2 - (\mu\varepsilon)^2]^{-1} >> \omega^2/\,\mu^2\,\sigma^2$

$$\omega^2 << \mu^2 \sigma^2 [(\mu_d \varepsilon_d)^2 - (\mu\varepsilon)^2]^{-1}$$

$$\omega^2 << \frac{\mu^2 \sigma^2}{(\mu_d \varepsilon_d)^2 - (\mu\varepsilon)^2}$$

$$\omega^2 << \frac{\sigma^2}{([\mu_d/\mu]\varepsilon_d)^2 - \varepsilon^2}$$

Now typically we will have $\mu = \mu_d = \mu_0$ in which case this says

$$\omega^2 << \frac{\sigma^2}{(\varepsilon_d)^2 - \varepsilon^2}$$

At this point I don't know what to say about ε . This result seems wrong to me. Well what is the *largest* that the denominator might be, because this controls the bound. Perhaps $\varepsilon = \varepsilon_0$ and $\varepsilon_d = 2.3\ \varepsilon_0$ so then

$$\omega^2 << \frac{\sigma^2}{(2.3)^2(\varepsilon_0)^2 - \varepsilon_0^2} = 1/(2.3)^2 * \sigma^2/\varepsilon_0^2 = 0.19\ \sigma^2/\varepsilon_0^2$$

$\omega << 0.43\ \sigma/\varepsilon_0$ // so this undoes the squaring I did above, regarding <<

$\omega << 0.43\ [5.81 \times 10^7$ mho/m]/ $[8.8541877 \times 10^{-12}$ farad/m]

$= 0.43\ (5.8/8.6)\ 10^{19}$ mho/m * m/farad // mho/F = sec^{-1}

$\approx 0.3 \times 10^{19}\ \text{sec}^{-1}$

$2\pi f << 0.3 \times 10^{19}$

$f << (1/2\pi)\ 0.3 \times 10^{19} = (1/2\pi)30 \times 10^{17} = 4.8 \times 10^{17} = 4.8 \times 10^8$ GHz

So as long as we keep the frequency below 1/2 billion GHz, our condition is met.

Let's start over on the above like this:

$$\beta^2 = \omega^2\mu\xi = \omega^2\mu\ [\varepsilon + \sigma/(j\omega)]$$

I argue below (2.2.3) that as long as f << one billion GHz, we can neglect the ε term. Then we have

$$\beta^2 \approx \omega^2\mu\ [\sigma/(j\omega)]$$

$|\beta^2| \approx \omega\mu\sigma$ // sec^{-1}* henry/m * mho/m = sec^{-1} ohm-sec * mho/m = $1/\text{m}^2$ OK

Then my I get

$\beta_d^2 << |\beta|^2$

$\omega^2 \mu_d \varepsilon_d << \omega \mu \sigma$

$\omega \mu_d \varepsilon_d << \mu \sigma$

$\omega << \mu\sigma/(\mu_d \varepsilon_d)$

Now let us suppose the $\mu_d = \mu$. This condition is then

$\omega << \sigma/(\varepsilon_d)$

Let's now evaluate both sides for copper, assuming roughly that $\varepsilon_d = \varepsilon_0$.

RHS = σ/ε_0 = [5.81 x 10^7 mho/m] / [8.8541877 x 10^{-12} farad/m]

= (5.8/8.6) 10^{19} mho/m * m/farad // mho/F = sec^{-1}

= (5.8/8.6) 10^{19} sec^{-1}

So again we get

$2\pi f << (5.8/8.6)\ 10^{19}$

$f << (58/8.6*2\pi)\ 10^{18} = 1.07 \times 10^{18} \approx 10^9$ GHz

But this is the same value and the same calculation as the first.

Conclusion: Assuming $\mu = \mu_d = \mu_0$ and $\varepsilon = \varepsilon_0$ and $\varepsilon_d = 2.3\ \varepsilon_0$ we conclude that if

$f << 5 \times 10^8$ GHz // note: typically $f << 100$GHz

then we will have

$\beta_d^2 << |\beta|^2$

4. Did I assume $\beta_d^2 << |\beta|^2$ somewhere in my Appendix D development? Answer is no!

Question: What assumptions did I make in doing the Appendix D solution?

D.1
(a) Only assumption here is ansatz of $e^{j(\omega t - \beta_d z)}$ for everything.
(b) no new assumptions here, so Summary equations are exact

(c) no new assumptions here
(d) Here I note that $\beta'^2 = \beta^2 - \beta_d^2$ and I say we *could* ignore the distinction between β and β', but I continue to use β' which maintains that distinction! So this is a little warning. In Chapter 2 everything is done with β, but App D shows that it is really β'^2 which appears in most equations.

D.2
(a) no new assumptions made
(b) no new assumptions made, but m = 0 gets treated as a special case
(c) no new assumptions made

D.3 that third H equation, no new assumptions made

D.4. no new assumptions made, and here I have the summary box!

So to answer the question: the only assumption make is $e^{j(\omega t - \beta_d z)}$. I do not assume β' = β, I do not assume that $\beta_d \ll \beta$ or anything like that. So maybe $e^{j(\omega t - \beta_d z)}$ is not an exactly realizable ansatz! [and that is my conclusion #2 above]

Appendix D summary (just for fun): Here is what App D does:

(1) makes the $e^{-j\beta_d z}$ ansatz for the z dependence of all functions

(2) does a PWE to replace azimuthal variable φ by m.

(3) writes out the Vector Helmholtz Equation for E in each coordinate

(4) writes out the div **E** = 0 equation as well

(5) derives a charge pumping boundary condition at r = a which relates E_r there to N_m which is the amount of charge in that mode. [but this condition is only used later for constant eval on a_m and K_m]

(6) solves first for $E_z(r,m)$ and shows it has the form $C_{zm} J_m(\beta' r)$ which agrees with Chap 2.

(7) finds the E_r solution using both its Helm and using div E = 0. Get solution for $E_r(r,m)$ as (D.2.11).

(8) finds the $E_\varphi(r,m)$ solution as (D.2 15). At this points, we have constants a_m and K_m .

5. My strange E_φ boundary condition.

I then DO make a new ansatz assumption which is that $E_\varphi = 0$ at r = a. How do I justify that? I really have no justification other than "it would be nice". This ansatz is used in part to determine those constant. I then end up with the "complete solution".

Comments on Appendix D:
- I have no external verification for this complete solution

- The E_z does agree with my Chapter 2, but THAT had no justification either, maybe a little Matick. Yes, Matick confirms my expression for $E_z = E(r)$ exactly, so that is worth a lot to me.
- my BC that $E_\varphi = 0$ at r = a seems totally *out of the blue*
- I think the charge-pumping other BC is basically OK.
- everything is suspect when m = 0, so that needs a solid review

Even when I make the assumption $E_\varphi = 0$ at r = a, I still get this weird solution for $m \neq 1$.

$$E_\varphi(r,m) = (1/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\, \left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right]$$

So it may be zero on the surface, but it is swirling around inside. Could the above expression be 0 and I failed to notice it? Is it possible that

$$\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} = \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)}$$

or

$$J_{m+1}(x)\,[\,1/J_{m+1}(x_a) + 1/J_{m-1}(x_a)\,] = 2mx^{-1}J_m(x)/\,J_{m-1}(x_a)$$

or

$$J_{m+1}(x)\,[J_{m-1}(x_a) + J_{m+1}(x_a)\,]/[\,J_{m+1}(x_a)\,J_{m-1}(x_a)\,] = 2mx^{-1}J_m(x)/\,J_{m-1}(x_a)$$

or

$$J_{m-1}(x_a)J_{m+1}(x)\,[J_{m-1}(x_a) + J_{m+1}(x_a)\,] = J_{m+1}(x_a)\,J_{m-1}(x_a)\,2mx^{-1}J_m(x)$$

The only Bessel rule I could use is this (only one with no derivatives)

$$J_{m+1}(x) = (2m/x)J_m(x) - J_{m-1}(x) \quad \Rightarrow \quad J_{m+1}(x) + J_{m-1}(x) = (2m/x)J_m(x)$$
$$J_{m+1}(x_a) = (2m/x_a)J_m(x_a) - J_{m-1}(x_a) \quad \Rightarrow \quad J_{m+1}(x_a) + J_{m-1}(x_a) = (2m/x_a)J_m(x_a)$$

Using the lower right we get

$$J_{m-1}(x_a)J_{m+1}(x)\,[(2m/x_a)J_m(x_a)\,] = J_{m+1}(x_a)\,J_{m-1}(x_a)\,2mx^{-1}J_m(x)$$

$$J_{m+1}(x)\,[(1/x_a)J_m(x_a)\,] = J_{m+1}(x_a)\,x^{-1}J_m(x)$$

$$J_{m+1}(x)\;\; J_m(x_a)/x_a = J_{m+1}(x_a)\,J_m(x)/x$$

This is clearly not true for general x, only when $x = x_a$.

<u>So no, and it is true that E_φ does swirl around inside the wire.</u> And I just happen to force it to be 0 on the surface. Again, no justification whatsoever!!!

Here is 100% confirmation from Maple (assuming my Appendix D result for $E_\varphi i$ is correct:

```
m := 3;
```

$$m := 3$$

```
alias(J = BesselJ);
```

$$I, J$$

```
f := x*diff(J(1,x),x) - x*J(0,x) + J(1,x);
```

$$f := x\left(\mathrm{J}(0,x) - \frac{\mathrm{J}(1,x)}{x}\right) - x\,\mathrm{J}(0,x) + \mathrm{J}(1,x)$$

```
simplify(%);
```

$$0$$

```
t1 := -2*m*(1/x)*J(m,x)/J(m-1,xa);
```

$$t1 := -6\,\frac{\mathrm{J}(3,x)}{x\,\mathrm{J}(2,xa)}$$

```
t2 := J(m+1,x)/J(m+1,xa);
```

$$t2 := \frac{\mathrm{J}(4,x)}{\mathrm{J}(4,xa)}$$

```
t3 := J(m+1,x)/J(m-1,xa);
```

$$t3 := \frac{\mathrm{J}(4,x)}{\mathrm{J}(2,xa)}$$

```
g := t1+t2+t3;
```

$$g := -6\,\frac{\mathrm{J}(3,x)}{x\,\mathrm{J}(2,xa)} + \frac{\mathrm{J}(4,x)}{\mathrm{J}(4,xa)} + \frac{\mathrm{J}(4,x)}{\mathrm{J}(2,xa)}$$

```
xa := 2.4;
```

$$xa := 2.4$$

```
x := 2;
```

$$x := 2$$

```
evalf(g);
```

$$-.2900307285$$

This shows
- how to use the alias idea to simplify everything
- the fact that Maple does really know about J_m relationships (the 0 after simplify)
- that $E_\varphi(r,m) \neq 0$. I test m = 3 use x = 2 and x_a = 2.4 and I DON'T get 0.

6. Would it be better to say maybe that $E_\varphi(r=0,m) = 0$?

How can you have an E field NOT obeying this condition? Consider:

$$\text{curl } \mathbf{E} = -\partial_t \mathbf{B} \quad \Leftrightarrow \quad \oint_C \mathbf{E} \bullet d\mathbf{s} = -\partial_t\left[\int_S \mathbf{B} \bullet d\mathbf{S}\right] \qquad (1.1.36)$$

Apply to tiny loop of radius r centered at r = 0:

$$E_\varphi\, 2\pi r = -j\omega\,(\pi r^2) B_z \quad \Rightarrow \quad E_\varphi = -j\omega(1/2)\, r\, B_z \rightarrow 0$$

Also, what possible direction can E_φ point in at the origin of a polar system? There is no vortex line there. So yes, I think $E_\varphi(r=0,m) = 0$ is a much better choice.

But, I see that my existing solution meets this condition already:

$$jE_\varphi(r,m) = - a_m\ x^{-1}\ J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x)$$

I can see at once that $a_0 = 0$. Then we have

$$jE_\varphi(r,0) = \left(\frac{K_0}{2} \right) J_1(x) \sim \left(\frac{K_0}{2} \right) x \to 0$$

$$jE_\varphi(r,m) = - a_m\ x^{-1}\ J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x)$$

$$\approx a_m\ x^{-1}(x/2)^m/m! + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) (x/2)^{m+1}/(m+1)! \qquad \text{for all } m \geq 0 \text{ in fact}$$

$$\approx a_m(1/2)\ (2/x)\ (x/2)^m/m! + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) (x/2)^{m+1}/(m+1)!$$

$$\approx a_m(1/2)\ (x/2)^{m-1}/m! + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) (x/2)^{m+1}/(m+1)! \qquad m > 0$$

For m = 1 this says, as x→ 0

$$jE_\varphi(r,1) = a_1(1/2) + \left(\frac{K_1}{2} + \frac{a_1}{1} \right) (x/2)^2/(2)! = a_1(1/2)$$

For m > 1 we get $jE_\varphi(r,m) = 0$ since both terms vanish for any coefficients.

So consideration of this limit r → 0 gives a very strange result : $jE_\varphi(r\to 0,m) = 0$ for all m except for m = 1 in which the limit is $a_1/2$. By my argument above, I think this requires that $a_1 = 0$.

But the point is this: requiring $jE_\varphi(r\to 0,m) = 0$ does not nail down anything other than maybe a_1 , so it is not a very valuable BC.

7. So now I am stuck with a new question: how to compute a_m and K_m ?

Question: Starting at equation (D.2.19), assume no errors were made above it, how do you determine the coefficients a_m and K_m ? Basically since I don't like my (D.2.19) BC, the tail end of this Appendix D has now fallen apart. All of App D is OK except for this final part.

Review: K_m came from the E_z solution which is just some constant in effect times the J solution of this equation, just a scaling factor. Constant a_m came from the fact that there is an arbitrary homo solution to the E_r equation that you have to add in! That is then where (D.2.11) comes from.

Here are my solutions without using any BC's (except I can see that $a_0 = 0$ is required in last below)

$$K_m \equiv 2j\,(\beta_d/\beta')\,C_{zm}\,. \qquad \text{(D.2.5)}$$

$$E_z(r,m) = C_{zm}\,J_m(\beta' r) \;=\; K_m\,(\beta'/\beta_d)(1/2j)\,J_m(\beta' r) \qquad \text{(D.1.27)}$$

$$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)\;. \qquad \text{(D.2.11)}$$

$$jE_\varphi(r,m) = -\,a_m\,x^{-1}\,J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad x = \beta' r\;. \qquad \text{(D.2.15)}$$

I do show above (D.4.1) that

$$I = -j(\beta'\beta_d)^{-1}\,2\pi\sigma\,\frac{K_0}{2}\,[\,x_a\,J_1(x_a)\,] \;=\; -j(\beta'\beta_d)^{-1}\,\pi\sigma\,K_0\,[\,x_a\,J_1(x_a)\,]$$

$$= -j(1/\beta_d)\,(1/\beta')\,\pi\sigma\,K_0\,[\beta' a\,J_1(\beta' a)\,]$$

$$= -j(1/\beta_d)\,\pi a\sigma\,K_0\,J_1(\beta' a)\,]$$

so this does determine K_0 and I have

$$K_0 = jI\,\beta_d/(\pi\sigma a) * 1/\,J_1(\beta' a)$$

So I think for the m=0 fields I get

$$E_z(r,0) \;=\; \{\,jI\,\beta_d/(\pi\sigma a) * 1/\,J_1(\beta' a)\}\,(\beta'/\beta_d)(1/2j)\,J_0(\beta' r)$$

$$= \{\,I/(2\pi\sigma a)\,\}\,(\beta')\,J_0(\beta' r)/\,J_1(\beta' a) \;= (1/2)\,(a\beta')\,I\,R_{dc}\,[\,J_0(\beta' r)/\,J_1(\beta' a)\,]$$

which compare to chapter 2

$$E(r) = [\mu I/(2\pi a)]\;\frac{J_0(\beta r)}{J_1(\beta a)}\;(-j\omega/\beta)$$

Now use approximations that $\beta^2 = -j\omega\mu\sigma$ (as used in Ch 2) and $\beta' = \beta$ to write

$$\beta'^2 = \;-j\omega\mu\sigma \qquad \Rightarrow \qquad \beta' = (1/\beta')[\,-j\omega\mu\sigma] \qquad \Rightarrow (1/\beta') \;= \beta' /\;[\,-j\omega\mu\sigma]$$

$$\Rightarrow\; (-j\omega/\beta') \;= \beta' / [\mu\sigma]$$

and using these facts, rewrite the Ch 2 result as

$$E(r) = [\mu I/(2\pi a)]\;\frac{J_0(\beta' r)}{J_1(\beta' a)}\;(-j\omega/\beta')$$

$$= [\mu I/(2\pi a)]\;\frac{J_0(\beta' r)}{J_1(\beta' a)}\;\beta' / [\mu\sigma]$$

$$= [I/(2\pi a\sigma)] \frac{J_0(\beta' r)}{J_1(\beta' a)} \beta'$$

and thus the Chapter 2 result agrees with the Appendix D result, and to get this agreement I assumed no boundary conditions at all except relating K_0 to I and setting $a_0 = 0$. Thus, the boundary condition part of Appendix D is NOT NEEDED to get a justification of the Chapter 2 result.

What about the other two fields for m = 0?

$$E_r(r,0) = \frac{K_0}{2} J_1(x) \quad . \qquad \text{(D.2.11)}$$

$$= (1/2) \{jI \beta_d/(\pi\sigma a) * 1/ J_1(\beta' a)\} J_1(\beta' r)$$

$$= (1/2) \{jI \beta_d/(\pi\sigma a) \ [J_1(\beta' r)/ J_1(\beta' a)]$$

$$= (1/2) \{jI (a\beta_d)/(\pi\sigma a^2) \ [J_1(\beta' r)/ J_1(\beta' a)]$$

$$= (j/2) (a\beta_d) I R_{dc} [J_1(\beta' r)/ J_1(\beta' a)]$$

where R_{dc} is the resistance per unit length of the wire. Agrees with my original box D.4.6.

Next we have

$$jE_\varphi(r,m) = - a_m \ x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x)$$

$$\partial(r,0) = C_{\varphi 0} J_1(\beta' r) \qquad \text{(D.2.17)}$$

where in this last, $C_{\varphi 0}$ is another unknown constant, it is not related to any other constants at this point.

8. What should I do about boundary conditions then for m > 0 ?

When I write

$$E_r(r=a,m) = (j\omega/\sigma) N_m \qquad \text{(D.2.18)}$$

which implies

$$a_m \ x_a^{-1} J_m(x_a) + \frac{K_m}{2} J_{m+1}(x_a) = (j\omega/\sigma) N_m \qquad (1)$$

I am merely trading into the new set of unknown constants N_m which are the charge moments which are unknown. So in a way, this BC doesn't really do anything for us.

Perhaps we really have to tie in with the exterior solution in order to nail down K_m and a_m .
This is something I could work on.

Another task would be to think again about those Maxwell equations being satisfied, where I do everything in terms of the a_m and K_m constants. Once again here is my solution for **E**(r.m)

$$E_z(r,m) = C_{zm}\, J_m(\beta' r) \;=\; K_m\,(\beta'/\beta_d)(1/2j)\, J_m(\beta' r) \qquad \text{(D.1.27)}$$

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x) \;. \qquad \text{(D.2.11)}$$

$$jE_\varphi(r,m) = -\, a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad x = \beta' r \;. \qquad \text{(D.2.15)}$$