App D Actions 2 PhL 11.30.13

This material is all in Appendix D now. I did a few checks to show new App D agrees with B calculations that were done here.

1. Our first task is to compute the B field 1

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I showed in App D Actions 1 that when I computed the corresponding B field, I obtained a pair of fields E and B which satisfies all four of Maxwell's equations, but that was only for m = 0.

Here I would like to repeat that same process but for general m. I want to do this without assuming any boundary conditions. My E fields from App D are stated before any BC's are applied as"

$$jE_\varphi(r,m) = -a_m\, x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad x = \beta' r \quad . \tag{D.2.15}$$

$$E_r(r,m) = a_m\, x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x) \quad . \tag{D.2.11}$$

$$E_z(r,m) = C_{zm} J_m(\beta' r) \tag{D.1.27}$$

$$K_m \equiv 2j\,(\beta_d/\beta')\, C_{zm} \tag{D.2.5}$$

From the last we get

$$C_{zm} = (1/2j)(\beta'/\beta_d)K_m = -j\,(\beta'/\beta_d)\frac{K_m}{2}$$

Thus without any assertion of boundary conditions, our E field App D solutions for general m > 0 are

$$E_z(r,m) = -j\,(\beta'/\beta_d)\frac{K_m}{2} J_m(x) \qquad x = \beta' r$$

$$E_r(r,m) = a_m\, x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x)$$

$$jE_\varphi(r,m) = -a_m\, x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)$$

1. Our first task is to compute the B field

Now you have to remember that there is an outside factor of

$$e^{j(\omega t - \beta_d z)}\, e^{jm\varphi}$$

so now we have two rules

$\partial_z \rightarrow -j\beta_d$
$\partial_\theta \rightarrow +jm$

We identify θ and φ as the same variable.

$\mathbf{B} = (-1/j\omega)\ \mathrm{curl}\ \mathbf{E}$

$\mathrm{curl}\ \mathbf{E} = \hat{\mathbf{r}}\ [\ r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\ [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\ ]$

$= \hat{\mathbf{r}}\ [\ r^{-1}jmE_z + j\beta_d E_\theta] + \hat{\boldsymbol{\theta}}\ [-j\beta_d E_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\ ]$

Maple has computed these three components, I won't copy them here right now but they have this form

$B = B_r(r)\ \hat{\mathbf{r}} + B_\theta(r)\ \hat{\boldsymbol{\theta}} + \mathbf{B}_z(r)\ \hat{\mathbf{z}}$

2. Our next task is to check the four Maxwell equations

(1) div **B** = 0.

$\mathrm{div}\ \mathbf{B} = r^{-1}\partial_r(rB_r) + r^{-1}\partial_\theta B_\theta + \partial_z B_z$
$= r^{-1}\partial_r(rB_r) + r^{-1}jmB_\theta - j\beta_d B_z$

Maple confirms that div **B** = 0, I get to use my unapply operator, seems good.

(2) div **E** = 0.

$\mathrm{div}\ \mathbf{E} = r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z$
$= r^{-1}\partial_r(rE_r) + r^{-1}jmE_\theta - j\beta_d E_z$

Again Maple confirms. We just have verification within each partial wave since it could be activated all by itself.

(3) curl **E** = -jω**B**, but this is how we produced B in the first place so OK.

(4) curl **B** = μ(σ+jωε) **E**

This one is the Big Enchilada. I should at the start make this replacement

$\mu(\sigma + j\omega\varepsilon) = (-1/j\omega)\ \beta^2$

so what I really need to show is this: $\beta'^2 + \beta_d^2 = \beta^2$

$$\text{curl } \mathbf{B} = (-1/j\omega)\,\beta^2\,\mathbf{E} \;=\; (-1/j\omega)(\,\beta'^2 + \beta_d^2)\,\mathbf{E}$$

We start then with

$$\text{curl } \mathbf{B} = \hat{\mathbf{r}}\,[\,r^{-1}\partial_\theta B_z - \partial_z B_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z B_r - \partial_r B_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rB_\theta) - r^{-1}\partial_\theta B_r\,]$$

$$= \hat{\mathbf{r}}\,[\,r^{-1}jmB_z + j\beta_d B_\theta] + \hat{\boldsymbol{\theta}}\,[-j\beta_d B_r - \partial_r B_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rB_\theta) - r^{-1}jmB_r\,]$$

We want to show that

$$\text{curl } \mathbf{B} + (1/j\omega)\,\beta^2\,\mathbf{E} \;= 0$$

or

$$[\text{curlB}]_z + (1/j\omega)\,\beta^2\,E_z \;= 0$$
$$[\text{curlB}]_r + (1/j\omega)\,\beta^2\,E_r \;= 0$$
$$[\text{curlB}]_{th} + (1/j\omega)\,\beta^2\,E_{th} \;= 0$$

Maple confirms all three of these equations! This is a major happy thing for me right now.

3. Here are the B fields read off from the Maple computation

$$B_z = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(\beta' r)$$

$$B_r = (j/\omega)(\,r^{-1})[\,(\beta'/\beta_d)\,m\,\frac{K_m}{2} - (\beta_d/\beta')a_m\,]\,J_m(\beta' r) \;+\; (j/\omega)\beta_d\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_{m+1}(\beta' r)$$

$$B_\theta = -(1/\omega)(\,r^{-1})[\,(\beta'/\beta_d)\,m\,\frac{K_m}{2} - (\beta_d/\beta')a_m\,]\,J_m(\beta' r) + \;(1/\omega)\,\frac{K_m}{2}\,(\beta_d + \beta'^2/\beta_d)\,J_{m+1}(\beta' r)$$

Rewrite to simplify if possible // these three match box (D.4.9) of new App D

$$B_z = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(\beta' r)$$

$$B_r = (j/\omega)(\,r^{-1})[\,(\beta'/\beta_d)\,m\,\frac{K_m}{2} - (\beta_d/\beta')a_m\,]\,J_m(\beta' r) \;+\; (j/\omega)\beta_d\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_{m+1}(\beta' r)$$

$$B_\theta = -(1/\omega)(\,r^{-1})[\,(\beta'/\beta_d)\,m\,\frac{K_m}{2} - (\beta_d/\beta')a_m\,]\,J_m(\beta' r) + \;(1/\omega)\beta'\,\frac{K_m}{2}\,(\beta_d/\beta' + \beta'/\beta_d)\,J_{m+1}(\beta' r)$$

or

$$B_z = (1/\omega)\beta'\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x)$$

$$B_r = j\,(1/\omega)\,\beta_d[\,(\beta'^2/\beta_d^2)\,m\,\frac{K_m}{2} - a_m\,]\,J_m(x)/x + j\,(1/\omega)\beta_d\,(\frac{K_m}{2} + \frac{a_m}{m})\,J_{m+1}(x)$$

$$B_\theta = -\,(1/\omega)\,\beta_d[\,(\beta'^2/\beta_d^2)\,m\,\frac{K_m}{2} - a_m\,]\,J_m(x)/x + (1/\omega)\,\beta_d\,\frac{K_m}{2}\,(1 + \beta'^2/\beta_d^2)\,J_{m+1}(x)$$

I don't know how large the constants are, so not sure what terms are large.

For m = 0 these say // these match (D.6.1)

$$B_z = (1/\omega)\,\beta'\,(\frac{K_0}{2})J_0(x)$$

$$B_r = j\,(1/\omega)\beta_d\,(\frac{K_0}{2})\,J_1(x)$$

$$B_\theta = (1/\omega)\,\beta_d\,\frac{K_0}{2}\,(1 + \beta'^2/\beta_d^2)\,J_1(x)$$

For $\beta >> \beta_d$ we can write these as

$$B_z = (1/\omega)\,\beta_d\,(\beta'/\beta_d)\,(\frac{K_0}{2})J_m(x) \quad \text{medium}$$

$$B_r = j\,(1/\omega)\beta_d\,(\frac{K_0}{2})\,J_{m+1}(x) \quad \text{small}$$

$$B_\theta \approx (1/\omega)\,\beta_d\,(\beta'^2/\beta_d^2)\,\frac{K_0}{2}\,J_1(x) \quad \text{large}$$

I know from actions 1 v2 that

$$K_0 = jI\,\beta_d/(\pi\sigma a) * 1/\,J_1(\beta' a) = jI\,(a\beta_d)/(\pi\sigma a^2) * 1/\,J_1(\beta' a) = j\,(a\beta_d)\,I\,R_{dc}/\,J_1(\beta' a)$$

So my m=0 B field components become

$$B_z = (1/2\omega)\,\beta'\,(K_0)J_0(x)$$
$$B_r = j\,(1/2\omega)\beta_d\,(K_0)\,J_1(x)$$
$$B_\theta = (1/2\omega)\,\beta_d\,K_0\,(1 + \beta'^2/\beta_d^2)\,J_1(x)$$

or

$$B_z = j\,(1/2\omega)\,\beta'\,(a\beta_d)\,I\,R_{dc}J_0(x)\,/\,J_1(\beta' a)$$
$$B_r = -\,(1/2\omega)\,\beta_d\,(a\beta_d)\,I\,R_{dc}\,J_1(x)\,/\,J_1(\beta' a)$$
$$B_\theta = j\,(1/2\omega)\,\beta_d\,(a\beta_d)\,I\,R_{dc}\,(1 + \beta'^2/\beta_d^2)\,J_1(x)\,/\,J_1(\beta' a)$$

Again in the $\beta >> \beta_d$ limit these become

$$B_z = j\,(1/2\omega)\,\beta'\,(a\beta_d)\,I\,R_{dc}\,J_0(x)\,/\,J_1(x_a) \quad \text{medium}$$
$$B_r = -\,(1/2\omega)\,\beta_d\,(a\beta_d)\,I\,R_{dc}\,J_1(x)\,/\,J_1(x_a) \quad \text{small}$$

$B_\theta =$ $j\,(1/2\omega)\,\beta'(\beta'/\beta_d)\,(a\beta_d)\;I\,R_{dc}\,(J_1(x)/J_1(x_a)$ large

Now I can rewrite the last as

$B_\theta \approx$ $j\,(1/2\omega)\,\beta^2\,a\;I\,(1/\pi\sigma a^2)\,(J_1(x)/J_1(x_a) = j\,(1/2\omega)\,[\omega^2\mu\sigma/j\omega]\,I\,(1/\pi\sigma a)\,(J_1(x)/J_1(x_a)$

$= ([\mu]\,I\,(1/2\pi a)\,(J_1(x)/J_1(x_a)$

and this is the Chapter 2 result.

Here then is my general-m summary of all fields

$$E_z(r,m) = -j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x) \qquad x = \beta' r$$

$$E_r(r,m) = a_m\,J_m(x)/x + \frac{K_m}{2}\,J_{m+1}(x)$$

$$jE_\varphi(r,m) = -a_m\,J_m(x)/x + \left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_{m+1}(x)$$

$$(\omega/\beta')B_z(r,m) = \left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x)$$

$$(\omega/j\beta_d)B_r(r,m) = \left[\,(\beta'^2/\beta_d^2)\,m\,\frac{K_m}{2} - a_m\,\right]J_m(x)/x + \left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_{m+1}(x)$$

$$(\omega/\beta_d)B_\theta(r,m) = -\left[\,(\beta'^2/\beta_d^2)\;m\,\frac{K_m}{2} - a_m\,\right]J_m(x)/x + \frac{K_m}{2}\,(1+\beta'^2/\beta_d^2)\,J_{m+1}(x)$$

And they satisfy the four Maxwell equations exactly. I don't know how to better state these fields.